Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff

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Abstract

For regularized hard potentials cross sections, the solution of the spatially homogeneous Boltzmann equation without angular cutoff lies in Schwartz's space $\mathcal{S}(\mathbb{R}^N)$. The proof is presented in full detail for the two-dimensional case, and for a moderate singularity of the cross section. Then we present those parts of the proof for the general case, where the dimension, or the strength of the singularity play an essential role.

1 Introduction

We consider in this work the spatially homogeneous Boltzmann equation (Cf. [3])

$$\frac{\partial f}{\partial t}(t,v) = Q(f,f)(t,v)\,, \tag{1}$$

where $f : \mathbb{R}^N \to \mathbb{R}_+$ is the nonnegative density of particles which at time t move with velocity v, and the bilinear operator in the right-hand side is defined by

$$Q(g,f)(v) = \int_{\mathbb{R}^N}\int_{S^{N-1}} \Big\{ f(v')\,g(v'_*) - f(v)\,g(v_*) \Big\}\, b(\cos\theta, v - v_*)\, d\sigma dv_*. \tag{2}$$

In this formula, v', v'_* and v, v_* are the velocities of a pair of particles before and after a collision,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,$$
$$v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

where $\sigma \in S^{N-1}$. Throughout the paper, θ denotes the angle between σ and $v - v_*$.

Our goal is to prove that for a large class of collision cross sections b, and for all initial data $f_{in} \geq 0$ with finite mass, energy and entropy, i.e. satisfying the

Assumption 1 :

$$\int_{\mathbb{R}^N} f_{in}(v) \left(1 + |v|^2 + |\log(f_{in}(v))|\right) \, dv < +\infty \,,$$

there exists a solution of the homogeneous Boltzmann equation (1) with $f(0, \cdot) = f_{in}$ such that when $t > 0$, $f(t, \cdot) \in \mathcal{S}(\mathbb{R}^N)$. Apart from certain technical conditions that are discussed below, the main assumption on b is that near $\theta = 0$, it looks like $|\theta|^{-\gamma}$, with $1 < \gamma < 3$. Such a behavior, which naturally appears when the interaction between the particles has a long range, is called "non cutoff". It means that all the grazing collisions (those for which θ is close to 0) are taken into account. Under such a condition, the collision operator is expected to behave essentially as a fractional power of the Laplacian:

$$Q(f, f) = -C_f \, (-\Delta)^{(\gamma-1)/2} f + \text{more regular terms} \,,$$

where $C_f > 0$ depends only on quantities which are somehow controlled.

Relevant existence results were obtained in [2], [13], and [8]. Previous works have demonstrated partial regularity under rather general conditions (Cf. [1], [5], [6], [9], [10], [13]) or C^∞ regularity as here, but under severe restrictions on the equation : namely, the cross section did not depend on $v - v_*$ (Maxwellian molecules assumptions) and the solution was radially symmetric (Cf. [4]).

In order to keep the paper rather short, we refer to the quoted papers for a more complete history of the problem, and for discussions on the physical relevance.

The precise conditions that we impose on the cross section $b = b(\cos\theta, w)$ are the following:

Assumption 2 : We suppose that

$$\inf_{w \in \mathbb{R}^N, \, \theta \in [-\delta, \delta]} \frac{|\theta|^\gamma \, b(\cos\theta, w)}{(1 + |w|^2)^\alpha} \geq K \tag{3}$$

for some $K > 0$, $\alpha \in]0, 1[$, $\gamma \in]1, 3[$, $\delta \in]0, \pi[$, and that for all $p \in \mathbb{N}$,

$$\sup_{w \in \mathbb{R}^N, \, \theta \in [-\pi, \pi]} \frac{|\theta|^{3-\varepsilon} \, |D_p b(\cos\theta, w)|}{(1 + |w|^2)^{r_p}} \leq C_p^1, \tag{4}$$

where $\varepsilon > 0$ is a given number, $r_p, C_p^1 > 0$ are given constants, and D_p is any derivative of order p *with respect to the variable* w.

Note that the usual regularized hard potentials without angular cutoff satisfy Assumption 2; those are cross sections of the form

$$b(\cos\theta, w) = b_1(w)\, b_2(\theta),$$

where b_1 is a smooth and strictly positive function such that $b_1(w) \sim_{w\to+\infty} |w|^\alpha$ for $\alpha \in]0,1[$ and b_2 is a function such that $b_2(\theta) \sim_{\theta\to 0} |\theta|^{-\gamma}$ for $\gamma \in]1,3[$.

We can now state our main theorem :

Theorem 1 : Let b be a cross section satisfying Assumption 2 and f_{in} be an initial datum satisfying Assumption 1. Then, there exists a solution to eq. (1), (2) with initial datum f_{in} lying in $L^\infty([t_0, +\infty[; \mathcal{S}(\mathbb{R}^N))$ for all $t_0 > 0$.

Let us now rapidly discuss the assumptions and the conclusion of this theorem. Note first that the initial data are only assumed to belong to the space of functions satisfying the natural bounds coming from physics (finite mass, energy and entropy). It is likely that the assumption of finite entropy can be somewhat relaxed (Cf. recent works by Villani (Cf. [11])). The assumptions on the dependence with respect to θ of the cross section are also quite satisfying, and probably close to being optimal (Cf. [1] to get an idea of what really optimal assumptions might be). The situation is however not so good as far as the kinetic part of the cross section (that is, its dependence with respect to $v - v_*$) is concerned. First, we basically assume that this dependence is smooth, and this is not true for inverse power laws (Cf. [3] for example). While for such a cross section (having a singularity near $v = v_*$) some smoothing effect certainly occurs (Cf. [1] for example), it is not clear whether a complete smoothing of the solution can appear (the study of the Landau equation with that kind of cross section would suggest that the complete smoothing should indeed appear, Cf. [7]). Secondly, we also assume that (the kinetic part of) the cross section is strictly positive. This is also not true for inverse power laws. The study of this problem in [1] suggests that this assumption of strict positivity could maybe be relaxed. Most probably however, to look for a result for "true" hard potentials (that is, coming from inverse power laws) would lead to tremendous technicalities (the proof of our theorem is already quite technical), which we leave to future works.

Finally, we would like to put the stress on the following facts: the conclusion of Theorem 1 most probably does not hold when soft potentials or Maxwellian molecules are concerned (no gain of moments is expected in such a situation) or when the singularity in the angular variable is removed (no gain of smoothness is expected in this case). Under our assumptions, we think that maybe the solution of the Boltzmann equation is even smoother than $\mathcal{S}$, it might belong to some Gevrey space for example.

Since our computations are rather long, we first present the proof of Theorem 1 in the case when the dimension is $N = 2$ and when in Assumption 2, eq. (4), the term

$|\theta|^{3-\varepsilon}$ is replaced by $|\theta|^{2-\varepsilon}$ (that is, the singularity in the angular variable is moderately strong). These simplifications enable to present complete proofs: this is done in Section 2. Then, in Section 3, we explain how to modify the proof to get the result in the general case. Finally, for the sake of completeness, we give in an appendix the proof of a (more or less standard) interpolation lemma which is crucial in our proof.

2 The simplified case

In this section, we prove Theorem 1 in the case when the dimension is $N = 2$ and when eq. (4) in Assumption 2 is replaced by

$$\sup_{w\in\mathbb{R}^2,\,\theta\in[-\pi,\pi]} \frac{|\theta|^{2-\varepsilon}\,|D_p b(\cos\theta, w)|}{(1+|w|^2)^{r_p}} \le C_p^1, \tag{5}$$

where (as in eq. (4)) $\varepsilon > 0$ is a given number, $r_p, C_p^1 > 0$ are given constants, and D_p is any derivative of order p *with respect to* w.

