Now, fix k_0 such that $\operatorname{Im} k_0 \neq 0$ and $\kappa - 2\operatorname{Re} k_0 \le -2n-2$. Next, fix N_0 such that $N_0 > -2n-2-(\kappa - 2\operatorname{Re} k_0) - 1$. Let $\Lambda_K^{N_0}$ denote Λ_K with the particular choice of $N = N_0$. Then $\Lambda_K^{N_0}(\phi)$ depends analytically on k where $\operatorname{Im} k \neq 0$ and $\kappa - 2\operatorname{Re} k > \kappa - 2\operatorname{Re} k_0 - 1$.

For all $\widehat{\phi} \in \mathcal{Q}$, we have $\Lambda_K(\phi) = c_n(\widehat{\Lambda}_K \,|\, \widehat{\phi})$. But for k satisfying $\kappa - 2\operatorname{Re} k > -2n-2$,

$$(\widehat{\Lambda}_K^{N_0} \,|\, \widehat{\phi}) = \int_{-\infty}^{\infty} \sum_{\alpha} \langle J(\lambda)E_{\alpha\lambda}, \widehat{\phi}E_{\alpha\lambda}\rangle (2|\lambda|)^n \, d\lambda,$$

where the right hand side depends analytically on k for $\operatorname{Im} k \neq 0$ and $\kappa - 2\operatorname{Re} k > \kappa - 2\operatorname{Re} k_0 - 1$. Hence, by analytic continuation, the statement of the theorem holds for all k satisfying $\operatorname{Im} k \neq 0$, $\kappa - 2\operatorname{Re} k > \kappa - 2\operatorname{Re} k_0 - 1$. So in particular the result holds for k_0, but k_0 was an arbitrary complex number satisfying $\operatorname{Im} k_0 \neq 0$, $\kappa - \operatorname{Re} k_0 \le -2n-2$. So the theorem is proved.

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Received July 21, 1999
Revised version August 22, 2000 (4349)

STUDIA MATHEMATICA 143 (3) (2000)

Sobolev embeddings with variable exponent

by

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Abstract. Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and let $p : \overline{\Omega} \to [1,\infty)$ be Lipschitz-continuous. We consider the generalised Lebesgue space $L^{p(x)}(\Omega)$ and the corresponding Sobolev space $W^{1,p(x)}(\Omega)$, consisting of all $f \in L^{p(x)}(\Omega)$ with first-order distributional derivatives in $L^{p(x)}(\Omega)$. It is shown that if $1 \le p(x) < n$ for all $x \in \Omega$, then there is a constant $c > 0$ such that for all $f \in W^{1,p(x)}(\Omega)$,

$$\|f\|_{M,\Omega} \le c\|f\|_{1,p,\Omega}.$$

Here $\|\cdot\|_{M,\Omega}$ is the norm on an appropriate space of Orlicz–Musielak type and $\|\cdot\|_{1,p,\Omega}$ is the norm on $W^{1,p(x)}(\Omega)$. The inequality reduces to the usual Sobolev inequality if $\sup_\Omega p < n$. Corresponding results are proved for the case in which $p(x) > n$ for all $x \in \Omega$.

1. Introduction. The most common assumptions in existence theorems for the Dirichlet boundary-value problem for the quasi-linear equation

$$-\sum_{i=1}^{n} D_i a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x)) = f(x), \quad x \in \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$, involve the polynomial growth of coefficients:

$$|a_i(x,\xi)| \le g(x) + c|\xi|^{q-1}, \quad g \in L^{q'}(\Omega),$$
$$\sum_{i=0}^{n} a_i(x,\xi)\xi_i \ge c_1|\xi|^p - c_2,$$

for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^{n+1}$.

Similarly, regularity problems for variational integrals $\int_\Omega F(\nabla u(x))\,dx$ are solved under the assumption

$$c_1|\xi|^p \le F(\xi) \le c_2(1+|\xi|)^q, \quad \xi \in \mathbb{R}^n.$$

2000 *Mathematics Subject Classification*: 46E35, 26D10.

The research was supported by a Royal Society Joint Project Grant. The second author was also partly supported by the grant no. 201/97/0744 of the Grant Agency of the Czech Republic.

If $p = q$ then the theory of Sobolev spaces $W^{1,p}(\Omega)$ provides a natural and efficient way of handling such questions (cf. [LU]). The situation dramatically changes when $p < q$ and then requires more careful considerations. A particular case appears when the rate of growth of the coefficients varies with $x \in \Omega$.

There has recently been increasing interest in partial differential equations and variational integrals with non-standard growth. Let us mention, for example, [G], [M1], [M2], [BMS] and [FS] for a large number of papers devoted to the regularity of variational problems with $p < q$. V. V. Zhikov [Zh] considers the variational integrals $\int_\Omega (1+|\nabla u(x)|^2)^{\alpha(x)}\,dx$. M. Růžička [R1], [R2] recently studied mathematical models of electrorheological fluids which involved non-linear systems with coefficients of variable rate of growth.

By analogy with the standard situation, a natural tool for the problems with variable growth of coefficients may be the theory of Sobolev spaces $W^{1,p(x)}(\Omega)$ based on generalised Lebesgue spaces $L^{p(x)}(\Omega)$.

