SMOOTHNESS PROPERTIES OF LIE GROUP SUBDIVISION SCHEMES *

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Abstract. Linear stationary subdivision rules take a sequence of input data and produce ever denser sequences of subdivided data from it. They are employed in multiresolution modeling and have intimate connections with wavelet and more general pyramid transforms. Data which naturally do not live in a vector space, but in a nonlinear geometry like a surface, symmetric space, or a Lie group (e.g. motion capture data), require different handling. One way to deal with Lie group valued data has been proposed by D. Donoho [3]: It is to employ a log-exponential analogue of a linear subdivision rule. While a comprehensive discussion of applications is given by Ur Rahman et al. in [9], this paper analyzes convergence and smoothness of such subdivision processes and show that the nonlinear schemes essentially have the same properties regarding C^1 and C^2 smoothness as the linear schemes they are derived from.

Key words. Nonlinear subdivision, C^1 and C^2 smoothness, Lie groups, Log-Exponential scheme

AMS subject classifications. 41A25, 26B05, 22E05, 68U05

1. Motivation. The handling of data from the multiscale perspective is a very active topic. One particular concept which underlies multiresolution methods is *subdivision*, which often means a shift-invariant refinement procedure. The need to extend subdivision to nonstandard data types comes from data which do not live in a vector space, but in various nonlinear geometries. Examples are e.g. unit vectors (headings), which live in the unit circle or a higher dimensional unit sphere; orientations of rigid bodies in space, which live in the group SO_3; and poses of rigid bodies, which live in the Euclidean motion group. It is no coincidence that the last two examples are *group*-valued — others can easily be found (see [9]).

While the numerical representation of such data types is usually no problem, and subdivision rules can be *defined* without too much effort, their analysis with regard to smoothness and approximation power is still far from complete. The aim of the present paper is to contribute to our knowledge of properties of nonlinear subdivision schemes: We give results on an important topic, namely the convergence of subdivision processes, and C^1 and C^2 smoothness of their limits. We completely skip applications and the relation to wavelet analysis here — the interested reader is referred to [9].

2. Introduction. G. de Rham [1] introduced the concept of *curve subdivision rule*, which means the refinement of a control polygon with the intent of generating a smooth curve in the limit. The current paper is initially concerned with linear stationary curve subdivision rules, like the cubic B-spline rule, where a polygon $(p_i)_{i\in\mathbb{Z}}$ defines another polygon $(Sp_i)_{i\in\mathbb{Z}}$ via

$$Sp_{2i} = p_i - \frac{1}{8}w_{i-1} + \frac{1}{8}w_i, \quad Sp_{2i+1} = p_i + \frac{1}{2}w_i, \quad \text{where } p_{j+1} = p_j + w_j. \tag{2.1}$$

It is known that repeated refinement of p leads to an ever denser sequence of polygons $S^j p$, which converge to the cubic B-spline curve whose control polygon is the original polygon p, or indeed any intermediate polygon $S^j p$. This subdivision rule is illustrated

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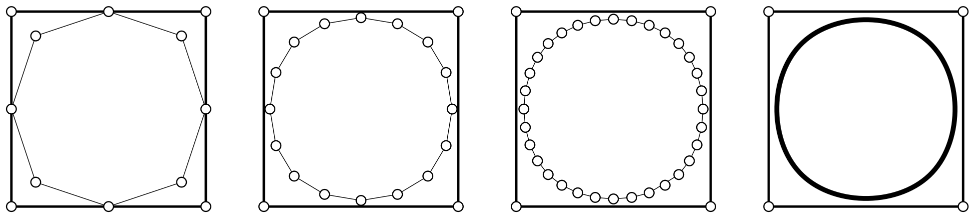

FIG. 1. *Applying the cubic B-spline rule S and its iterates S^2, S^3 to a periodic sequence p_i (from left). At right, the limit curve is shown.*

by Fig. 1. A classical question in the theory of linear stationary subdivision rules is convergence to a continuous limit, and smoothness of that limit. Literature references which cover this well established theory are [4, 14, 6].

2.1. Previous Work. In recent years there has been some progress in the smoothness analysis of nonlinear subdivision rules which are *analogous* to linear ones: By replacing affine combinations by geodesic combinations one defines subdivision rules in surfaces and Riemannian manifolds. A similar construction using one-parameter subgroups instead of geodesic lines yields subdivision rules in Lie groups. Another way of constructing a nonlinear subdivision scheme which operates in a surface is to subdivide linearly, and project the result back onto the surface under consideration. These and other ways of perturbing linear curve subdivision rules are the topic of [12, 13, 11]. Those papers contain applications in Computer Graphics as well as a general analysis of C^1 smoothness and a more restricted analysis of C^2 smoothness of subdivision rules in nonlinear geometries. The method of analysis is to establish a *proximity condition* between the nonlinear rule and the linear rule it is derived from. The same idea is applied in [15] where C^k smoothness of interpolatory subdivision schemes operating in the sphere S^n and the group SO_n are studied. [7] deals with C^1 smoothness analysis in the regular multivariate case.

The present paper studies log-exponential analogues of linear subdivision rules. This idea originally was proposed by [3] and is also the basis of the comprehensive discussion of multiscale representations of Lie-valued and manifold-valued data in [9].

For references to other contributions to nonlinear subdivision processes, whose methods are different from the present work, the interested reader is referred to [12] and [11], and especially to [9]. To the knowledge of the authors, the present paper together with [11] and [15] is the only source of results concerning smoothness higher than C^1.

2.2. Subdivision in Lie groups. The aim of the present paper is to study log-exponential analogues of linear subdivision rules. This way of creating a subdivision rule in Lie groups works as follows: An expression in affine space which involves the addition of a vector to a point, like "$p_{i+1} = p_i + w_i$", is turned into the analogous expression "$p_{i+1} = p_i \circ \exp v_i$" in the Lie group $(G, \circ)$, where p_i, p_{i+1} are points (i.e., elements of G), and v_i is a tangent vector (i.e., an element of the corresponding Lie algebra $\mathfrak{g} = T_{\mathbb{I}} G$).