When velocities are restricted to two dimensions, it is possible to parametrize the pre- and postcollisional velocities by a rotation of the relative velocity $v - v_*$:

$$v' = \frac{v+v_*}{2} + R_\theta\left(\frac{v-v_*}{2}\right), \tag{6}$$

$$v'_* = \frac{v+v_*}{2} - R_\theta\left(\frac{v-v_*}{2}\right), \tag{7}$$

where R_θ denotes a rotation by the angle θ. Then the integral over S^{N-1} in (2) can be replaced by $\int_{-\pi}^{\pi} d\theta$, which simplifies many of the subsequent calculations.

Thanks to the change of variable $\theta \mapsto \theta \pm \pi$, which exchanges v' and v'_*, the collision operator can be written

$$\begin{aligned} Q(f,f)(v) = \int_{\mathbb{R}^2}\int_{-\pi/2}^{\pi/2} &\Big\{ f(v')\,f(v'_*) - f(v)\,f(v_*) \Big\} \\ &\times \Big[b(\cos\theta, v-v_*) + b(\cos(\theta-\pi), v-v_*)\,1_{[0,\pi/2]}(\theta) \\ &+ b(\cos(\theta+\pi), v-v_*)\,1_{[-\pi/2,0]}(\theta) \Big]\, d\theta dv_*. \end{aligned}$$

As a consequence, it is enough (as far as the Boltzmann equation $\partial_t f = Q(f,f)$ is concerned) to assume that $b(\cos(\cdot), v-v_*)$ has its support included in $[-\pi/2, \pi/2]$.

We shall use in the sequel the following (easy) consequences of eq. (5) (C_p^2 and C_p^3 are constants which depend only on C_p^1) :

$$\sup_{w\in\mathbb{R}^2} \frac{\left| D_p \int_{-\pi/2}^{\pi/2} \left[b\Big(\cos\theta, \frac{w}{\cos\frac{\theta}{2}}\Big) \frac{1}{(\cos\frac{\theta}{2})^2} - b(\cos\theta, w) \right] d\theta \right|}{(1+|w|^2)^{r_p}} \le C_p^2, \tag{8}$$

$$\sup_{w\in\mathbb{R}^2,\, \theta\in[-\pi/2,\pi/2]} \frac{|\theta|^{2-\varepsilon} \left| D_p b\Big(\cos\theta, \frac{w}{\cos\frac{\theta}{2}}\Big) \right|}{(1+|w|^2)^{r_p}} \le C_p^3. \tag{9}$$

The proof of Theorem 1 (under the assumptions of this section) runs as follows: in Section 2.1, we split quantities like

$$\int_{\mathbb{R}^2} (D_k Q(g,f))(v)\, D_k f(v)\, dv$$

(where D_k is any derivative of order k) in a certain number of terms. Each term is then estimated in Section 2.2. Finally, we gather the estimates in Section 2.3 to conclude the proof of our theorem.

2.1 Decomposition

First, we observe that (6) and (7) imply the following useful formulas :

$$v = \frac{1}{\cos\frac{\theta}{2}}\, R_{-\frac{\theta}{2}} v' + \tan\frac{\theta}{2}\, R_{\frac{\pi}{2}} v_*, \tag{10}$$

$$v' - v = \sin\frac{\theta}{2}\, R_{\frac{\theta}{2}+\frac{\pi}{2}}(v - v_*), \tag{11}$$

$$v' - v_* = \cos\frac{\theta}{2}\, R_{\frac{\theta}{2}}(v - v_*). \tag{12}$$

It is in these formulas that the simplifications related to the two-dimensional case are obvious: For a *fixed* angle of deflection θ, all relations between velocities before and after collision are given by some fixed linear operator, and hence these transformations are smooth.

Denoting by τ_h the translation operator (that is, $\tau_h f(x) = f(x+h)$), we write down the invariance of Q with respect to translations in the form

$$(\tau_h Q(g,f))(v) = Q(\tau_h g, \tau_h f)(v)\,.$$

In the following we denote any generic differential operator of order k by D_k, and when it is necessary to indicate which variables it is acting on, we write $D_{k;v}$, for example.

A consequence of the translation invariance is that a Leibnitz formula holds for the collision operator: for any derivative D_k of order k, there exist derivatives D_l of order l such that

$$(D_k Q(g,f))(v) = Q(g, D_k f)(v) + \sum_{l=0}^{k-1} C_k^l \, Q(D_{k-l} g, D_l f)(v),$$

and

$$\int_{\mathbb{R}^2} (D_k Q(g,f))(v) \, D_k f(v) \, dv = \int_{\mathbb{R}^2} D_k f(v) \, Q(g, D_k f)(v) \, dv + \sum_{l=0}^{k-1} C_k^l \, B_{kl}, \quad (13)$$

where

$$B_{kl} = \int_{\mathbb{R}^2} D_k f(v) \, Q(D_{k-l} g, D_l f)(v) \, dv. \quad (14)$$

We then use the pre/post-collisional change of variables $v, v_*, \theta \mapsto v', v'_*, -\theta$, and get

$$\begin{aligned} B_{kl} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} D_k f(v) \Big\{ & D_l f(v') \, D_{k-l} g(v'_*) \\ & - D_l f(v) \, D_{k-l} g(v_*) \Big\} \, b(\cos\theta, v - v_*) \, d\theta dv_* dv \quad (15) \end{aligned}$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} \Big\{ D_k f(v') - D_k f(v) \Big\} D_l f(v) \, D_{k-l} g(v_*) \, b(\cos\theta, v - v_*) \, d\theta dv_* dv.$$

The next step in our calculation is to carry out the change of variables $w = v'$ (with the variables θ and v_* fixed; this has been used eg. in [1]). From (10) and (12), we get the following formulas,

$$w = \frac{v + v_*}{2} + R_\theta \left(\frac{v - v_*}{2} \right), \quad (16)$$

$$v = \frac{1}{\cos\frac{\theta}{2}} \, R_{-\frac{\theta}{2}} w + \tan\frac{\theta}{2} \, R_{\frac{\pi}{2}} v_*, \quad (17)$$

$$|w - v_*| = \cos\frac{\theta}{2} \, |v - v_*|, \quad (18)$$

$$dw = (\cos\frac{\theta}{2})^2 \, dv, \quad (19)$$

which then give

$$B_{kl} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} D_k f(w)\, D_l f\big(\frac{1}{\cos\frac{\theta}{2}}\, R_{-\frac{\theta}{2}} w + \tan\frac{\theta}{2}\, R_{\frac{\pi}{2}} v_*\big)$$

$$\times\, D_{k-l} g(v_*)\, b\big(\cos\theta, \frac{w - v_*}{\cos\frac{\theta}{2}}\big)\, \frac{1}{(\cos\frac{\theta}{2})^2}\, d\theta dv_* dw$$

$$- \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} D_k f(v)\, D_l f(v)\, D_{k-l} g(v_*)\, b(\cos\theta, v - v_*)\, d\theta dv_* dv.$$

Writing then v instead of w and using the notation

$$\tilde{v} = \frac{1}{\cos\frac{\theta}{2}}\, R_{-\frac{\theta}{2}} v + \tan\frac{\theta}{2}\, R_{\frac{\pi}{2}} v_*,$$

we get

$$B_{kl} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} D_k f(v)\, D_{k-l} g(v_*)$$

$$\times \Big[D_l f(\tilde{v})\, b\big(\cos\theta, \frac{v - v_*}{\cos\frac{\theta}{2}}\big)\, \frac{1}{(\cos\frac{\theta}{2})^2} - D_l f(v)\, b(\cos\theta, v - v_*) \Big]\, d\theta dv_* dv.$$

Next, the terms B_{kl} are split as $B_{kl} = B_{kl}^1 + B_{kl}^2$, where

$$B_{kl}^1 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} D_k f(v)\, [D_l f(\tilde{v}) - D_l f(v)]\, D_{k-l} g(v_*)$$

$$\times\, b\big(\cos\theta, \frac{v - v_*}{\cos\frac{\theta}{2}}\big)\, \frac{1}{(\cos\frac{\theta}{2})^2}\, d\theta dv_* dv$$

$$B_{kl}^2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} D_k f(v)\, D_l f(v)\, D_{k-l} g(v_*)$$

$$\times \Big[b\big(\cos\theta, \frac{v - v_*}{\cos\frac{\theta}{2}}\big)\, \frac{1}{(\cos\frac{\theta}{2})^2} - b(\cos\theta, v - v_*) \Big]\, d\theta dv_* dv.$$