Let Ω be a non-empty open bounded set in $\mathbb{R}^n$ and let $p : \Omega \to [1,\infty]$ be a measurable function. Set $\Omega_1 = \{x \in \Omega : p(x) < \infty\}$ and $\Omega_\infty = \Omega \setminus \Omega_1$. For every measurable function f on Ω we define

$$\varrho_p(f) = \max\Big\{ \int_{\Omega_1} |f(x)|^{p(x)}\,dx,\ \operatorname*{ess\,sup}_{x\in\Omega_\infty} |f(x)|\Big\} \tag{1.1}$$

and

$$\|f\|_{p,\Omega} = \inf\{\lambda > 0 : \varrho_p(f/\lambda) \le 1\}.$$

The functional ϱ_p is a *convex modular*, i.e. $\varrho_p \ge 0$, $\varrho_p(f) = 0$ if, and only if, $f = 0$, $\varrho_p(-f) = \varrho_p(f)$, ϱ_p is convex, and $\|\cdot\|_{p,\Omega}$ is a norm on the set $L^{p(x)}(\Omega) = \{f : \varrho_p(f/\lambda) < \infty$ for some $\lambda > 0\}$. The set $L^{p(x)}(\Omega)$ endowed with the norm $\|\cdot\|_{p,\Omega}$ is a Banach space called a *generalised Lebesgue space*. If p is finite a.e. then $L^{p(x)}(\Omega)$ is a particular case of the so-called *Orlicz–Musielak space* (cf. [Mu]) $L^M(\Omega)$ which consists of all measurable functions f on Ω such that $\int_\Omega M(x, \lambda|f(x)|)\,dx < \infty$ for some $\lambda > 0$. Here the function $M : \Omega\times[0,\infty) \to [0,\infty)$ is such that $M(\cdot,t)$ is measurable for every $t \ge 0$ and for a.a. $x \in \Omega$ the function $M(x,\cdot)$ is continuous, non-decreasing, convex and such that $M(x,0) = 0$, $M(x,t) > 0$ for $t > 0$ and $M(x,t) \to \infty$ as $t \to \infty$. The norm in $L^M(\Omega)$ is given by

$$\|f\|_{M,\Omega} = \inf\Big\{\lambda > 0 : \int_\Omega M(x, |f(x)|/\lambda)\,dx \le 1\Big\}.$$

The corresponding Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $f \in L^{p(x)}(\Omega)$ such that all generalised derivatives $D_i f$, $i = 1,\dots,n$, belong to $L^{p(x)}(\Omega)$. Endowed with the norm

$$\|f\|_{1,p,\Omega} = \|f\|_{p,\Omega} + \|\nabla f\|_{p,\Omega}$$

it forms a Banach space.

If $p(x) \equiv p$ then $L^{p(x)}(\Omega)$ coincides with the classical Lebesgue space $L^p(\Omega)$ and the norms in both spaces are equal. Therefore there is no confusion in the notation of the norm. The generalised Lebesgue space $L^{p(x)}(\Omega)$ shares numerous properties with the Lebesgue space. However, there is one essential difference: in general, $L^{p(x)}(\Omega)$ is not invariant with respect to translation (cf. [KR, Ex. 2.9]). This is a cause of difficulties in questions related to convolutions, to continuity of functions in the mean in $L^{p(x)}(\Omega)$ and to boundedness of the Hardy–Littlewood maximal operator.

All these difficulties are reflected in the theory of Sobolev spaces $W^{1,p(x)}(\Omega)$. For instance, the density of smooth functions in $W^{1,p(x)}(\Omega)$ remains an open problem. It is not known whether the well known equality $H = W$ by N. G. Meyers and J. Serrin [MS] (see also [DL]) has a counterpart in spaces with variable exponent $p(x)$. A partial result for p satisfying a certain local monotonicity condition was proved by the authors in [ER].

Another range of questions without satisfactory answer concerns the Sobolev inequality and embedding theorems. We define the Sobolev conjugate exponent p^* by

$$p^*(x) = \frac{np(x)}{n - p(x)}, \qquad x \in \Omega.$$

O. Kováčik and J. Rákosník showed that, in general, the Sobolev space $W^{1,p(x)}(\Omega)$ is not embedded in $L^{p^*(x)}(\Omega)$ (see [KR, Ex. 3.2]). They also proved the following approximate result for continuous functions p (cf. [KR, Thm. 3.3]).

THEOREM 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n > 1)$ and let $p : \overline{\Omega} \to [1, n)$ be continuous. Let $0 < \varepsilon < (n-1)^{-1}$ and let q be a measurable function satisfying $1 \le q(x) \le p^*(x) - \varepsilon$ for $x \in \Omega$. Then there exists a constant $c > 0$ such that*

$$\|f\|_{q,\Omega} \le c\|f\|_{1,p,\Omega}, \qquad f \in C_0^\infty(\Omega).$$

The proof is based on the use of an approximation by step functions and of a partition of unity; as a result, the constant c, in general, blows up when $\varepsilon \to 0$. Let us note that Example 3.2 in [KR] is based on a discontinuous function p. A similar counterexample involving a continuous function p is not known.

M. Růžička recently proved another interesting result by considering the level sets of p and using the power series expansion of the exponential function.