Even if our results are valid for general finite-dimensional Lie groups, we initially work with matrix groups only. In that case the exponential function coincides with the matrix exponential function. The reader may think of G as the group SO_3 of rotations, or the group GL_n of invertible linear mappings, or the group SE_n of Euclidean congruence transformations.

The B-spline subdivision scheme "S" of (2.1) is by this analogy turned into the following subdivision scheme "T" for groups:

$$Tp_{2i} = p_i \exp(-\frac{1}{8}v_{i-1} + \frac{1}{8}v_i), \quad Tp_{2i+1} = p_i \exp(\frac{1}{2}v_i), \quad \text{where } p_{j+1} = p_j e^{v_j}. \quad (2.2)$$

In the abelian Lie group $(\mathbb{R}^n, +)$, we have $\exp(v) = v$ and $p \circ q = p + q$. so in that special case the newly defined scheme T equals the original scheme S. Our way of defining a log-exp analogue is different from the one in [9]. We briefly discuss the differences and similarities between these definitions in §6.

2.3. Contents. We organize our paper as follows: First we recall basic definitions and results for linear subdivision schemes, including derived schemes, norms, and convergence. After that we define a log-exponential analogue of a linear scheme. Section 4 is devoted to the Taylor expansion of expressions which involve the matrix exponential function. The topic of §5 is proximity of linear schemes and their nonlinear analogues, and shows convergence and C^1 and C^2 smoothness of the log-exponential analogue of linear schemes which fulfill certain technical conditions. The latter look rather unexpected, but are fulfilled for the majority of subdivision rules with C^2 limits, namely for those where the well known method of smoothness analysis via the method of Laurent series works (it is nice but perhaps a coincidence that an example of those few schemes which can be shown to possess C^2 limits without fitting into that theory still fulfills the technical condition). Section 6 remarks on the limitations of and alternatives to our method. Finally we briefly illustrate log-exp subdivision by means of an example for rigid body motions, and show how to apply Theorems 5, 6 for certain subdivision rules.

3. Curve subdivision rules. In general, a linear stationary curve subdivision rule "S" with dilation factor 2 has the form

$$Sp_j = \sum\nolimits_{i\in\mathbb{Z}} a_{j-2i} p_i \quad \text{for all } j \in \mathbb{Z}. \quad (3.1)$$

It takes the polygon $(p_i)_{i\in\mathbb{Z}}$ and maps it to the polygon $(Sp_i)_{i\in\mathbb{Z}}$. The coefficient sequence a_j is the *mask* of the scheme, and is always assumed to be nonzero only for finitely many j. Equation (3.1) actually consists of two rules, one for computing the even points of Sp, and another one for computing the odd points. We consider only affinely invariant rules, which means that $\sum_j a_{2j} = \sum_j a_{2j+1} = 1$. Thus, (3.1) can also be written in the form

$$Sp_{2i} = p_i + \sum\nolimits_{j\neq 0} a_{-2j}(p_{i+j} - p_i), \quad Sp_{2i+1} = p_i + \sum\nolimits_{j\neq 0} a_{1-2j}(p_{i+j} - p_i). \quad (3.2)$$

We would like to express (3.2) in terms of the difference vectors $w_i = p_{i+1} - p_i$. With the elementary relation $p_{i+j} - p_i = w_i + \cdots + w_{i+j-1}$ and analogous for p_{i-j} we get

$$Sp_{2i} = p_i + \sum\nolimits_{j>0} \big(w_{i+j-1} \sum\nolimits_{l\geq j} a_{-2l} - w_{i-j} \sum\nolimits_{l\geq j} a_{2l}\big) \quad (3.3)$$

$$Sp_{2i+1} = p_i + \sum\nolimits_{j>0} \big(w_{i+j-1} \sum\nolimits_{l\geq j} a_{1-2l} - w_{i-j} \sum\nolimits_{l\geq j} a_{1+2l}\big). \quad (3.4)$$

3.1. The log-exponential analogue. We consider a finite-dimensional real Lie group G with its Lie algebra $\mathfrak{g} = T_{\mathbb{I}}G$. Mostly we can without loss of generality consider matrix Lie groups, but both the definition of log-exponential schemes below and our smoothness results are valid for general Lie groups. The exponential mapping is denoted by $\exp : \mathfrak{g} \to G$. For matrix Lie groups, we also write $\exp(A) = e^A = \sum_{k\geq 0} A^k/k!$. For each Lie group, there is a neighbourhood of the identity element $\mathbb{I} \in G$ where the exponential mapping is a diffeomorphism and has an inverse, then called the logarithm. We now define the log-exp analogue of (3.3)–(3.4) as follows.

Definition 1. *Let $p : \mathbb{Z} \to G$ be a sequence, such that for all i, $\log(p_i^{-1} p_{i+1})$ is defined. Then difference vectors v_i of successive points p_i, p_{i+1} are defined by*

$$p_{i+1} = p_i \exp(v_i), \quad \textit{i.e.}, \ v_i = \log(p_i^{-1} p_{i+1}). \tag{3.5}$$

The log-exponential analogue T of the linear rule S of (3.3)–(3.4) is given by

$$Tp_{2i} = p_i \exp \sum\nolimits_{j>0} \big(v_{i+j-1} \sum\nolimits_{l\geq j} a_{-2l} - v_{i-j} \sum\nolimits_{l\geq j} a_{2l}\big) \tag{3.6}$$

$$Tp_{2i+1} = p_i \exp \sum\nolimits_{j>0} \big(v_{i+j-1} \sum\nolimits_{l\geq j} a_{1-2l} - v_{i-j} \sum\nolimits_{l\geq j} a_{1+2l}\big). \tag{3.7}$$

The definition of T is what we get when we substitute the operation "point+vector" in (3.3) and (3.4) by "point times exponential of vector".