2.2 Estimates

We introduce the weighted L^p and Sobolev spaces and their following norms:

Definition 1 *: For all $p \in [1, +\infty[$ and $r > 0$, we define the space $L^p_r(\mathbb{R}^N)$ by its norm*

$$||f||^p_{L^p_r(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |f(x)|^p\, (1 + |x|^2)^{rp/2}\, dx. \tag{20}$$

For all $k \in \mathbb{N}$ and $r > 0$, we also define the weighted Sobolev space $H^k_r(\mathbb{R}^N)$ by its norm

$$||f||^2_{H^k_r(\mathbb{R}^N)} = \sum_{|\alpha|\le k} \int_{\mathbb{R}^N} |\partial_\alpha f(x)|^2 \,(1+|x|^2)^r \, dx. \tag{21}$$

We begin with an estimate of the terms B^2_{kl} ($k \in \mathbb{N}$, $l = 0, 1, .., k-1$).

Lemma 1 *: For $k \in \mathbb{N}$, $l = 0, 1, .., k-1$, one has the estimate*

$$|B^2_{kl}| \le C^2_p \, ||g||_{L^1_{2r_p}} \, ||f||^2_{H^k_{r_p}}, \tag{22}$$

for some constant $C^2_p > 0$.

Proof :We recall that

$$B^2_{kl} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} D_k f(v) \, D_l f(v) \, D_{k-l} g(v_*) \tag{23}$$

$$\times \left[b\Big(\cos\theta, \frac{v-v_*}{\cos\frac{\theta}{2}}\Big) \, \frac{1}{(\cos\frac{\theta}{2})^2} - b(\cos\theta, v-v_*) \right] d\theta dv_* dv,$$

so that (after $k-l$ integrations by part, and denoting by $D_{k-l;2}$ a derivative of order $k-l$ with respect to the second variable ($v - v_*$))

$$\begin{aligned} |B^2_{kl}| \le & \Bigg| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} D_k f(v) \, D_l f(v) \, g(v_*) \\ & \times D_{k-l;2} \int_{-\pi/2}^{\pi/2} \left[b\Big(\cos\theta, \frac{v-v_*}{\cos\frac{\theta}{2}}\Big) \, \frac{1}{(\cos\frac{\theta}{2})^2} - b(\cos\theta, v-v_*) \right] d\theta \, dv_* dv \Bigg| \\ \le & C^2_p \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |D_k f(v)| \, |D_l f(v)| \, g(v_*) \, (1+|v-v_*|^2)^{r_p} \, dv \, dv_* \\ \le & C^2_p \, ||g||_{L^1_{2r_p}} \, ||f||^2_{H^k_{r_p}}. \end{aligned}$$

□

We now study the term B^1_{kl}. Integrating by parts $k-l$ times with respect to the variable v_*, we get

$$
\begin{aligned}
B^1_{kl} =& (-1)^{k-l} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} D_k f(v)\, g(v_*) \\
& \times\ D_{k-l;v_*} \left([D_l f(\tilde{v}) - D_l f(v)]\, b\big(\cos\theta, \frac{v-v_*}{\cos\frac{\theta}{2}}\big) \right) \frac{1}{(\cos\frac{\theta}{2})^2}\, d\theta dv_* dv \qquad (24) \\
=& \sum_{m=0}^{k-l} (-1)^m\, C^m_{k-l}\, B^1_{klm},
\end{aligned}
$$

with

$$
B^1_{klm} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} D_k f(v)\, g(v_*)\, D_{m;v_*}\, [D_l f(\tilde{v}) - D_l f(v)]
$$

$$
\times D_{k-l-m;v_*}\, b\big(\cos\theta, \frac{v-v_*}{\cos\frac{\theta}{2}}\big)\, \frac{1}{(\cos\frac{\theta}{2})^2}\, d\theta dv_* dv.
$$

In those formulas, $D_{r;v_*}$ denotes a derivative of order r with respect to the variable v_*.

Lemma 2 *: We suppose that $k \in \mathbb{N}$, $l = 0,..,k-1$, and $m = 1,..,k-l$. Then there exists a constant $C^4_{klm} > 0$ such that*

$$
|B^1_{klm}| \le C^4_{klm}\, ||g||_{L^1_{2\,r_{k-l-m}}}\, ||f||^2_{H^k_{r_{k-l-m}}}.
$$

Proof: Note first that for a given function h, one has (for $m \ge 1$)

$$
|D_{m;v_*}\, [h(\tilde{v}) - h(v)]| \le |\sin\frac{\theta}{2}|^m |D_m h(\tilde{v})|.
$$

Then,

$$
\begin{aligned}
|B^1_{klm}| \le & \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{-\pi/2}^{\pi/2} |D_k f(v)|\, g(v_*)\, |\sin\frac{\theta}{2}|^m\, |D_{m+l} f(\tilde v)| \\
& \qquad \times \ \frac{1}{|\cos\frac{\theta}{2}|^{k-l-m+2}} \left| D_{k-l-m} b\big(\cos\theta, \frac{v-v_*}{\cos\frac{\theta}{2}}\big)\right| d\theta dv_* dv \\
\le & \int_{-\pi/2}^{\pi/2} |\sin\frac{\theta}{2}|^m \frac{|\theta|^{\varepsilon-2}}{|\cos\frac{\theta}{2}|^{k-l-m+2}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{2}\, g(v_*) \left(|D_k f(v)|^2 + |D_{m+l} f(\tilde v)|^2 \right) \\
& \qquad \times C^3_{k-l-m}\, (1+|v-v_*|^2)^{r_{k-l-m}}\, dv dv_* d\theta \\
\le & \frac{C^3_{k-l-m}}{2} \int_{-\pi/2}^{\pi/2} |\sin\frac{\theta}{2}|^m \frac{|\theta|^{\varepsilon-2}}{|\cos\frac{\theta}{2}|^{k-l-m+2}}\, d\theta \\
& \qquad \times \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(v_*)\, |D_k f(v)|^2\, (1+|v|^2)^{r_{k-l-m}}\, (1+|v_*|^2)^{r_{k-l-m}}\, dv dv_* \\
& + \frac{C^3_{k-l-m}}{2} \int_{-\pi/2}^{\pi/2} |\sin\frac{\theta}{2}|^m \frac{|\theta|^{\varepsilon-2}}{|\cos\frac{\theta}{2}|^{k-l-m+2}} \\
& \times \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(v_*)\, |D_{m+l} f(\tilde v)|^2\, (1+(\cos\frac{\theta}{2})^2\, |\tilde v - v_*|^2)^{r_{k-l-m}}\, (\cos\frac{\theta}{2})^2\, d\tilde v dv_* d\theta,
\end{aligned} \tag{25}
$$

where the factor $(\cos\frac{\theta}{2})^2$ in the last term is simply the Jacobian in the transformation $v \longrightarrow \tilde v$ (see formula (12)). Finally, we see (recalling that $m+l \le k$) that

$$
|B^1_{klm}| \le C^4_{klm}\, ||g||_{L^1_{2\, r_{k-l-m}}}\, ||f||^2_{H^k_{r_{k-l-m}}},
$$

with

$$
C^4_{klm} = C^3_{k-l-m} \int_{-\pi/2}^{\pi/2} |\sin\frac{\theta}{2}|^m \frac{|\theta|^{\varepsilon-2}}{|\cos\frac{\theta}{2}|^{k-l-m+2}}\, d\theta,
$$

and Lemma 2 is proven. □

We now turn to the case when $m=0$. We have to estimate the term

$$
\begin{aligned}
B^1_{kl0} = & \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{-\pi/2}^{\pi/2} D_k f(v)\, g(v_*)\, [D_l f(\tilde v) - D_l f(v)] \\
& \times\ D_{k-l;v_*}\, b\big(\cos\theta, \frac{v-v_*}{\cos\frac{\theta}{2}}\big)\, \frac{1}{(\cos\frac{\theta}{2})^2}\, d\theta dv_* dv.
\end{aligned}
$$