THEOREM 1.2 ([R1, Prop. 2.19]). *Let p be such that $1 \le p_1 < p(x) \le p_2 < n$ for all $x \in \Omega$ and let all the sets $\Omega_q := \{x \in \Omega : p(x) > q\}$,*

$p_1 \le q < p_2$, *have Lipschitz boundary. Moreover, let*

$$\int_{p_1}^{p_2} c(q)^{q^*}\,dq < \infty, \tag{1.2}$$

where $c(q)$ *is the constant of the embedding of* $W^{1,q}(\Omega_q)$ *in* $L^{q^*}(\Omega_q)$, *i.e.* $\|f\|_{q^*,\Omega_q} \le c(q)\|f\|_{1,q,\Omega_q}$ *for* $f \in W^{1,q}(\Omega_q)$. *Then there exists* $c > 0$ *such that*

$$\int_\Omega \frac{|f(x)|^{p^*(x)}}{\log(2+|f(x)|)}\,dx \le c\Big[1+\Big(\int_\Omega (|f(x)|^{p(x)}+|\nabla f(x)|^{p(x)})\,dx\Big)^{p_2^*/p_2}\Big] \tag{1.3}$$

holds for $f \in W^{1,p(x)}(\Omega)$.

Our aim in this paper is to prove inequalities of Sobolev type under the assumption that p is a Lipschitz function. For example, we show that if Ω has a Lipschitz boundary and $p \in C^{0,1}(\overline{\Omega})$ is such that $1 \le p(x) < n$ for $x \in \Omega$, $b > 4-1/n$ and $w(x) = \min\{(n-p(x))^{bp^*(x)}, 1\}$, $M(x,t) = t^{p^*(x)}w(x)$ for $x \in \Omega$, $t \ge 0$, then there exists a constant $c > 0$ such that the inequality

$$\|f\|_{M,\Omega} \le c\|f\|_{1,p,\Omega} \tag{1.4}$$

holds for all $f \in W^{1,p(x)}(\Omega)$. If $\sup_\Omega p < n$, then the weight function w is bounded below and above by positive constants and therefore can be omitted. The inequality (1.4) then has the usual form $\|f\|_{p^*,\Omega} \le c\|f\|_{1,p,\Omega}$. In this case also the inequality $\|f\|_{p^*,\Omega} \le c\|\nabla f\|_{p,\Omega}$ holds for all functions $f \in W^{1,p(x)}(\Omega)$ with $\operatorname{supp} f \subset \Omega$. The method of proof depends upon local estimates in sets in which the oscillation of p is small. Corresponding results are provided for the situation in which $p(x) > n$ for all $x \in \Omega$. To conclude, we present some examples to illustrate what may go wrong if the assumptions are weakened.

To compare the three results mentioned above we first note that each concerns a different class of functions p. The function p in Theorem 1.1 is assumed only continuous but the target space is rather far from the desired optimal case. The function p in Theorem 1.2 can be even discontinuous but there is the logarithmic defect on the left-hand side of (1.3). On the other hand, Lipschitz (and even C^∞) functions p do not, in general, satisfy the assumptions of Theorem 1.2 since their level sets Ω_q need not have a Lipschitz boundary. If p is a Lipschitz function such that all the level sets Ω_q have Lipschitz boundary and (1.2) holds then p satisfies the assumptions of all three assertions and (1.4) gives the best result.

2. Preliminaries. Throughout the paper Ω will be a non-empty, open, bounded subset in $\mathbb{R}^n$, $n \in \mathbb{N}$, and p will be a measurable function on Ω with values in $[1,\infty]$. By saying that Ω has a *Lipschitz boundary* we mean that

the boundary $\partial\Omega$ is locally described by Lipschitz-continuous functions (see, e.g., [KJF, Def. 6.2.2] and the proof of Theorem 4.1 below). For a measurable set $E \subset \mathbb{R}^n$ the symbols $|E|$ and χ_E stand for the n-dimensional Lebesgue measure and for the characteristic function of E, respectively. By $D_i f$, $i = 1,\dots,n$, we denote the generalised derivative of a function f with respect to x_i and by ∇ we denote the (generalised) gradient, $\nabla = (D_1,\dots,D_n)$. The classes of all Lipschitz functions on $\overline{\Omega}$ and of all smooth functions on $\mathbb{R}^n$ with compact support in Ω will be denoted by $C^{0,1}(\overline{\Omega})$ and by $C_0^\infty(\Omega)$, respectively.

Let us recall some basic properties of the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, defined in the Introduction, which will be frequently used in this paper. We refer for further results to [Hu] and [KR].

Every function $f \in L^{p(x)}(\Omega)$ such that $0 < \|f\|_{p,\Omega} < \infty$ satisfies

$$\varrho_p(f/\|f\|_{p,\Omega}) \le 1 \tag{2.1}$$

(cf. [KR, (2.9)]). There is equality in (2.1) if p is bounded.

If $p(x) \le q(x)$ a.e. in Ω and $|\Omega| < \infty$ then

$$\|f\|_{p,\Omega} \le (|\Omega|+1)\|f\|_{q,\Omega}, \quad f \in L^{q(x)}(\Omega) \tag{2.2}$$

(see [KR, Thm. 2.8]).