3.2. Derived subdivision schemes and generating functions. We recall some tools which are convenient in the smoothness analysis of linear subdivision schemes. For more details, the reader is referred to [4, 6, 14]. With the difference operator Δ defined by $\Delta p_i = p_{i+1} - p_i$ the derived schemes S_k, if they exist, are recursively defined by $S_0 = S$ and $S_k(\Delta p) = 2\Delta S_{k-1} p$. This implies the commutation relation $S_k \Delta^k = 2^k \Delta^k S$. It is customary to define the generating functions $a(z)$, $p(z)$, $Sp(z)$, and $\Delta p(z)$ of the mask a, the sequence p_i, the subdivided sequence Sp_i, and the sequence of differences Δp_i, respectively. For example, we have $a(z) = \sum a_i z^i$. The function $a(z)$ is called the symbol of S. Equation (3.1) translates to $Sp(z) = a(z)p(z^2)$, whereas the definition of the difference sequence reads $\Delta p(z) = (z^{-1} - 1)p(z)$. The commutation relation $S_1 \Delta p = 2\Delta Sp$ immediately implies that the symbol $a^{[1]}(z)$ of the derived scheme S_1 equals $a^{[1]}(z) = 2za(z)/(1+z)$. It follows that the derived schemes up to order k exist if and only if the symbol $a(z)$ has the factor $(1+z)^k$.

We are not going to need it, but we would like to demonstrate how to rewrite (3.3)–(3.4) in terms of generating functions: Define the subdivision operators D and U by $Dp_{2i} = Dp_{2i+1} = p_i$ and $Sp = Dp + U\Delta p$ (the last equation being shorthand for (3.3)–(3.4)). Then $Dp(z) = (1+z)p(z^2)$ and $U\Delta p(z) = Sp(z) - Dp(z)$, which in terms of generating functions means $u(z)\Delta p(z^2) = a(z)p(z^2) - (1+z)p(z^2) \implies u(z) = z^2(a(z) - 1 - z)/(1 - z^2)$. We already know that this division works out, because (3.3)–(3.4) is true; but divisibility of $a(z) - 1 - z$ by $1 - z^2$ also follows from the relations $a(1) = 2$ and $a(-1) = 0$, which state affine invariance.

3.3. Convergence and smoothness. The theory of convergence and smoothness of linear stationary curve subdivision rules of finite mask can be considered more or less complete (see e.g. the surveys [4, 6, 14]). For a sequence $(p_i)_{i\in\mathbb{Z}}$ and any previously chosen Euclidean norm $\|\cdot\|$ in $\mathbb{R}^n$, we use the notation

$$\|p\| := \sup\nolimits_i \|p_i\|, \quad d(p) := \|\Delta p\|. \tag{3.8}$$

Recall that the norm of a subdivision operator S, and indeed the norm of any iterated operator S^k is defined by $\|S^k\| := \sup_{\|p\|=1} \|S^k p\|$ and can be computed as $\|S^k\| = \max_{j\in\{1,\dots,2^k\}} \sum_{i\in\mathbb{Z}} |\mathrm{coeff}(z^{j-2^k i}, a(z)a(z^2)\cdots a(z^{2^{k-1}}))|$. This operator norm does not depend on the choice of norm in $\mathbb{R}^n$. The limit curve $S^\infty p$ of the sequence of ever denser polygons p, Sp, $S^2p, \ldots$ is defined as follows: We consider the sequence of functions f, Sf, $S^2f, \ldots,$ such that S^kf is linear in each interval $[2^{-k}i, 2^{-k}(i+1)]$, and $S^kf(2^{-k}i) = S^kp_i$. Then $S^\infty p(t) := \lim S^kf(t)$ for all $t \in \mathbb{R}$.

In case $\|S_1^k\| < 2^k$ for some integer $k > 0$, the limit curve $S^\infty p$ is continuous for all p. If the same is true for a power of S_2 or even S_3, then limit curves possess C^1 or even C^2 smoothness. For instance, if S is the cubic B-spline scheme, all derived schemes have norm 1, so we can let $k = 1$, and limit curves are C^2.

4. Miscellaneous facts concerning the exponential function. This section is concerned with Taylor expansions of expressions which involve the matrix exponential function. One topic is the deviation of the exponential function from the identity mapping for small vectors; another topic is how to write Taylor expansions such that later we can easily give upper bounds.

4.1. Choosing neighbourhoods $U, \widetilde{U}$ of small vectors and "small points". This subsection shows how to select a small neighbourhood U of the zero vector in the Lie algebra $\mathfrak{g}$, such that certain inequalities needed later are true. Via the exponential function, this neighbourhood is turned into a small neighbourhood $\widetilde{U}$ of the identity element $\mathbb{I} \in G$. We first define auxiliary functions ρ_j by letting

$$\rho_j(v) = \sum\nolimits_{k\geq 0} v^k/(k+j)! \implies e^v = \sum\nolimits_{k=0}^{j-1} v^k/\,k! + v^j\rho_j(v). \tag{4.1}$$

The argument v in $\rho_j(v)$ can be a matrix, including the case of 1×1 matrices, i.e., real numbers. If we use a matrix norm with $\|AB\| \leq \|A\| \cdot \|B\|$, then it is obvious that $\|\rho_j(v)\| \leq \rho_j(\|v\|) \leq \exp(\|v\|)$. As $\exp : \mathfrak{g} \to G$ is a local diffeomorphism, there exists a neighbourhood U of the zero vector $0 \in \mathfrak{g}$ and constants $\gamma_1, \gamma_2 > 0$ such that

$$v \in U \implies \gamma_1\|v\| \leq \|e^v - \mathbb{I}\| \leq \gamma_2\|v\|. \tag{4.2}$$

In fact, using Equation (4.1) it is not difficult to give U, γ_1, γ_2 explicitly: The equations $e^v = \mathbb{I} + v\rho_1(v) = \mathbb{I} + v + v^2\rho_2(v)$ imply that

$$\|v\| - \|v\|^2\rho_2(\|v\|) \leq \|e^v - \mathbb{I}\| \leq \rho_1(\|v\|) \cdot \|v\|. \tag{4.3}$$

We choose e.g. U as the ball of radius r, and $\gamma_1 = 1 - r\rho_2(r)$, $\gamma_2 = \rho_1(r)$. If $r < \ln 2 \approx 0.693$, then $\|e^v - \mathbb{I}\| \leq r\rho_1(r) = e^r - 1 < 1$, so we are within the convergence radius of the logarithm series, and $\exp|U$ is a diffeomorphism. For $r = .5$ we get $\gamma_1 \approx 0.7025$, $\gamma_2 \approx 1.297$. Further, there is $\gamma_3 > 0$ such that for all $u, v \in U$,

$$\|(e^{-u} + e^v - 2\mathbb{I}) - (v - u)\| \leq \gamma_3 \max(\|u\|, \|v\|)^2. \tag{4.4}$$

This is easy to see, when we compute $(e^{-u} + e^v - 2\mathbb{I}) - (v - u) = -u^2\rho_2(u) + v^2\rho_2(v)$. We only have to choose the neighbourhood U as before, e.g. as a ball of radius r, and then let $\gamma_3 = 2\rho_2(r)$.