Lemma 3 *: There exists a constant $C^5_{kl} > 0$ such that,*

$$
|B^1_{kl0}| \le C^5_{kl}\, ||g||_{L^1_{2\, r_{k-l+1}}}\, ||f||^2_{H^k_{r_{k-l}+\frac{1}{2}}}.
$$

Proof: We first note that

$$D_l f(\tilde v) - D_l f(v) = (\tilde v - v)\cdot \int_0^1 \nabla D_l f((1-s)\,v + s\,\tilde v)\,ds.$$

Then,

$$\begin{aligned}
|B^1_{kl0}| &= \Big| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{-\pi/2}^{\pi/2} D_k f(v)\, g(v_*) \left[(\tilde v - v)\cdot \int_0^1 \nabla D_l f((1-s)\,v + s\,\tilde v)\,ds \right] \\
&\qquad \times D_{k-l} b\Big(\cos\theta, \frac{v-v_*}{\cos\frac{\theta}{2}}\Big)\,\frac{1}{(\cos\frac{\theta}{2})^{k-l+2}}\,d\theta dv_* dv \Big| \\
&\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{-\pi/2}^{\pi/2} g(v_*)\,\frac{1}{2}\left[|D_k f(v)|^2 + \int_0^1 |\nabla D_l f((1-s)\,v + s\,\tilde v)|^2\,ds \right] \\
&\qquad \times |\tilde v - v|\, C^3_{k-l}\,(1+|v-v_*|^2)^{r_{k-l}}\,|\theta|^{\varepsilon-2}\,\frac{1}{(\cos\frac{\theta}{2})^{k-l+2}}\,d\theta dv_* dv.
\end{aligned}$$

Using formula (11), we see that

$$|\tilde v - v| = |\sin\frac{\theta}{2}|\,|\tilde v - v_*|,$$

so that

$$\begin{aligned}
|B^1_{kl0}| \le& \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{-\pi/2}^{\pi/2} g(v_*)\,|D_k f(v)|^2\,|\tilde v - v_*|\,(1+|v-v_*|^2)^{r_{k-l}} \\
&\qquad \times\ C^3_{k-l}\,|\sin\frac{\theta}{2}|\,|\theta|^{\varepsilon-2}\,\frac{1}{(\cos\frac{\theta}{2})^{k-l+2}}\,d\theta dv_* dv \\
&+ \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{-\pi/2}^{\pi/2} g(v_*)\,|\tilde v - v_*|\,\int_0^1 |\nabla D_l f((1-s)\,v + s\,\tilde v)|^2\,ds \qquad (26)\\
&\qquad \times\ C^3_{k-l}\,(1+|v-v_*|^2)^{r_{k-l}}\,|\sin\frac{\theta}{2}|\,|\theta|^{\varepsilon-2}\,\frac{1}{(\cos\frac{\theta}{2})^{k-l+2}}\,d\theta dv_* dv.
\end{aligned}$$

We now introduce the variable $u = (1-s)\,v + s\,\tilde v$. Its Jacobian is given by the formula

$$du = (1 + s^2\,(\tan\frac{\theta}{2})^2)\,dv\,.$$

Then,

$$|B^1_{kl0}| \le \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} g(v_*)\, |D_k f(v)|^2 \left(\big(1 + |\tan\frac{\theta}{2}|\big)\, |v_*| + \frac{1}{\cos\frac{\theta}{2}}\, |v| \right)$$
$$\times\, C^3_{k-l}\, (1+|v|^2)^{r_{k-l}}\, (1+|v_*|^2)^{r_{k-l}}\, \frac{|\sin\frac{\theta}{2}|\, |\theta|^{\varepsilon-2}}{(\cos\frac{\theta}{2})^{k-l+2}}\, d\theta dv_* dv$$
$$+ \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} \int_0^1 g(v_*)\, |\nabla D_l f(u)|^2\, |\tilde{v} - v_*|\, (1 + |v - v_*|^2)^{r_{k-l}}$$
$$\times\, C^3_{k-l}\, \frac{|\sin\frac{\theta}{2}|\, |\theta|^{\varepsilon-2}}{(\cos\frac{\theta}{2})^{k-l+2}}\, \frac{ds}{1 + s^2\, (\tan\frac{\theta}{2})^2}\, d\theta dv_* du. \tag{27}$$

Noticing that

$$v = \left((1-s)\, Id + \frac{s}{\cos\frac{\theta}{2}}\, R_{-\frac{\theta}{2}} \right)^{-1} \left[u - s\, \tan\frac{\theta}{2}\, R_{\frac{\pi}{2}} v_* \right],$$

we see that

$$|v|^2 \le 4 \left| u - s\, \tan\frac{\theta}{2}\, R_{\frac{\pi}{2}} v_* \right|^2$$
$$\le 8\, (|u|^2 + |v_*|^2).$$

In the same way,

$$\tilde{v} = \left(s\, Id + (1-s)\, \cos\frac{\theta}{2}\, R_{\frac{\theta}{2}} \right)^{-1} \left[u + (1-s)\, \sin\frac{\theta}{2}\, R_{\frac{\theta}{2}+\frac{\pi}{2}} v_* \right],$$

so that

$$|\tilde{v}|^2 \le 4 \left| u + (1-s)\, \sin\frac{\theta}{2}\, R_{\frac{\theta}{2}+\frac{\pi}{2}} v_* \right|^2 \le 8\, (|u|^2 + |v_*|^2). \tag{28}$$

Then, we obtain the estimate

$$|B^1_{kl0}| \le C^5_{kl}\, ||g||_{L^1_{2\, r_{k-l}+1}}\, ||f||^2_{H^k_{r_{k-l}+\frac{1}{2}}},$$

with (for example)

$$C^5_{kl} = \frac{C^3_{k-l}}{2} \int_{-\pi/2}^{\pi/2} \left(1 + |\tan\frac{\theta}{2}| + \frac{1}{\cos\frac{\theta}{2}} + 2\sqrt{8}\, (18)^{r_{k-l}} \right) \frac{|\sin\frac{\theta}{2}|\, |\theta|^{\varepsilon-2}}{(\cos\frac{\theta}{2})^{k-l+2}}\, d\theta. \tag{29}$$

This concludes the proof of Lemma 3.

Finally, we estimate the main term (that is, the term $\int Q(g, D_k f)(v)\, D_k f(v)\, dv$, which is crucial for the gain of smoothness). The computations are done here for any dimension N, since they are identical for all dimensions.

Lemma 4 *: Let $N \geq 2$. There exists a constant $C_6 > 0$ depending only on f_{in} and a constant $C_7 > 0$ such that*

$$- \int_{\mathbb{R}^N} Q(g, D_k f)(v)\, D_k f(v)\, dv \leq -C_6\, ||D_k f||^2_{H^{(\gamma-1)/2}} + C_7\, ||f||_{L^1_2}\, ||f||^2_{H^k_2} .$$

Proof: We compute (for all $g \geq 0$)

$$\begin{aligned} - \int Q(g,&D_k f)(v)\, D_k f(v)\, dv \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} b\, g(v_*)\, D_k f(v)\, (D_k f(v') - D_k f(v))\, d\sigma dv_* dv \\ &= - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} b\, g(v_*) \left(D_k f(v') - D_k f(v) \right)^2 d\sigma dv_* dv \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} b\, g(v_*) \left((D_k f(v'))^2 - (D_k f(v))^2 \right) d\sigma dv_* dv. \end{aligned}$$

Then we note that thanks to Assumption 2 (more precisely, to eq. (3)), we have

$$b(\cos\theta, v - v_*) \geq 1_{\theta \in]-\delta,\delta[} |\theta|^{-\gamma},$$

so that according to Corollary 2.1 and Proposition 3 of [1],

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} b\, g(v_*) \left(D_k f(v') - D_k f(v) \right)^2 d\sigma dv_* dv \geq C_g\, ||D_k f||_{H^{(\gamma-1)/2}} \ ,$$

with C_g depending only on (an upper bound of) the entropy and on the L^1_1norm of g. But those quantities are controlled by the same quantities for f_{in} when $g = f$ and f satisfies the Boltzmann equation under our assumptions.