Hölder's inequality [KR, Thm. 2.1]. *Define the conjugate function* p' *by*

$$p'(x) = \begin{cases} p(x)/(p(x)-1) & \text{if } 1 < p(x) < \infty, \\ \infty & \text{if } p(x) = 1. \end{cases}$$

Then all $f \in L^{p(x)}(\Omega)$ *and* $g \in L^{p'(x)}(\Omega)$ *satisfy the inequality*

$$\int_\Omega |f(x)g(x)|\,dx \le c_p\|f\|_{p,\Omega}\|g\|_{p',\Omega}, \tag{2.3}$$

where

$$c_p = \|\chi_{\Omega_1}\|_{\infty,\Omega} + \|\chi_{\Omega_\infty}\|_{\infty,\Omega} + \operatorname*{ess\,sup}_{x,y\in\Omega}\Big(\frac{1}{p(x)} - \frac{1}{p(y)}\Big) \in [1,3].$$

Let us mention that in order to simplify some estimates we have defined the modular ϱ_p in a way slightly different from that in [KR]. It is easy to see that both definitions lead to equivalent norms and that the assertions (2.1)–(2.3) have in both cases the same form.

Lemma 2.1. *Let* $\gamma \in L^\infty(\Omega)$ *be such that* $1 \le \gamma(x)p(x) \le \infty$ *for a.a.* $x \in \Omega$. *Let* $f \in L^{p(x)}(\Omega)$, $f \ne 0$. *Then*

$$\|f\|^\beta_{\gamma p,\Omega} \le \||f|^\gamma\|_{p,\Omega} \le \|f\|^\alpha_{\gamma p,\Omega} \quad \text{if } \|f\|_{\gamma p,\Omega} \le 1, \tag{2.4}$$

$$\|f\|^\alpha_{\gamma p,\Omega} \le \||f|^\gamma\|_{p,\Omega} \le \|f\|^\beta_{\gamma p,\Omega} \quad \text{if } \|f\|_{\gamma p,\Omega} \ge 1, \tag{2.5}$$

where $\alpha = \operatorname{ess\,inf}_{x\in\Omega}\gamma(x)$, $\beta = \operatorname{ess\,sup}_{x\in\Omega}\gamma(x)$. *In particular, if* $\gamma(x) =$ const *then*

$$\|\,|f|^\gamma\,\|_{p,\Omega} = \|f\|^\gamma_{\gamma p,\Omega}.$$

Proof. According to (2.1), we have $\varrho_{\gamma p}(f/\|f\|_{\gamma p,\Omega}) \le 1$, which yields

$$1 \ge \int\limits_{\{\gamma(x)p(x)<\infty\}} \left(\frac{|f(x)|}{\|f\|_{\gamma p,\Omega}}\right)^{\gamma(x)p(x)} dx \tag{2.6}$$

$$\ge \int\limits_{\{p(x)<\infty\}} \left(\frac{|f(x)|^{\gamma(x)}}{\operatorname*{ess\,sup}_{x\in\Omega}\|f\|^{\gamma(x)}_{\gamma p,\Omega}}\right)^{p(x)} dx$$

and $\operatorname{ess\,sup}_{\gamma(x)p(x)=\infty}|f(x)|/\|f\|_{\gamma p,\Omega} \le 1$. The last inequality implies

$$\operatorname*{ess\,sup}_{p(x)=\infty}|f(x)|^{\gamma(x)} \le \operatorname*{ess\,sup}_{x\in\Omega}\|f\|^{\gamma(x)}_{\gamma p,\Omega},$$

which together with (2.6) yields $\varrho_p(|f|^\gamma/\operatorname{ess\,sup}_{x\in\Omega}\|f\|^{\gamma(x)}_{\gamma p,\Omega}) \le 1$. Thus

$$\|\,|f|^\gamma\,\|_{p,\Omega} \le \operatorname*{ess\,sup}_{x\in\Omega}\|f\|^{\gamma(x)}_{\gamma p,\Omega}.$$

This proves the first inequality in (2.4) and the second inequality in (2.5).

Similarly, $\varrho_p(|f|^\gamma/\|\,|f|^\gamma\,\|_{p,\Omega}) \le 1$. Hence

$$1 \ge \int\limits_{\{p(x)<\infty\}} \left(\frac{|f(x)|^{\gamma(x)}}{\|\,|f|^\gamma\,\|_{p,\Omega}}\right)^{p(x)} dx$$

$$\ge \int\limits_{\{\gamma(x)p(x)<\infty\}} \left(\frac{|f(x)|}{\operatorname*{ess\,sup}_{x\in\Omega}\|\,|f|^\gamma\,\|^{1/\gamma(x)}_{p,\Omega}}\right)^{\gamma(x)p(x)} dx,$$

and $\operatorname{ess\,sup}_{p(x)=\infty}|f(x)|^{\gamma(x)}/\|\,|f|^\gamma\,\|_{p,\Omega} \le 1$, which yields

$$\operatorname*{ess\,sup}_{\gamma(x)p(x)=\infty}|f(x)| \le \operatorname*{ess\,sup}_{x\in\Omega}\|\,|f|^\gamma\,\|^{1/\gamma(x)}_{p,\Omega}.$$

Thus we have $\varrho_{\gamma p}(|f|/\operatorname{ess\,sup}_{x\in\Omega}\|\,|f|^\gamma\,\|^{1/\gamma(x)}_{p,\Omega}) \le 1$, and

$$\|f\|_{\gamma p,\Omega} \le \operatorname*{ess\,sup}_{x\in\Omega}\|\,|f|^\gamma\,\|^{1/\gamma(x)}_{p,\Omega}.$$