Any such neighbourhood U defines a neighbourhood $\widetilde{U}$ of $\mathbb{I} \in G$ via $\widetilde{U} := \exp(U)$. Clearly, $\exp|U$ and $\log|\widetilde{U}$ are diffeomorphisms inverse to each other.

Example 1. As an example, we give the neighbourhoods U and $\widetilde{U}$ for the group $G = \mathrm{SO}_3$ of rotations, and the Frobenius norm $\|v\|^2 = \mathrm{tr}(v^Tv)$: It is well known that then $\mathfrak{g}$ is the space of skew-symmetric matrices, $\|gvg^{-1}\| = \|v\|$ and $\exp(gvg^{-1}) = g\exp(v)g^{-1}$ for all $g \in \mathrm{SO}_3$. For any skew-symmetric v there is $g \in \mathrm{SO}_3$ such that

$$gvg^{-1} = \begin{bmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \exp(gvg^{-1}) = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

With $\|v\|^2 = 2x^2 < r^2$ we see that $\widetilde{U}$ is the set of rotations whose angle does not exceed $r/\sqrt{2}$. The maximum angle for which the estimates above are valid, is 28.08°.

4.2. A technical lemma concerning Taylor expansions. The purpose of this subsection is to provide a result which is later used in the comparison of linear subdivision rules with the log-exp analogues. We define the functions $P(p, q;\, u_1, \ldots;\, v_1, \ldots;\, w_1, \ldots;\, r_1 \ldots)$ and $Q(p, q;\, u_1, \ldots;\, v_1, \ldots;\, w_1, \ldots;\, r_1, \ldots)$ by

$$P := p \exp\big(\sum_{j>0}(u_j \sum_{l\ge j} a_{-2l} - v_j \sum_{l \ge j} a_{2l})\big) - q \exp\big(\sum_{j>0}(w_j \sum_{l \ge j} a_{1-2l} - r_j \sum_{l\ge j} a_{1+2l})\big),$$

$$\begin{aligned} Q := p &+ \textstyle\sum_{j>0} \big(a_{-2j}(pe^{u_1}\cdots e^{u_j} - p) + a_{2j}(pe^{-v_1}\cdots e^{-v_j} - p)\big) \\ &- q - \textstyle\sum_{j>0}\big(a_{1-2j}(qe^{w_1}\cdots e^{w_j} - q) + a_{1+2j}(qe^{-r_1}\cdots e^{-r_j} - q)\big). \end{aligned}$$

The functions P and Q have been designed such that differences like $\Delta Sp_{2i} = Sp_{2i+1} - Sp_{2i}$ can be expressed in terms of either P or Q:

$$\Delta S p_{2i} = -Q(p_i, p_i; v_i, v_{i+1}, \ldots; v_{i-1}, v_{i-2}, \ldots; v_i, v_{i+1}, \ldots; v_{i-1}, v_{i-2}, \ldots), \tag{4.5}$$
$$\Delta T p_{2i} = -P(p_i, p_i; v_i, v_{i+1}, \ldots; v_{i-1}, v_{i-2}, \ldots; v_i, v_{i+1}, \ldots; v_{i-1}, v_{i-2}, \ldots), \tag{4.6}$$
$$\Delta S p_{2i-1} = Q(p_i, p_{i-1}; v_i, v_{i+1}, \ldots; v_{i-1}, v_{i-2}, \ldots; v_{i-1}, v_i, \ldots; v_{i-2}, v_{i-3}, \ldots), \tag{4.7}$$
$$\Delta T p_{2i-1} = P(p_i, p_{i-1}; v_i, v_{i+1}, \ldots; v_{i-1}, v_{i-2}, \ldots; v_{i-1}, v_i, \ldots; v_{i-2}, v_{i-3}, \ldots). \tag{4.8}$$

We are going to derive a second order Taylor expansion of $P - Q$. Using that $(\mathbb{I} + u_1)\cdots(\mathbb{I} + u_j) - \mathbb{I}$ equals $u_1 + \cdots + u_j$ up to first order, we get

$$\begin{aligned} P \overset{(1)}{=} Q \overset{(1)}{=} p\Big(\mathbb{I} + \textstyle\sum_{j>0}\big(a_{-2j}(u_1 + \cdots + u_j) + a_{2j}(-v_1 - \cdots - v_j)\big)\Big) \qquad (4.9)\\ - pe^x\Big(\mathbb{I} + \textstyle\sum_{j>0}\big(a_{1-2j}(w_1 + \cdots w_j) + a_{1+2j}(-r_1 - \cdots - r_j)\big)\Big), \quad \text{where } q = pe^x. \end{aligned}$$

Apparently the second order Taylor polynomial involves no terms linear in u_i, v_i, w_i, r_i. Further it is obvious that when expanding e^x, both linear and quadratic terms involving x cancel. If we let $p = \mathbb{I}$, then the second order Taylor polynomial of $P - Q$ reads

$$\begin{aligned} &\frac{1}{2}\big(\sum_{j>0}(u_j \sum_{l\ge j} a_{-2l} - v_j \sum_{l\ge j} a_{2l})\big)^2 - \frac{1}{2}\big(\sum_{j>0}(w_j \sum_{l\ge j} a_{1-2l} - r_j \sum_{l \ge j} a_{1+2l})\big)^2 \qquad (4.10)\\ &+ \frac{1}{2}\sum_{j>0}\Big(a_{1-2j}\big(\sum_{l=1}^{j} w_l^2 + 2\sum_{1\le i<l\le j} w_k w_l\big) + a_{1+2j}\big(\sum_{l=1}^{j} r_l^2 + 2\sum_{1\le i<l\le j} r_k r_l\big) \\ &- a_{-2j}\big(\sum_{l=1}^{j} u_l^2 + 2 \sum_{1\le i<l\le j} u_k u_l\big) - a_{2j}\big(\sum_{l=1}^{j} v_l^2 + 2\sum_{1\le i<l\le j} v_k v_l\big)\Big). \end{aligned}$$