On the other hand, according to Corollary 1.2 of [1] (cancellation lemma), we know that

$$\frac{1}{2} \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} b\, g(v_*) \left((D_k f(v'))^2 - (D_k f(v))^2 \right) d\sigma dv_* dv \right|$$

$$\leq C_7\, ||g||_{L^1_2}\, ||(D_k f)^2||_{L^1_2} \leq C_7\, ||g||_{L^1_2}\, ||f||^2_{H^k_2} .$$

This concludes the proof of lemma 4.

2.3 A differential inequality

In this section, we denote by C any strictly positive constant which can be replaced by a smaller strictly positive constant, and by D any constant which can be replaced by a larger one.

Gathering the estimates of the previous section (that is, formula (13) and Lemmas 1 to 4) and summing with respect to all derivatives of order k, we see that for f solution of the Boltzmann equation under our assumptions (and supposing without loss of generality that the sequence r_k is nondecreasing and is such that $r_0 \geq 2$), we get the differential inequality :

$$\frac{d}{dt}||f||^2_{H^k} \leq -C\,||f||^2_{H^{k+(\gamma-1)/2}} + D\,||f||^2_{H^k_{r_k+1/2}}\,||f||_{L^1_{2\,r_k+1}}.$$

Using Proposition 1 of the appendix, and supposing that $k \geq N$, we see that for some $\delta \in]0,1[$ and some $s > 0$,

$$\frac{d}{dt}||f||^2_{H^k} \leq -C\,||f||^2_{H^{k+(\gamma-1)/2}} + D\,||f||^{2-\delta}_{H^{k+(\gamma-1)/4}}\,||f||^{1+\delta}_{L^1_s}.$$

According to [14], for example, we can suppose that for all $s > 1$, the quantity $||f||_{L^1_s}$ is bounded on all compact sets of $]0,+\infty[$ (that is, all moments in L^1 are immediately gained). Then, we obtain

$$\frac{d}{dt}||f||^2_{H^k} \leq -C\,||f||^2_{H^{k+(\gamma-1)/2}} + D\,||f||^{2-\delta}_{H^{k+(\gamma-1)/4}}.$$

For all $\varepsilon > 0$, one can find $D_\varepsilon > 0$ such that the inequality

$$(1+|\xi|^2)^{k+(\gamma-1)/4} \leq \varepsilon\,(1+|\xi|^2)^{k+(\gamma-1)/2} + D_\varepsilon\,(1+|\xi|^2)^{-N-1}$$

holds. Then,

$$||f||^{2-\delta}_{H^{k+(\gamma-1)/4}} = \left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2\,(1+|\xi|^2)^{k+(\gamma-1)/4}\,d\xi \right)^{1-\frac{\delta}{2}}$$

$$\leq \left(\varepsilon \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2\,(1+|\xi|^2)^{k+\frac{\gamma-1}{2}}\,d\xi + D_\varepsilon \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2\,(1+|\xi|^2)^{-N-1}\,d\xi \right)^{1-\frac{\delta}{2}}$$

$$\leq \varepsilon^{1-\frac{\delta}{2}} \left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2\,(1+|\xi|^2)^{k+\frac{\gamma-1}{2}}\,d\xi \right)^{1-\frac{\delta}{2}} + D_\varepsilon^{1-\frac{\delta}{2}}\,m^{2-\delta},$$

where m is the mass $\int f\,dv$ of f. Finally, we obtain the differential inequality

$$\frac{d}{dt}||f||^2_{H^k} \leq -C\,||f||^2_{H^{k+(\gamma-1)/2}} + D\,\varepsilon^{1-\frac{\delta}{2}}\,||f||^{2-\delta}_{H^{k+(\gamma-1)/2}} + D\,D_\varepsilon^{1-\frac{\delta}{2}}\,m^{2-\delta},$$

so that there exists $D' > 0$ such that

$$\frac{d}{dt}||f||^2_{H^k} \le -\frac{C}{2}\,||f||^2_{H^{k+(\gamma-1)/2}} + D'.$$

Using (for example) Jensen's inequality, we see that

$$||f||^{2+\frac{\gamma-1}{k+N+1}}_{H^k} = \left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2\,(1+|\xi|^2)^k\,d\xi \right)^{1+\frac{\gamma-1}{2(k+N+1)}}$$

$$\le \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2\,(1+|\xi|^2)^{k+\frac{\gamma-1}{2}}\,d\xi \;\times \left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2\,(1+|\xi|^2)^{-N-1}\,d\xi \right)^{\frac{\gamma-1}{2(k+N+1)}}$$

$$\le m^{\frac{\gamma-1}{k+N+1}}\,||f||^2_{H^{k+(\gamma-1)/2}},$$

so that the differential inequality can be rewritten

$$\frac{d}{dt}\left(||f||^2_{H^k} \right) \le -\frac{C}{2}\,m^{\frac{\gamma-1}{k+N+1}} \left(||f||^2_{H^k} \right)^{1+\frac{\gamma-1}{2(k+N+1)}} + D'.$$

Using a standard argument (first used by Nash for parabolic equations), we see that for all k big enough (and consequently for all k), f lies in H^k as soon as $t > 0$. By interpolation (thanks to Proposition 1 for example), we see that (still when $t > 0$), f lies in H^k_s for all $k, s > 0$, and therefore lies in $\mathcal{S}$.

Note that in the previous computation, one should use approximate solutions of the Boltzmann equation in order to give a completely rigorous proof. For example, solutions of the equation

$$\partial_t f_\varepsilon = Q(f_\varepsilon, f_\varepsilon) + \varepsilon\,\Delta_v f_\varepsilon,$$
$$f_\varepsilon(0,\cdot) = f_{in} * \phi_\varepsilon,$$

where ϕ_ε is a sequence of mollifiers, can be used. This point does not lead to any difficulties.

Thus, we conclude the proof of Theorem 1 (in the particular case when $N = 2$ and when (4) is replaced by (5) in Assumption 2).

3 The general case

In this section, we explain how to modify the proofs described in the previous section to get Theorem 1 in any dimension and for all cross sections satisfying Assumption 2.

3.1 Higher Singularity

We briefly explain here how to prove Theorem 1 when the (angular part of) the cross section has a higher singularity, that is when (4) holds but not necessarily (5).

Note first that the term B^2_{kl} can be treated exactly as before, and the same is true for the terms B^1_{klm} for $m \neq 0, 1$.

The two last terms (B^1_{kl0} and B^1_{kl1}) can be treated simultaneously. Let us concentrate for example on the case when $m = 0$. The quantity that we wish to estimate is

$$B^1_{kl0} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\pi/2}^{\pi/2} D_k f(v)\, g(v^*)\, [D_l f(\tilde{v}) - D_l f(v)]\, b\, d\theta dv^* dv.$$

Then, the term $D_l f(\tilde{v}) - D_l f(v)$ can be bounded by $|\theta|^{\gamma-1+\eta}$ (with $\eta > 0$ small, and at least small enough for $\gamma - 1 + \eta < 2$ to hold) times a fractional derivative of f, of order $l + \gamma - 1 + \eta \leq k + \gamma - 2 + \eta$. Note that a symmetrisation by parity with respect to the variable θ is necessary to get such an estimate.