If $\alpha > 0$, this proves the second inequality in (2.4) and the first one in (2.5). If $\alpha = 0$ and $\|\,|f|^\gamma\,\|_{p,\Omega} \le 1$, then $\|f\|_{\gamma p,\Omega} \le 1$ by the first inequality (2.4) and the second one in (2.4) holds trivially. If $\alpha = 0$ and $\|\,|f|^\gamma\,\|_{p,\Omega} \ge 1$, then $\|f\|_{\gamma p,\Omega} \ge 1$ by the second inequality (2.5) and the first inequality in (2.5) follows. ■

3. A Sobolev inequality

LEMMA 3.1. *Let* $p \in C^{0,1}(\overline{\Omega})$ *and let* q, r *be such that*

$$1 < r \le p(x) \le q < \min\{n, r^*\}, \qquad x \in \Omega. \tag{3.1}$$

Then there exists $c > 0$ *such that*

$$\|f\|_{p^*,\Omega} \le c\|\nabla f\|_{p,\Omega} \tag{3.2}$$

for all $f \in W^{1,p(x)}(\Omega)$ *with* $\operatorname{supp} f \subset \Omega$. *The constant* c *satisfies the estimate*

$$c \le \max\{1, [c_0(n-q)^{-2}]^a\} \tag{3.3}$$

where $c_0 > 0$ *depends on* $|\Omega|$, n, p, *and* $a = (r'-n')/(q'-n')$.

Proof. Let $f \in W^{1,p(x)}(\Omega)$ be such that $\operatorname{supp} f \subset \Omega$ and $\|f\|_{1,p,\Omega} \le 1$. Since $W^{1,p(x)}(\Omega) \subset W^{1,r}(\Omega)$ we can assume without loss of generality that f is absolutely continuous on almost all closed segments in Ω parallel to the coordinate axes and that for a.a. $x \in \Omega$ the classical derivatives $\partial f(x)/\partial x_i$, $i = 1, \dots, n$, exist and coincide with the corresponding generalised derivatives $D_i f(x)$.

Following the standard idea of the proof of the Sobolev inequality we set

$$\gamma(x) = \frac{p^*(x)}{n'} = \frac{(n-1)p(x)}{n-p(x)}, \qquad x \in \Omega. \tag{3.4}$$

Note that, by (3.1), γ is a Lipschitz function satisfying

$$1 < \frac{(n-1)r}{n-r} \le \gamma(x) \le (n-1)\max\left\{\frac{r}{n-2r}, \frac{q}{n-q}\right\} < \infty. \tag{3.5}$$

For $i = 1, \dots, n$ and for a.a. $x \in \Omega$ we have

$$|D_i(|f(x)|^{\gamma(x)})| \le \frac{n(n-1)|D_i p(x)|}{(n-p(x))^2}|f(x)|^{\gamma(x)}|\log|f(x)|| + \gamma(x)|f(x)|^{\gamma(x)-1}|D_i f(x)|.$$

By integrating this inequality along segments in Ω parallel to the x_i axis and then over Ω_i, the projection of Ω onto the hyperplane $x_i = 0$, we obtain

$$\int\limits_{\Omega_i}\max_{x_i}|f(x)|^{\gamma(x)}\,dx_i' \le \frac{n(n-1)L}{(n-q)^2}\int\limits_\Omega |f(x)|^{\gamma(x)}|\log|f(x)||\,dx + \frac{(n-1)q}{n-q}\int\limits_\Omega |f(x)|^{\gamma(x)-1}|\nabla f(x)|\,dx \tag{3.6}$$

where $x_i' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, L is the Lipschitz constant for p and the supremum is taken over all x_i such that $(x_i', x_i) \in \Omega$ for some $x_i' \in \Omega_i$.

The second term on the right-hand side of (3.6) can be estimated using the Hölder inequality (2.3):

$$\int_\Omega |f(x)|^{\gamma(x)-1}|\nabla f(x)|\,dx \le c_p \|\,|f|^{\gamma-1}\|_{p',\Omega}\,\|\nabla f\|_{p,\Omega}. \tag{3.7}$$

To estimate the first term on the right-hand side of (3.6) we have to handle the disturbing logarithm. We distinguish the cases when $|f(x)|\le 1$ and $|f(x)|>1$. Using the relations

$$\sup_{0<t\le 1} t|\log t| = e^{-1}, \quad \sup_{t>1} t^{-\varepsilon}\log t = (e\varepsilon)^{-1}, \quad \varepsilon>0,$$

and the Hölder inequality we obtain

$$\begin{aligned}\int_{\{|f(x)|\le 1\}} |f(x)|^{\gamma(x)}\big|\log|f(x)|\big|\,dx &= \int_{\{|f(x)|\le 1\}} |f(x)|^{\gamma(x)-1}|f(x)|\big|\log|f(x)|\big|\,dx\\ &\le c_p e^{-1}\|\,|f|^{\gamma-1}\|_{p',\Omega}\|1\|_{p,\Omega},\end{aligned} \tag{3.8}$$

$$\begin{aligned}\int_{\{|f(x)|>1\}} |f(x)|^{\gamma(x)}\big|\log|f(x)|\big|\,dx &= \int_{\{|f(x)|>1\}} |f(x)|^{\gamma(x)-1}|f(x)|^{1+\varepsilon}|f(x)|^{-\varepsilon}\big|\log|f(x)|\big|\,dx\\ &\le c_p(e\varepsilon)^{-1}\|\,|f|^{\gamma-1}\|_{p',\Omega}\|\,|f|^{1+\varepsilon}\|_{p,\Omega}.\end{aligned} \tag{3.9}$$