We could rewrite this formula as a linear combination of products of exactly two of the variables u_i, $v_j, \ldots$. We will not write down that expression, but note that the *sum of coefficients* is given by

$$\begin{aligned} s = \textstyle\sum_{i,j>0}\big(\sum_{m\ge i, n\ge j}&(a_{-2m}a_{-2n} - a_{1-2m}a_{1-2n} + a_{2m}a_{2n} - a_{1+2m}a_{1+2n} \\ &+ 2(a_{1-2m}a_{1+2n} - a_{-2m}a_{2n})) + \eta_{ij}\textstyle\sum_{m\ge j}(a_{1-2m} - a_{-2m} + a_{1+2m} - a_{2m})\big); \\ \text{where } \eta_{ij} = 2 \text{ or } 1 \text{ or } 0,\ &\text{if } j > i \text{ or } i = j \text{ or } j < i,\ \text{resp.} \qquad (4.11) \end{aligned}$$

LEMMA 2. *Consider a matrix Lie group G, a sequence $p_i : \mathbb{Z} \to G$, the linear curve subdivision rule S and its log-exponential analogue T. Assume that the symbol of S has the property*

$$a'(-1)(1 - a'(1)) = a''(-1). \tag{4.12}$$

Then the second order Taylor polynomial of $\Delta S p_{2i} - \Delta T p_{2i}$ has the general form

$$p_i\big(\sum\nolimits_{k,l} \alpha_{k-i,l-i} \Delta v_k \cdot v_l + \sum\nolimits_{k,l} \beta_{k-i,l-i} v_k \cdot \Delta v_l\big), \tag{4.13}$$

when we express differences $\Delta S p_{2i}$ and so on via (4.5)–(4.8). *The coefficients α_{kl} and β_{kl} depend only the symbol of S. An analogous statement is true for $\Delta S p_{2i-1} - \Delta T p_{2i-1}$.*

Proof. By shift invariance of the subdivision algorithms S and T it is sufficient to consider $i = 0$, and further without loss of generality we let $p_i = \mathbb{I}$. We use (4.5)–(4.8) to expand $\Delta S p_{2i} - \Delta T p_{2i}$ and and to compute its second order Taylor expansion. This leads to (4.10), but with the appropriate substitutions according to (4.5)–(4.8). In any case it has the general form $\sum x_{kj} v_k v_j$, where $\sum_{k,j} x_{kj}$ is given by the expression s of (4.11). Some manipulations show that $s = 0 \iff (\sum_{j>0} j(a_{1-2j} - a_{1+2j}))^2 - (\sum_{j>0} j(a_{-2j} - a_{2j}))^2 = \sum_{j>0} j^2(a_{1-2j} + a_{1+2j} - a_{-2j} - a_{2j}) \iff (\sum_{j\in\mathbb{Z}} j a_{1-2j})^2 - (\sum_{j\in\mathbb{Z}} j a_{-2j})^2 = \sum_{j\in\mathbb{Z}} j^2(a_{1-2j} - a_{-2j})$. We convert both the left and right hand side of this equation into an expression involving the values and derivatives of the generating function $a(z)$ and get $(1 - a'(-1))(1 - a'(1))/4 = (1 - a'(1) - a''(-1))/4$, i.e., condition (4.12). Now that we know that $\sum_{k,l} x_{kl} = 0$, we can compute

$$\begin{aligned}\sum\nolimits_{k,l\in\mathbb{Z}} x_{kl} v_k v_l &= \sum\nolimits_{k,l\in\mathbb{Z}} x_{kl}\big(v_k(v_l - w) + (v_k - w)w + w^2\big) \\ &= \sum\nolimits_{k,l\in\mathbb{Z}} x_{kl}\big((v_k - w)w + v_k(v_l - w)\big),\end{aligned} \tag{4.14}$$

where $w \in \mathfrak{g}$ is arbitrary. We choose $w = v_0$ and replace the terms $v_l - v_0$, $v_k - v_0$ by expressions involving Δv_l, Δv_k, respectively — e.g. in the case $l > 0$, $v_l - v_0 = \Delta v_0 + \cdots + \Delta v_{l-1}$. This shows the statement of the lemma. When we compare (4.14) with (4.13), we see that $\alpha_{kl} = 0$ whenever $l \neq 0$, so the sum in (4.13) actually reads $\sum_k \alpha_{k-i,0} \Delta v_k \cdot v_i + \sum_{k,l} \beta_{k-i,l-i} v_k \cdot \Delta v_l$. ☐

Remark. Condition (4.12) is fulfilled for all subdivision rules one usually thinks of when discussing schemes which enjoy C^2 smoothness. This is because a standard method of smoothness analysis in the linear case is by derived schemes, so we usually require that derived schemes up to order 3 exist. Then $a'(-1) = a''(-1) = 0$, and (4.12) is fulfilled trivially.

5. Smoothness Analysis. This section first establishes proximity between a linear subdivision scheme S and its nonlinear analogue T, and then proceeds to show smoothness, invoking the general theory which relates proximity inequalities and smoothness of limit curves, and which is contained in [12, 11].

5.1. Proximity inequalities. Proposition 3 below states a 'zero order' proximity condition. This name is the same as used in [13, 11] and means a bound on the distance of points Tp_i from Sp_i, i.e., a quantification of closeness of the nonlinear subdivision scheme T and the linear scheme S it is derived from. We still assume that

G is a matrix group. Further we assume that the sequence $(p_i)_{i\in\mathbb{Z}}$ of input data is bounded. We will later get rid of both assumptions.