Then, there exists a constant number C^5_{kl} such that (for some $s > 0$)

$$|B^1_{kl0}| \leq C^5_{kl}\, ||g||_{L^1_s}\, ||f||^2_{H^{k+\gamma-2+\eta}_s},$$

and the differential inequality of Section 2.3 becomes

$$\frac{d}{dt} ||f||^2_{H^k} \leq -C\, ||f||^2_{H^{k+\frac{\gamma-1}{2}}} + D\, ||f||^2_{H^{k+\gamma-2+\eta}_s}\, ||f||_{L^1_s}.$$

Finally, we choose $\eta > 0$ such that

$$\gamma - 2 + \eta < \frac{\gamma - 1}{2}.$$

Note that this is possible since $\gamma < 3$.

The rest of the computation is similar to what has been done in Section 2.

3.2 Higher dimensions

It remains to prove that Theorem 1 also holds when $N > 2$. Most of the ideas used in the two-dimensional computation carry over unchanged in higher dimensions. However, in two dimensions, the representation of the pre- and postcollisional velocities by rotations in a fixed plane makes the consequent calculations much easier. The new difficulties that arise in higher dimensions come from the difficulty in finding smooth representations of the parameter space ($\mathbb{R}^3 \times S^2$ in dimension 3) in terms of pre- and post-collisional velocities.

We show here how to carry out the partial integrations in the expression B_{kl} (see eq. (15)); this shows all the essential differences. We restrict ourselves to dimension 3 for the sake of simplicity.

The corresponding expression in three dimensions is

$$\begin{aligned}&\int_{\mathbb{R}^3} D_k f(v) Q(D_{k-l}g, D_l f)(v)\,dv \\ &\qquad= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} D_k f(v') D_l f(v) D_{k-l}g(v_*) b(\cos\theta, v-v_*)\,dv dv_* d\sigma \\ &\qquad\quad- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} D_k f(v) D_l f(v) D_{k-l}g(v_*) b(\cos\theta, v-v_*)\,dv dv_* d\sigma.\end{aligned} \tag{30}$$

The first term in (30) can be rewritten using the transformation of the "cancellation lemma" (see [1]). If v_* and σ are kept fixed, we make the change of variables $v' \mapsto v$; the Jacobian for this transformation is $\det\left(\frac{\partial v'}{\partial v}\right) = \frac{1}{4}\cos^2\frac{\theta}{2}$. In higher dimensions, this changes to $C\cos^{N-1}\frac{\theta}{2}$.

N.B. The angle θ *always* denotes the angle between the relative velocities before and after a collision, exactly as defined in the introduction. Here the expression is computed for a fixed σ, which is slightly at variance with the expression (16) – (19), where the change of variable is carried out for a fixed angle of rotation. This explains the factor $\cos^2\frac{\theta}{2}$.

In this way we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} D_k f(v') D_l f(v) D_{k-l}g(v_*) b(\cos\theta, v-v_*)\,dv dv_* d\sigma = \\ &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} D_k f(v') D_l f(v) D_{k-l}g(v_*) \frac{4}{(\cos(\theta/2))^2} b\left(\cos\theta, \frac{v'-v_*}{\cos(\theta/2)}\right) dv' dv_* d\sigma\,.\end{aligned} \tag{31}$$

We can now change names of the variables, writing v instead of v', and instead of v writing $\tilde{v} = v + z$, where

$$z = |v - v_*|\,\tan(\theta/2)\,\Omega\,, \tag{32}$$

and where Ω is a unit vector which is orthogonal to $v - v_* \equiv w$, and in the plane spanned by $v - v_*$ and σ (this is well defined, modulo a sign, when $\theta \neq 0$). The result is

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} D_k f(v') D_l f(v) D_{k-l}g(v_*) b(\cos\theta, w)\,dv dv_* d\sigma = \\ &\quad\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} D_k f(v) D_l f(\tilde{v}) D_{k-l}g(v_*) \frac{4}{(\cos(\theta/2))^2} b\big(\cos\theta, \frac{w}{\cos(\theta/2)}\big)\,dv dv_* d\sigma\,.\end{aligned} \tag{33}$$

To simplify the notation, write

$$\tilde{b}(\cos\theta, w) = \frac{4}{(\cos(\theta/2))^2} b\big(\cos\theta, \frac{w}{\cos(\theta/2)}\big) \tag{34}$$

and hence (30) becomes

$$\begin{aligned}
&\int_{\mathbb{R}^3} D_k f(v) Q(D_{k-l} g, D_l f)(v)\, dv \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} D_k f(v) D_{k-l} g(v_*)\big(D_l f(\tilde{v}) - D_l f(v)\big)\tilde{b}(\cos\theta, w)\, dv dv_* d\sigma \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} D_k f(v) D_{k-l} g(v_*) D_l f(v) \\
&\qquad\qquad \times \Big(\tilde{b}(\cos\theta, w) - b(\cos\theta, w)\Big)\; dv dv_* d\sigma.
\end{aligned} \tag{35}$$

Setting $D_l f = F$, the inner integral in the first term of (35) can be written

$$\begin{aligned}
&\int_{S^2} \big(D_l f(\tilde{v}) - D_l f(v)\big)\tilde{b}(\cos\theta, v - v_*)\, dv dv_* d\sigma = \\
&\quad = \int_0^{\pi/2} \left(\int_{S^1} \big(F(v + z(\theta, \varphi, w)) - F(v)\big)\, d\varphi\right) \tilde{b}(\cos\theta, w)\, \sin\theta\, d\theta.
\end{aligned} \tag{36}$$

Then we write $v - v_* = w = (w_1, w_2, w_3)^t$. If we suppose that w_3, the third component of w, is positive, one possible representation of $z = z(\theta, \varphi, w)$ is

$$z = \frac{\tan(\theta/2)}{\sqrt{w_2^2 + w_3^2}} \begin{pmatrix} \big(w_2^2 + w_3^2\big)\cos(\varphi) \\ -w_3\sqrt{w_1^2 + w_2^2 + w_3^2}\sin(\varphi) + w_1 w_2 \cos(\varphi) \\ w_2\sqrt{w_1^2 + w_2^2 + w_3^2}\sin(\varphi) + w_1 w_3 \cos(\varphi) \end{pmatrix}. \tag{37}$$

Differentiating the innermost integral of (36) under the integral sign $k - l$ times with respect to w gives integrands of the form

$$D_{1;w} F(v - z) = D_{1;w} z\, D_{1;z} F(v - z)\,, \tag{38}$$

$$D_{2;w} F(v - z) = D_{2;w} z\, D_{1;z} F(v - z) + (D_{1;w} z)^2\; D_{2;z} F(v - z)\,, \tag{39}$$

$$\begin{aligned}
D_{3;w} F(v - z) =& D_{3;w} z D_{1;z}\, F(v - z) + D_{2;w} z D_{1;w} z D_{2;z}\, F(v - z) \\
&+ (D_{1;w} z)^3\, D_{3;z}\, F(v - z)\,,
\end{aligned} \tag{40}$$

and so on (recall that $D_{j;z}$ denotes a generic j-th order derivative with respect to the (components of) z). The problem comes from the factors deriving from the chain rule: $D_{j;w} z$ is homogeneous of degree $1 - j$ in the variables w, and so this seems to introduce new singularities into the problem. Of course this is a problem only for small w, and so now we assume that the equation has already been split into a part with $|w| < 1$, which we go on to study, and a remaining part where this is no problem. We write

$$F(v+z) = G(v,z) + P_{j-1}(v,z)\,,$$

where $P_{j-1}(v,z)$ is the Taylor polynomial of degree $j-1$, defined as

$$P_{j-1}(v,z) = \sum_{0\le j_1+j_2+j_3<j} \left[\frac{j!}{j_1!j_2!j_3!}\partial_{v_1}^{j_1}\partial_{v_2}^{j_2}\partial_{v_3}^{j_3}F(v)\right] z_1^{j_1} z_2^{j_2} z_3^{j_3}\,.$$

Now, $D_{j',z}G(v,z)$ involves derivatives of F up to order j', and vanishes at least to order $j-j'$ at $z=0$. That means that if $F(v+z)$ is replaced by $G(v,z)$ in terms like (40), then the singularities emerging from the chain rule are cancelled, and

$$D_{j,w}\int_{S^1} G(v,z)\,d\varphi \tag{41}$$

involves only derivatives of $F(v-z)$ up to order j, multiplied by bounded functions of w. It also has a factor $(\tan(\theta/2))^j$, which helps in cancelling the singularity of $\tilde b$ near $\theta = 0$.