According to (3.1), f satisfies the classical Sobolev inequality

$$\|f\|_{r^*,\Omega}\le c(n,r)\|\nabla f\|_{r,\Omega}.$$

Taking $\varepsilon = r^*/q-1$ we have $q(1+\varepsilon)=r^*$ and hence, by Lemma 2.1 and (2.2),

$$\begin{aligned}\|\,|f|^{1+\varepsilon}\|_{p,\Omega} = \|f\|^{1+\varepsilon}_{p(1+\varepsilon),\Omega} &\le (|\Omega|+1)^{1+\varepsilon}\|f\|^{1+\varepsilon}_{q(1+\varepsilon),\Omega}\\ &\le (|\Omega|+1)^{1+\varepsilon}c(n,r)^{1+\varepsilon}\|\nabla f\|^{1+\varepsilon}_{r,\Omega}\\ &\le c(|\Omega|,n,r,\varepsilon)\|\nabla f\|^{r^*/q}_{p,\Omega}.\end{aligned} \tag{3.10}$$

From (3.6)–(3.9) and (3.10) we conclude that

$$\begin{aligned}\int_{\Omega_i}\max_{x_i}|f(x)|^{\gamma(x)}\,dx_i' &\le c(|\Omega|,n,p,r,q)(1+\|\nabla f\|_{p,\Omega}+\|\nabla f\|^{r^*/q}_{p,\Omega})\|\,|f|^{\gamma-1}\|_{p',\Omega}\\ &\le 3c(|\Omega|,n,p,r,q)\|\,|f|^{\gamma-1}\|_{p',\Omega}.\end{aligned}$$

Using the well known Gagliardo inequality we obtain

$$\begin{aligned}\int_\Omega |f(x)|^{p^*}\,dx &\le \int_\Omega\Big(\prod_{i=1}^n \max_{x_i}|f(x)|^{p^*/n}\Big)dx\\ &\le \prod_{i=1}^n\Big(\int_{\Omega_i}\max_{x_i}|f(x)|^{\gamma(x)}\,dx_i'\Big)^{1/(n-1)}\\ &\le [c_0(n-q)^{-2}\|\,|f|^{\gamma-1}\|_{p',\Omega}]^{n'}\\ &\le [K\|\,|f|^{\gamma-1}\|_{p',\Omega}]^{n'},\end{aligned} \tag{3.11}$$

where $c_0>0$ depends on $|\Omega|$, n, p, r and q, and $K=\max\{1,c_0(n-q)^{-2}\}$. Setting $g=|f|^{\gamma-1}$ we can rewrite the estimate (3.11) in the form

$$\int_\Omega g(x)^{p'(x)}\,dx \le [K\|g\|_{p',\Omega}]^{n'}. \tag{3.12}$$

If $\|f\|_{p^*,\Omega}\ge 1$ then, by Lemma 2.1 and (3.5),

$$\|\,|f|^{\gamma-1}\|_{p',\Omega}\ge \|f\|^{\alpha}_{p^*,\Omega}\ge 1, \tag{3.13}$$

where $\alpha=(n-1)r/(n-r)-1=n'/(r'-n')>0$, and we use the convexity of the modular $\varrho_{p'}$ to obtain

$$\int_\Omega\Big(\frac{g(x)}{[K\|g\|_{p',\Omega}]^{n'/q'}}\Big)^{p'(x)}\le 1.$$

Hence $\|g\|_{p',\Omega}\le K^{n'/q'}\|g\|^{n'/q'}_{p',\Omega}$, i.e. $\|g\|_{p',\Omega}\le K^{n'/(q'-n')}$. Using (3.13) we obtain

$$\|f\|_{p^*,\Omega}\le K^a, \tag{3.14}$$

where $a=(r'-n')/(q'-n')$.

If $\|f\|_{p^*,\Omega}<1$, then (3.12) holds as well since $K\ge 1$ and $a>0$. ∎

Note that the constant c_0 depends on ε and blows up when ε tends to zero, i.e. when q tends to r^*. That is why the last inequality required in (3.1) is strict.

The assumption $\inf p\ge r>1$ was important for the estimates (3.13) and (3.14). If this condition is not satisfied we have to proceed in a slightly different way:

LEMMA 3.2. *Let $p\in C^{0,1}(\overline{\Omega})$ and q be such that*

$$1\le p(x)\le q<\frac{2n}{n+1}, \quad x\in\Omega. \tag{3.15}$$

Then there exists $c>0$ such that (3.2) holds for all $f\in W^{1,p(x)}(\Omega)$ with $\operatorname{supp} f\subset\Omega$. The constant c satisfies the estimate (3.3) where $c_0>0$ depends *on* $|\Omega|$, n, p, *and*

$$a = \frac{n-q}{2n-q(n+1)}. \tag{3.16}$$

Proof. We repeat the proof of Lemma 3.1 up to the estimate (3.11). Now, assume that $\|f\|_{p^*,\Omega} \ge 1$. By Lemma 2.1 and (3.15),

$$1 \le \| |f|^{\gamma-1}\|_{p',\Omega} \le \|f\|^{\beta}_{p^*,\Omega},$$

where $\beta = (n-1)q/(n-q) - 1 = n'/(q'-n') > 0$, and (3.11) implies

$$\int_\Omega |f(x)|^{p^*(x)}\,dx \le [K\|f\|^{\beta}_{p^*,\Omega}]^{n'}. \tag{3.17}$$