PROPOSITION 3. *Suppose that S is the subdivision scheme of* (3.3)–(3.4) *and T is its log-exponential analogue in a matrix group G according to* (3.6)–(3.7). *Consider a sequence $p:\mathbb{Z}\to G$ which is bounded with respect to some matrix norm. Then there is a constant $C>0$ and a neighbourhood $\widetilde{U}$ of the identity $\mathbb{I}\in G$ such that closeness of successive points in the sense of $p_{i+1}^{-1}p_i\in\widetilde{U}$ implies $\|Sp-Tp\|\le Cd(p)^2$.*

Proof. Since there are two different expressions for Sp_i and Tp_i depending on whether i is even or odd, we have to consider two cases: to give an upper bound for $\|Tp_{2i}-Sp_{2i}\|$, and the same for $\|Tp_{2i-1}-Sp_{2i-1}\|$. We restrict ourselves to the first case, the second one being completely analogous. The computation which led to (4.9) showed that Sp_{2i} and Tp_{2i} have the same first order Taylor expansion, so $Sp_{2i}-Tp_{2i}$ is a linear combination of remainder terms of the form $w^2\rho_2(w)$, where w is combined from the vectors v_k. As $\|v_k\|$ is bounded, so are $\|w\|$ and $\rho_2(\|w\|)$ for any vector w which occurs in this way. Consequently, $\|Sp_{2i}-Tp_{2i}\|\le C\sup_j\|v_j\|^2$. What we actually need is an upper bound in terms of $d(p)$ instead of $\|v_i\|$. This is done as follows: As $\|p_j\|$ is bounded, so is the sequence of inverses: $\|p_j^{-1}\|\le\alpha$ with $\alpha>0$. We now observe Equation (4.2), take the neighbourhood $U=\log(\widetilde{U})$ from there, and compute

$$\|v_i\|\le\gamma_1^{-1}\|e^{v_i}-\mathbb{I}\|=\gamma_1^{-1}\|p_i^{-1}(p_{i+1}-p_i)\|\le\gamma_1^{-1}\alpha\, d(p). \tag{5.1}$$

It follows that $\|Sp_{2i}-Tp_{2i}\|$ is bounded by a constant times $d(p)^2$, as required. □

The next result together with Lemma 2 is the main contribution of the present paper. It is rather technical, and establishes a "first order" proximity condition between a linear subdivision scheme and its nonlinear log-exponential analogue. Later it is used to show C^2 smoothness of limit curves.

PROPOSITION 4. *Under the additional assumption that derived schemes of orders 2,3 exist, Proposition 3 allows the conclusion $\|\Delta Tp-\Delta Sp\|\le C\big(d(p)d(\Delta p)+d(p)^3\big)$ (with different constant C and neighbourhood $\widetilde{U}$).*

Proof. As derived schemes S_1, S_2, S_3 exist, the symbol symbol $a(z)$ of the subdivision scheme S has a factor $(1+z)^3$, so $a'(-1)=a''(-1)=0$ and (4.12) is fulfilled. Thus we can directly apply Lemma 2 and conclude that

$$\Delta Sp_{2i}-\Delta Tp_{2i}=p_i\big(\sum\nolimits_{k,l}\alpha_{k-i,l-i}\Delta v_k\cdot v_l+\sum\nolimits_{k,l}\beta_{k-i,l-i}v_k\cdot\Delta v_l+R_3\big),$$

where R_3 means higher order terms in the Taylor series. A similar statement is true for $\Delta Sp_{2i-1}-\Delta Tp_{2i-1}$. As $\|p_i\|<\beta$, the norm $\|\Delta Sp_{2i}-\Delta Tp_{2i}\|$ is bounded by β times the norm of the series. According to (4.1), R_3 is a linear combination of terms of the form $w^3\rho_3(w)$, where w is combined from the vectors v_k, and the coefficients of this linear combination depend only on the mask $a(z)$. It follows that there are constants $C',C''>0$ such that

$$\|\Delta Sp_{2i}-\Delta Tp_{2i}\|\le C'\sup\|v_i\|\sup\|\Delta v_i\|+C''\sup\|v_i\|^3.$$

What we actually need is an upper bound for $\Delta Sp_{2i}-\Delta Tp_{2i}$ in terms of $d(p)$ and $d(\Delta p)$ instead of $\|v\|$ and $\|\Delta v\|$. With (5.1) we can eliminate $\|v\|$ and replace it by $d(p)$. In order to get rid of $\|\Delta v\|$ we use (4.4):

$$\begin{aligned}\|\Delta v_i\|&=\|v_{i+1}-v_i\|\le\|e^{v_{i+1}}-2\mathbb{I}+e^{-v_i}\|+\gamma_3\sup\nolimits_j\|v_j\|^2\\&\le\alpha\|p_{j+2}-2p_{j+1}+p_j\|+\gamma_3\alpha^2\gamma_1^{-2}d(p)^2\le\alpha d(\Delta p)+\gamma_3\alpha^2\gamma_1^{-2}d(p)^2.\end{aligned}$$

Here α is an upper bound of $\|p_i^{-1}\|$, analogous to the previous proof. So we finally have shown that $\|\Delta S p_{2i} - \Delta T p_{2i}\|$ is bounded as required. The computation for the odd case $\Delta S p_{2i-1} - \Delta T p_{2i-1}$ is analogous. □

5.2. Convergence and smoothness of limit curves. We put together the various results obtained so far and formulate Theorems 5, 6 below, which state that the log-exponential analogue T of a linear subdivision scheme S essentially has the same smoothness properties as S, if derived schemes S_1, S_2, S_3 are appropriately bounded.