But the inner integral of (36) also has a term

$$D_{j,w}\int_{S^1} P(v,z)\,d\varphi\,. \tag{42}$$

It is clear that after the integration over φ only terms where $j_1+j_2+j_3 \equiv 2m$ is even remain, because $z(\theta,\phi,w_1,w_2,w_3) = -z(\theta,\phi+\pi,w_1,w_2,w_3)$. Moreover, any monomial $z_1^{j_1}z_2^{j_2}z_3^{j_3}$ can be written as a sum of terms of the form $(a_1z_1+a_2z_2+a_3z_3)^{j_1+j_2+j_3}$, where the a_i are suitably chosen real numbers. Using formulas like $A\cos(\varphi)+B\sin(\varphi) = \sqrt{A^2+B^2}\sin(\varphi+\psi)$, where $\psi = \psi(A,B)$, we can take the expressions from (37) and write

$$\begin{aligned} a_1z_1+a_2z_2+a_3z_3 &= \tan(\theta/2) \\ &\times \sin(\varphi+\psi)\sqrt{\begin{matrix} a_1^2(w_2^2+w_3^2)+a_2^2(w_1^2+w_3^2)+a_3^2(w_1^2+w_2^2) \\ \qquad -2a_1w_1a_3w_3-2a_1w_1a_2w_2-2a_2w_2a_3w_3\end{matrix}}\,,\end{aligned}$$

where ψ is a rather complicated expression of w and the a_i. This expression, which can be verified by straightforward calculations, shows that the singularities that come from the parametrisation are just apparent, and that

$$\begin{aligned}&\int_{S^1}(a_1z_1+a_2z_2+a_3z_3)^{2m}\,d\varphi = (\tan(\theta/2))^{2m} \\ &\times\left(\begin{matrix} a_1^2(w_2^2+w_3^2)+a_2^2(w_1^2+w_3^2)+a_3^2(w_1^2+w_2^2) \\ \qquad -2a_1w_1a_3w_3-2a_1w_1a_2w_2-2a_2w_2a_3w_3\end{matrix}\right)^m \int_0^{2\pi}\sin^{2m}(\varphi)\,d\varphi\,,\end{aligned}$$

i.e. a polynomial of order $2m$ in the components of w, and at this point the choice of parametrisation in (37) is no longer visible. We can conclude that (42) is zero, because it is a j-th order derivative of a polynomial of order at most $j-1$.

Thus we see that the proof of Lemma 2 can be translated to this case, with very little change. Of course all the calculations done here assume that the function F is sufficiently regular, but this can be achieved by a density argument, just because the involved operations do no introduce any unnecessarily high order of differentiation.

To conclude, we also comment on the changes of variables as in (25), (26) and (27). We have already seen that for $N=3$,

$$dv = \frac{1}{4}\cos^2\frac{\theta}{2}\, d\tilde{v},$$

though it was then expressed in v' and v. The variable u in (27) is simply a convex combination of v and $\tilde{v}$. With notation (32), we get

$$\begin{aligned} u = (1-s)\, v + s\,\tilde{v} &= v_* + w + s\,|v-v_*|\, \tan(\theta/2)\,\Omega \\ &= v + s\,|v-v_*|\, \tan(\theta/2)\,\Omega\,. \end{aligned}$$

It is hence easy to compute the Jacobian corresponding to the one in (27), and also to verify that (28) holds.

This concludes the proof of Theorem 1 in all cases.

Appendix : Interpolations

Definition 2 *: Let $k \in \mathbb{R}_+$ and $p \in \mathbb{N}$. We denote by $H^k_p(\mathbb{R}^N)$ the weighted Sobolev space of functions f such that*

$$||f||^2_{H^k_p(\mathbb{R}^N)} = \sum_{|\alpha|\le p} \int_{\xi\in\mathbb{R}^N} (1+|\xi|^2)^k\, |\widehat{(\cdot)^\alpha f}(\xi)|^2\, d\xi < +\infty.$$

The quantity $||f||_{H^k_p(\mathbb{R}^N)}$ defines the natural norm on $H^k_p(\mathbb{R}^N)$, and endows it with a structure of Hilbert space.

Note that when $k \in \mathbb{N}$, then H^k_p is nothing else than the space defined in (21), endowed with an equivalent norm.

We prove here the following result of interpolation, used in Section 2.3.

Proposition 1 *: Let $k \in \mathbb{R}_+$, $p \in \mathbb{N}$ and $\varepsilon > 0$. Then, there exists a constant number $K_{k,p,\varepsilon,N} > 0$ such that for all $f \in \mathcal{D}(\mathbb{R}^N)$,*

$$||f||_{H^k_p(\mathbb{R}^N)} \leq K_{k,p,\varepsilon,N}\, ||f||^{2^{-\frac{1}{2}-\frac{1}{2}J}}_{L^1_{2p\,(\frac{k}{\varepsilon}+1)+\frac{N}{4}+1}}\, ||f||^{2^{-\frac{1}{2}-\frac{1}{2}J}}_{H^N_0(\mathbb{R}^N)}\, ||f||^{1-2^{-1-J}}_{H^{k+\varepsilon}_0(\mathbb{R}^N)},$$

where

$$J = \lfloor \left(\log(k\varepsilon^{-1}+1)\right) / \log 2 \rfloor$$

and where $\lfloor x \rfloor$ denotes the largest integer smaller than x.

We begin by the proof of the following result:

Lemma 5 *: Let $k \in \mathbb{R}_+$, $p \in \mathbb{N}$ and $\varepsilon > 0$. Then, there exists a constant $K'_{k,p} > 0$ such that for all $f \in \mathcal{D}(\mathbb{R}^N)$,*

$$||f||^2_{H^k_p(\mathbb{R}^N)} \leq K'_{k,p}\, ||f||_{H^{k-\varepsilon}_{2p}(\mathbb{R}^N)}\, ||f||_{H^{k+\varepsilon}_0(\mathbb{R}^N)}.$$

Proof: We write down

$$\begin{aligned} ||f||^2_{H^k_p(\mathbb{R}^N)} &= \sum_{|\alpha|\leq p} \int_{\xi\in\mathbb{R}^N} (1+|\xi|^2)^k\, |\widehat{(\cdot)^\alpha f}(\xi)|^2\, d\xi \\ &\leq \sum_{|\alpha|\leq p} \left| \int_{\xi\in\mathbb{R}^N} (1+|\xi|^2)^k\, (\partial_\alpha \hat{f})(\xi)\, (\partial_\alpha \bar{\hat{f}})(\xi)\, d\xi \right| \\ &\leq \sum_{|\alpha|\leq p} \left| \int_{\xi\in\mathbb{R}^N} \partial_\alpha \Big((1+|\cdot|^2)^k\, \partial_\alpha \hat{f} \Big)(\xi)\, \bar{\hat{f}}(\xi)\, d\xi \right|. \end{aligned}$$

Then, we notice that

$$\partial_\alpha((1+|\xi|^2)^k\, g(\xi)) = \sum_{\beta\leq\alpha} P_{\alpha-\beta}(\xi)\, (1+|\xi|^2)^{k+|\beta|-|\alpha|}\, \partial_\beta g(\xi),$$

where the $P_{\alpha-\beta}$ are polynomials of degree $|\alpha|-|\beta|$.