Using the convexity of the modular ϱ_{p^*} and the inequality $n' \le p^*$ we obtain

$$\int_\Omega \left(\frac{|f(x)|}{K\|f\|^{\beta}_{p^*,\Omega}}\right)^{p^*(x)} dx \le 1,$$

i.e.

$$\|f\|_{p^*,\Omega} \le K\|f\|^{\beta}_{p^*,\Omega}. \tag{3.18}$$

According to the assumption (3.15) we have $n'/(q'-n') < 1$, and (3.18) implies

$$\|f\|_{p^*,\Omega} \le K^a, \tag{3.19}$$

where a satisfies (3.16). Since $K \ge 1$, the estimate (3.19) is satisfied also in the case when $\|f\|_{p^*,\Omega} < 1$. ■

THEOREM 3.1. *Let* $p \in C^{0,1}(\overline{\Omega})$ *and let* q *be such that*

$$1 \le p(x) \le q < n, \quad x \in \Omega.$$

Then there exists a constant $c > 0$ *such that*

$$\|f\|_{p^*,\Omega} \le c\|\nabla f\|_{p,\Omega}$$

for all $f \in W^{1,p(x)}(\Omega)$ *with* $\operatorname{supp} f \subset \Omega$.

Proof. The function p can be extended to a Lipschitz-continuous function on $\mathbb{R}^n$ preserving the Lipschitz constant L and the upper and lower bounds. Indeed, following the idea of E. J. McShane [McS, Theorem 1] we define $\widetilde{p}(x) = \inf\{p(y) + L|x-y| : y \in \Omega\}$ for $x \in \mathbb{R}^n \setminus \Omega$ and truncate the function $\widetilde{p}$ by $\sup_{x\in\Omega} p(x)$. We shall denote the extended function again by p.

Let $r_1 = \inf_{x\in\Omega} p(x) < r_2 < q_1 < r_3 < q_2 < \ldots < r_m < q_{m-1} < q_m = \sup_{x\in\Omega} p(x)$ be such that $1/r_j - 1/q_j < 1/n$ for $j = 1,\ldots,m$. Moreover, let $q_1 < 2n/(n+1)$ if $r_1 = 1$. There exist bounded open sets $G_1, \ldots, G_m$ such that $\overline{\Omega} \subset \bigcup_{j=1}^m G_j$ and $r_j \le p(x) \le q_j$ for $x \in G_j$. Let $\{\varphi\}_{j=1}^m$ be a partition of unity on $\overline{\Omega}$ subordinate to $\{G_j\}_{j=1}^m$, i.e. $\varphi_j \in C_0^\infty(G_j)$, $0 \le \varphi_j \le 1$, $\sum_{j=1}^m \varphi_j(x) = 1$ for $x \in \overline{\Omega}$.

Let $f \in W^{1,p(x)}(\Omega)$ be such that $\operatorname{supp} f \subset \Omega$. We extend the function f by zero outside Ω, still denote it by f and set $f_j = f\varphi_j$. For each f_j we can use Lemma 3.1 or 3.2 and we obtain

$$\begin{aligned}\|f\|_{p^*,\Omega} &\le \sum_{j=1}^m \|f_j\|_{p^*,\Omega\cap G_j} \le \sum_{j=1}^m c_j\|\nabla f_j\|_{p^*,\Omega\cap G_j} \\ &\le \sum_{j=1}^m \sup_{x\in\Omega}|\varphi_j(x)|\,\|f\|_{1,p,\Omega} = c\,\|f\|_{1,p,\Omega}.\end{aligned} \tag{3.20}$$

It suffices to prove that there exists $c_0 > 0$ independent of f such that

$$\|f\|_{p,\Omega} \le c_0\|\nabla f\|_{p,\Omega}.$$

Let us assume, to the contrary, that there exists a sequence of functions $f_k \in W^{1,p(x)}(\Omega)$ with $\operatorname{supp} f_k \subset \Omega$ such that

$$k\|\nabla f_k\|_{p,\Omega} < \|f_k\|_{p,\Omega} = 1. \tag{3.21}$$

By the Hölder inequality (cf. [KR, Corollary 2.2]), there is a constant $c(p) > 0$ such that for every $g \in L^{p(x)}(\Omega)$,

$$\|g\|_{p,\Omega} \le c(p)\|g\|^{\mu}_{1,\Omega}\,\|g\|^{\nu}_{p^*,\Omega}, \tag{3.22}$$

where

$$\mu = \inf_{x\in\Omega}\frac{p^*(x)-p(x)}{p(x)(p^*(x)-1)} \ge \frac{q}{nq-n+q} > 0 \quad\text{and}\quad \nu \ge 0.$$