THEOREM 5. *Consider a matrix Lie group G and a subdivision scheme S with is affinely invariant and has finite mask, so its log-exponential analogue T according to* (3.6)–(3.7) *is defined. Then:*

1. *For any sequence $p : \mathbb{Z} \to G$, the polygons $Tp, T^2p, \ldots$ converge to a continuous limit curve $T^\infty p$, provided the points p_i are close enough, and the linear scheme S itself is convergent.*
2. *If the derived scheme S_2 exists and $\|(S_1)^k\| < 2^{k/2}$, $\|(S_2)^k\| < 2^k$, for some integer $k > 0$, then all continuous curves $T^\infty p$ enjoy C^1 smoothness.*
3. *If the derived scheme S_3 exists and $\|(S_1)^k\| < 2^{k/3}$, $\|(S_1)^k\|\|(S_2)^k\| < 2^k$, $\|(S_3)^k\| < 2^k$ for some integer $k > 0$, then all continuous curves $T^\infty p$ enjoy C^2 smoothness.*

Proof. We first show that we can without loss of generality consider a restricted class of polygons: Finiteness of the mask implies that for any compact interval $[a, b]$ the limit curve $T^\infty|[a, b]$ is determined by a finite number of points p_i. As statement (1) explicitly mentions points p_i which are close together, and the convergence assumption in statements (2) and (3) implicitly does the same, we can without loss of generality consider points p_i which lie in an arbitrarily small neighbourhood of a point $p \in G$. As the log-exponential subdivision schemes are invariant with respect to left multiplication in the group, we can without loss of generality assume that $p = \mathbb{I}$. Especially we can assume that the sequence p_i is bounded.

Now statement 1 is Theorem 3 of [12]. The proximity condition required there is our Proposition 3. Statement 2 is Theorem 6 of [12], applied to the k-th iterate of the scheme S, again using the same proximity condition. The decay rates μ_0, μ_1 employed in [12] are defined by Equations (31) and (32) in that paper, and the inequalities required by the cited theorem translate exactly to the conditions given above.

Statement 3 is analogous to Theorems 7 and 8 of [11], the difference being that that paper considers other analogues of linear rules, where the required proximity condition of Proposition 4 is shown only for a certain class of factorizable schemes, whereas we show it for all schemes where S_3 exist. The rest is exactly the same. The notation is slightly different from ours, e.g., [11, Definition 11] uses coefficients $\widetilde{\mu}_i = \frac{1}{N^m}\|S_{i+1}^m\|$, whereas the inequalities given above directly relate to the norms of derived schemes S_1, S_2, S_3, without introducing coefficients $\widetilde{\mu}_i$. □

THEOREM 6. *Theorem 5 extends to finite-dimensional Lie groups (which are not necessarily matrix groups).*

Proof. This is basically because all such Lie groups can be realized locally as a subgroup of GL_n. For the convenience of the reader, we give a more detailed argument: By Ado's theorem (e.g. Theorem 3.17.8 of [10]), there is $n > 0$ and an analytic Lie subgroup $H \leq \mathrm{GL}_n$, not necessarily embedded but immersed, and an isomorphism $\psi : \mathfrak{g} \to \mathfrak{h}$ of the Lie algebras of G, H, respectively. The universal cover $\widetilde{G}$ of G has the same Lie algebra as G, and there is a homomorphism $\widetilde{\phi} : \widetilde{G} \to H$ with $d\widetilde{\phi}_{\mathbb{I}} = \psi$ and $\exp_H(\psi(v)) = \widetilde{\phi}(\exp_{\widetilde{G}}(v))$. The natural homomorphism $\pi : \widetilde{G} \to G$ is a local isomorphism with $d\pi_{\mathbb{I}} = \mathrm{id}$, so $\phi = \widetilde{\phi}\pi^{-1}$ is defined locally around $\mathbb{I} \in G$. As $\psi = d\phi$

is regular, ϕ is a local isomorphism $G \to H$.

In order to prove the statement of the theorem, we follow the proof of Theorem 5 to see that we can restrict ourselves to a a small neighbourhood of the identity element. As all constructions are invariant with respect to local Lie group isomorphisms, and so are the notions of convergence and smoothness, we employ ϕ to switch from the group G to the group H, and Theorem 5 applies. ☐

6. Discussion.

On the generality of our results. The most unsatisfactory statement in Theorem 5 perhaps is that the limit curves of T exist and are continuous provided input data are 'close enough', without further quantification of this closeness. A more precise statement is given below. Smoothness is solved in a more satisfactory way: whenever convergence happens, the result is smooth, provided the other conditions are met. As to how many C^2 curve subdivision schemes fulfill these conditions, we can give an easy answer only if we restrict ourselves beforehand to such schemes where derived schemes S_1, S_2, S_3 exist (as subdivision schemes without a derived scheme S^3 only rarely produce C^2 limits, this is a reasonable restriction). The answer to the percentage of linear schemes where the present paper applies then is as follows: In the linear case, the spectral radii ρ_i of S_i must fulfill $\rho_j < 2$ for $j = 1, \dots, k+1$ to guarantee C^k smoothness ($k > 0$). In the present paper we additionally require the stronger bounds $\rho_1 < 2^{1/2}$ for C^1 smoothness and $\rho_1 < 2^{1/3}$, $\rho_1\rho_2 < 2$ for C^2 smoothness. Unfortunately we do not know yet if these bounds are artifacts of the method of proof or have deeper significance.

Further comments on convergence. It is possible to make the words 'provided points p_i are close enough' in Theorem 5 more precise. In order to prove convergence towards $T^\infty p$ in the interval $[a, b]$ we need to consider only the finitely many points which contribute to that segment. They constitute a bounded sequence, and Proposition 3 yields $C > 0$ such that $\|Sp_j - Tp_j\| < Cd(p)^2$. According to the proof of Theorems 2 and 3 of [12], we now consider $\mu := \|S^k\|/2^k < 1$ and choose δ such that $\mu + 2C\delta < 1$. Then $\|p_i - p_{i+1}\| < \delta$ is enough for convergence.

Unfortunately δ typically is rather small, so this does not address the problem of convergence for coarse control polygons. Even if our experience shows that it is hard to find an example of a polygon such that subdivision does not converge, the entire question is an important topic of future research where almost no results are available as yet.