Introducing constants $K_{\alpha,\beta} > 0$ such that

$$P_{\alpha-\beta}(\xi)\, (1+|\xi|^2)^{-|\alpha|+|\beta|} \leq K_{\alpha,\beta},$$

we obtain the estimate

$$||f||^2_{H^k_p(\mathbb{R}^N)} \le \sum_{|\alpha|\le p} \sum_{\beta\le\alpha} \left| \int_{\xi\in\mathbb{R}^N} P_{\alpha-\beta}(\xi)\,(1+|\xi|^2)^{k-|\alpha|+|\beta|}\,\partial_{\alpha+\beta}\hat{f}(\xi)\,\bar{\hat{f}}(\xi)\,d\xi \right|$$
$$\le \sum_{|\alpha|\le p} \sum_{\beta\le\alpha} \int_{\xi\in\mathbb{R}^N} \left| P_{\alpha-\beta}(\xi)\,(1+|\xi|^2)^{k-|\alpha|+|\beta|-\varepsilon} \right|^{1/2} |\partial_{\alpha+\beta}\hat{f}(\xi)|$$
$$\times \left| P_{\alpha-\beta}(\xi)\,(1+|\xi|^2)^{k-|\alpha|+|\beta|+\varepsilon} \right|^{1/2} |\hat{f}(\xi)|\,d\xi$$
$$\le \sum_{|\alpha|\le p} \sum_{\beta\le\alpha} K^2_{k,\alpha,\beta} \left(\int_{\xi\in\mathbb{R}^N} (1+|\xi|^2)^{k+\varepsilon}\,|\hat{f}(\xi)|^2\,d\xi \right)^{1/2}$$
$$\times \left(\int_{\xi\in\mathbb{R}^N} (1+|\xi|^2)^{k-\varepsilon}\,|\,\widehat{(\cdot)^{\alpha+\beta}f}(\xi)|^2\,d\xi \right)^{1/2},$$

and the lemma is proven. □

Proof of proposition 1: We apply Lemma 5 to get

$$||f||_{H^k_p(\mathbb{R}^N)} \le (K'_{k,p})^{1/2}\,||f||^{1/2}_{H^{k-\varepsilon}_{2p}(\mathbb{R}^N)}\,||f||^{1/2}_{H^{k+\varepsilon}_0(\mathbb{R}^N)}$$
$$\le (K'_{k,p})^{1/2}\,(K'_{k-\varepsilon,2p})^{1/4}\,||f||^{1/4}_{H^{k-3\varepsilon}_{4p}(\mathbb{R}^N)}\,||f||^{3/4}_{H^{k+\varepsilon}_0(\mathbb{R}^N)},$$

and then, by induction, for all $J\in\mathbb{N}$,

$$||f||_{H^k_p(\mathbb{R}^N)} \le \prod_{j=0}^{J} (K'_{k-(2^j-1)\,\varepsilon,2^j\,p})^{2^{-j-1}}\,||f||^{2^{-J-1}}_{H^{k-(2^{J+1}-1)\,\varepsilon}_{2^{J+1}\,p}(\mathbb{R}^N)}\,||f||^{1-2^{-J-1}}_{H^{k+\varepsilon}_0(\mathbb{R}^N)}.$$

We then consider the smallest $J\in\mathbb{N}$ such that $2^{J+1}\ge\frac{k}{\varepsilon}+1$, that is,

$$\hat{J} = \lfloor \big(\log(k\varepsilon^{-1}+1)\big)\,/\log 2 \rfloor\,.$$

Denoting

$$K''_{k,p,\varepsilon} = \prod_{j=0}^{\hat{J}} (K'_{k-(2^j-1)\,\varepsilon,2^j\,p})^{2^{-j-1}},$$

we obtain the inequality

$$||f||_{H^k_p(\mathbb{R}^N)} \le K''_{k,p,\varepsilon}\,||f||^{2^{-1-\hat{J}}}_{L^2_{2p\,(\frac{k}{\varepsilon}+1)}}\,||f||^{1-2^{-1-\hat{J}}}_{H^{k+\varepsilon}_0(\mathbb{R}^N)}. \qquad (43)$$

We now use the following lemma :

Lemma 6 *: Let $r \in \mathbb{R}_+$. Then, there exists a constant $K'''_{r,N} > 0$ such that for all $f \in \mathcal{D}(\mathbb{R}^N)$,*

$$||f||^2_{L^2_r(\mathbb{R}^N)} \leq K'''_{r,N}\, ||f||_{L^1_{r+\frac{N}{4}+1}}\, ||f||_{H^N_0(\mathbb{R}^N)}.$$

Proof of lemma 6 : We write $x = (x_1, .., x_N)$, $t = (t_1, .., t_N)$. For all $\alpha > 0$,

$$\int (1+|x|^2)^r\, |f(x)|^2\, dx = \int (1+|x|^2)^{r+\alpha} f(x)\, \frac{f(x)}{(1+|x|^2)^\alpha}\, dx$$

$$= \int (1+|x|^2)^{r+\alpha} f(x)\, \int_{-\infty}^{x_1} .. \int_{-\infty}^{x_N} \partial_{1..N}\left(\frac{f(t)}{(1+|t|^2)^\alpha}\right) dt_1..dt_N\, dx$$

$$\leq \int_{\mathbb{R}^N} (1+|x|^2)^{r+\alpha}|f(x)|\, dx\, \int_{\mathbb{R}^N} \left|\partial_{1..N}\left(\frac{f(t)}{(1+|t|^2)^\alpha}\right)\right| dt.$$

But

$$\left|\partial_{1..N}\left(\frac{f(t)}{(1+|t|^2)^\alpha}\right)\right| \leq \sum_{|\beta|\leq N} |\partial_\beta f(t)|\, |P_\beta(t)|\, (1+|t|^2)^{-\alpha-(N-|\beta|)},$$

where P_β is a polynomial of degree of order $N - |\beta|$. We introduce a constant $K^{(4)}_\beta > 0$ such that

$$|P_\beta(t)|\, (1+|t|^2)^{-\alpha-(N-|\beta|)} \leq K_\beta\, (1+|t|^2)^{-\alpha-(N-\beta)/2},$$

we see that

$$\int (1+|x|^2)^r\, |f(x)|^2\, dx \leq \int_{\mathbb{R}^N} (1+|x|^2)^{r+\alpha}|f(x)|\, dx$$

$$\times \sum_{|\beta|\leq N} K^{(4)}_\beta \int_{\mathbb{R}^N} |\partial_\beta f(t)|\, (1+|t|^2)^{-\alpha-(N-\beta)/2}\, dt$$

$$\leq \sum_{|\beta|\leq N} K^{(4)}_\beta \left(\int_{\mathbb{R}^N} (1+|u|^2)^{-2\alpha}\, du\right)^{1/2}$$

$$\times \int_{\mathbb{R}^N} (1+|x|^2)^{r+\alpha}|f(x)|\, dx \left(\int_{\mathbb{R}^N} |\partial_\beta f(t)|^2\, dt\right)^{1/2}.$$

The first integral is finite for all $4\alpha > N$. We conclude by taking $\alpha = N/4 + 1$.

We now end the proof of Proposition 1. We use eq. (43). We get

$$||f||_{H^k_p(\mathbb{R}^N)} \leq K''_{k,p,\varepsilon}\, (K'''_{2p\,(\frac{k}{\varepsilon}+1),N})^{2^{-\frac{1}{2}-\frac{1}{2}\,\hat{j}}}\, ||f||^{2^{-\frac{1}{2}-\frac{1}{2}\,\hat{j}}}_{L^1_{2p\,(\frac{k}{\varepsilon}+1)+\frac{N}{4}+1}}$$

$$\times ||f||_{H_0^N(\mathbb{R}^N)}^{2^{-\frac{1}{2}-\frac{1}{2}\hat{j}}} ||f||_{H_0^{k+\varepsilon}(\mathbb{R}^N)}^{1-2^{-1-\hat{j}}}.$$

We get proposition 1 by denoting

$$K_{k,p,\varepsilon,N} = K''_{k,p,\varepsilon} \, (K'''_{2p\,(\frac{k}{\varepsilon}+1),N})^{2^{-\frac{1}{2}-\frac{1}{2}\hat{j}}}.$$

□

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