Alternative log-exponential analogues. The definition of the nonlinear subdivision rule T by Equations (3.6)–(3.7) is analogous to Equations (3.3)–(3.4). It would also have been possible to define another log-exponential analogue which is related directly to (3.2): As a substitute for the difference $p_{i+k} - p_i$ we define $v_{ik} := \log(p_i^{-1} p_{i+k})$. The nonlinear analogue of (3.2) would then be given by $\widetilde{T}p_{2i} = p_i \exp \sum_{j \neq 0} a_{-2j} v_{ij}$, $\widetilde{T}p_{2i+1} = p_i \exp \sum_{j \neq 0} a_{1-2j} v_{ij}$. The difference vectors $v_i = \log(p_{i+1}^{-1} p_i)$ we employ in the present paper are independent of each other and determine the input data p_i, if one single point p_0 is given. The difference vectors v_{ij} of course also determine the input data together with a single point p_0, but they are not independent: by construction we have the relation $\exp(v_{i,j}) = \exp(v_{i,k}) \exp(v_{i+k,j-k})$ for all i, j, k. Smoothness analysis is also possible for the alternative definition, and indeed in the multivariate case we expect the alternative definition to be easier to use.

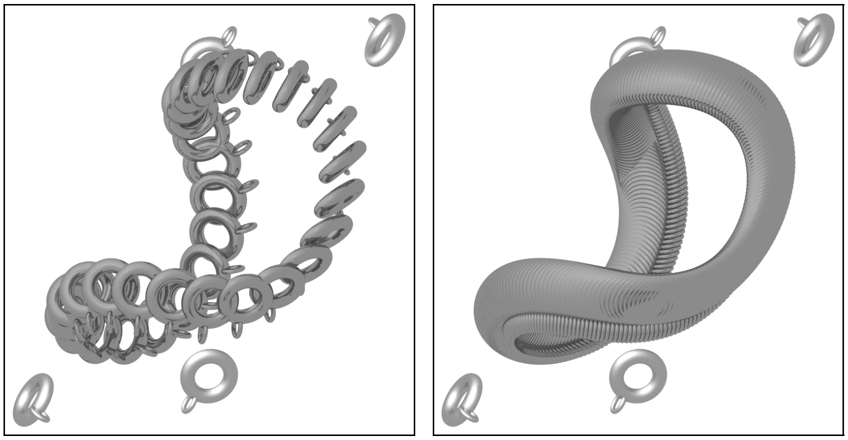

FIG. 2. *The log-exponential analogue T of the cubic B-spline subdivision rule in the group* SE_3*. We apply 3, resp., 6 rounds of T to a periodic sequence of poses of a rigid body.*

7. Examples. In a first example we show some subdivision rules which fulfill the requirements of Theorems 5, 6. The two following examples are of a different nature. One demonstrates how log-exponential subdivision works in the Euclidean motion group, and the other one considers some schemes which do not fit into the formalism at all.

Example 2. We have already mentioned that the B-spline subdivision schemes, which have the symbol $a(z) = (1+z)^{n+1}/(2z)^n$, possess derived schemes up to S_3 if $n \geq 3$, and that further $\|S_j\| = 1$ for $j = 1, 2, 3$. Consequently Theorems 5, 6 apply. In contrast to that nice behaviour, consider the interpolatory 4-point scheme of [2] with $a(z) = (1+z)^4(4z-1-z^2)/16z^3$. Again S_1, S_2, S_3 exist, and $\|S_1^2\| = 17/16 < 2$, $\|S_2^2\| < 4$, so the continuity and C^1 parts of Theorem 5 apply. However the condition on $\|S_3^k\|$ is not fulfilled, because that would imply C^2 smoothness of limit curves for the linear scheme (which is known to be not the case). Consider now another interpolatory scheme: The scheme of [5] with symbol $(1+z)^3(52z - 15z^3 + 16 + 16z^2)/(16z)^2$ has $\|S_3^2\| < 2^2$, $\|S_1^2\|\|S_2^2\| < 2^2$, $\|S_1^2\| < 2^{2/3}$, as already mentioned in [11]. This means that the C^2 part of Theorem 5 applies to that scheme and its log-exponential analogue.

Example 3. We demonstrate log-exponential subdivision by means of the group SE_3 of Euclidean motions. Here an element of the Lie algebra $\mathfrak{g}$ is the velocity vector field of a translational or rotational or helical motion, and $\exp(v)$ is the motion we get when we follow the flow of this time-invariant velocity vector field from time $t = 0$ to time $t = 1$.

The position of a rigid body $\mathbb{R}^3$ is defined by a pair (a, A), where $a \in \mathbb{R}^3$ is a translation vector and the matrix $A \in \mathrm{SO}_3$ gives the rotational component of that pose. A point x in the coordinate frame connected to the body has position $Ax + a$ in 3-space. The transformation $x \mapsto Ax + a$ can also be written in form of a matrix multiplication: $\left[\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & a \\ 0 & A \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right]$. Thus SE_3 is a matrix group, consisting of all block matrices in $\mathbb{R}^{4\times 4}$ of the form $\left[\begin{smallmatrix} 1 & a \\ 0 & A \end{smallmatrix}\right]$ with $A \in \mathrm{SO}_3$ and $a \in \mathbb{R}^3$. Its Lie algebra is consequently given by the matrices $\left[\begin{smallmatrix} 0 & v \\ 0 & V \end{smallmatrix}\right]$ with V skew-symmetric. Figure 1 shows the application of the cubic B-spline scheme S of (2.1) to a periodic sequence of points. Figure 2 illustrates its log-exponential analogue T defined by (2.2).

Example 4. It is known that existence of higher derived schemes $S_2, \ldots$ is not necessary for smoothness. For instance the subdivision scheme S with the mask $a(z) = (1+z)(1+z^2)^n/2^n$ produces C^{n-1} limit curves [8], but does not admit any higher derived schemes. Note that S satisfies the condition (4.12), even if Theorems 5, 6 do not apply. The authors do now know if this is a coincidence.

8. Conclusion. We have studied a natural nonlinear Lie group analogue of curve subdivision schemes, which serves as a tool in the multiresolution analysis of certain nonstandard data types, namely Lie group valued data. The analogy is based on the fact that the exponential function in a Lie group defines an analogue of the operation "point plus vector". We showed conditional convergence of such a nonlinear scheme, as well as C^1 and C^2 smoothness of limit curves, if certain technical conditions concerning norms of derived schemes are met. This establishes a key property, namely smoothness, of subdivision processes useful for dealing with such data.

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