It follows from (3.21) that the sequence $\{f_k\}$ is bounded in $W^{1,p(x)}(\Omega)$. Since $W^{1,p(x)}(\Omega)$ is embedded in $W^{1,1}(\Omega)$, $\{f_k\}$ is also bounded in $W_0^{1,1}(\Omega)$. There is a compact embedding of $W_0^{1,1}(\Omega)$ in $L^1(\Omega)$ and so $\{f_k\}$ contains an L^1-Cauchy subsequence, denoted again by $\{f_k\}$. Using (3.21), (3.22) and (3.20) we obtain

$$\|f_k - f_l\|_{p,\Omega} \le c(p)\|f_k - f_l\|^{\mu}_{1,\Omega}\|f_k - f_l\|^{\nu}_{p^*,\Omega} \le c(p)\|f_k - f_l\|^{\mu}_{1,\Omega}\cdot(4c)^{\nu}.$$

Thus $\{f_k\}$ is a Cauchy sequence in $L^{p(x)}(\Omega)$ and converges to a function f in $L^{p(x)}(\Omega)$. Using the definition of the generalised derivative and passing to the limit for $k \to \infty$ we conclude that $\nabla f = 0$ a.e. in Ω. Hence f is constant on Ω and therefore $f = 0$, which contradicts (3.21). ■

4. Extension operator. Theorem 3.1 concerns functions from Sobolev spaces $W^{1,p(x)}(\Omega)$ with compact support in Ω, i.e. functions which can be extended by zero outside Ω. The embedding properties of Sobolev spaces on domains strongly depend on the shape of the domain. One way of handling this obstacle is to consider the class of so-called extension domains. These are domains Ω for which there exists a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^n)$. We shall construct such an extension operator for $W^{1,p(x)}(\Omega)$ with a Lipschitz domain Ω using the reflection method due to M. Hestenes [H]. It is natural that the case of $W^{1,p(x)}$ also involves the question of a proper extension of p.

LEMMA 4.1. *Let* $-\infty \le a_i < b_i \le \infty$, $i = 1,\ldots,n-1$, $0 < b_n \le \infty$, $Q_+ = (a_1,b_1)\times\ldots\times(a_{n-1},b_{n-1})\times(0,b_n)$ *and let* $p : Q_+ \to [1,\infty)$ *be a measurable function. Let* $f \in W^{1,p(x)}(Q_+)$. *Define the extension* Ef *to* $Q = (a_1,b_1)\times\ldots\times(a_{n-1},b_{n-1})\times(-b_n,b_n)$ *by*

$$Ef(x) = \begin{cases} f(x',x_n), & (x',x_n)\in Q_+,\\ f(x',-x_n), & (x',-x_n)\in Q_+.\end{cases}$$

Define Ep *analogously. Then* $Ef \in W^{1,Ep(x)}(Q)$ *and*

$$\|Ef\|_{p,Q} \le 2\|f\|_{p,Q_+}, \quad \|\nabla(Ef)\|_{p,Q} \le 2\|\nabla f\|_{p,Q_+}.$$

Proof. Since $f, D_i f \in L^1_{\rm loc}(Q_+)$, $i = 1,\ldots,n$, we know from the classical result that

$$D_i(Ef) = E(D_i f), \quad i = 1,\ldots,n-1,$$

and

$$D_n(Ef)(x',x_n) = \begin{cases} D_n f(x',x_n), & (x',x_n)\in Q_+,\\ -D_n f(x',-x_n), & (x',-x_n)\in Q_+.\end{cases}$$

The assertion follows immediately. ■

A mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ is called *bi-Lipschitz* if there exists a constant L, $1 \le L < \infty$, such that

$$L^{-1}|x-y| \le |T(x)-T(y)| \le L|x-y|, \quad x,y\in\mathbb{R}^n.$$

To prove the extension theorem for Lipschitz domains we shall need the following property of bi-Lipschitz mappings.

LEMMA 4.2. *Let* $p : \Omega \to [0,\infty)$ *be measurable. Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *be a bi-Lipschitz mapping,* $G = T^{-1}(\Omega)$, *and let* $f \in W^{1,p(x)}(\Omega)$. *Set* $g = f\circ T$ *and* $q = p\circ T$. *Then* $g \in W^{1,q(x)}(G)$ *and*

$$\|g\|_{1,q,G} \le c\|f\|_{1,p,\Omega},$$

where $c > 0$ *depends only on* n *and on the Lipschitz constant* L *for* T *and* T^{-1}.

Proof. Let $\Omega' \subset\subset \Omega$ be a bounded subdomain and let $G' = T^{-1}(\Omega')$. Then $f \in W^{1,1}(\Omega')$. By the classical result (see [Z, Thm. 2.2.2]), $g \in W^{1,1}(G')$ and

$$\nabla f(T(x))\cdot dT(x,\xi) = \nabla g(x)\cdot\xi \tag{4.1}$$

for all $\xi\in\mathbb{R}^n$ and a.a. $x\in G'$. Since Ω' was arbitrary we conclude that (4.1) holds for a.a. $x \in G$. Hence

$$|\nabla g(x)| \le L|\nabla f(T(x))| \quad \text{for a.a. } x\in G. \tag{4.2}$$

Let

$$\lambda > L^{1+n}\|\nabla f\|_{p,\Omega}. \tag{4.3}$$

Then (4.2) and the estimates of the Jacobian, $L^{-n} \le JT(x) \le L^n$ for a.a. $x\in\mathbb{R}^n$, imply

$$\begin{aligned}\int_G \left(\frac{|\nabla g(x)|}{\lambda}\right)^{q(x)} dx &\le \int_G (L\lambda^{-1}|\nabla f(T(x))|)^{p(T(x))}\,dx\\ &\le \int_G L^n(L\lambda^{-1}\|\nabla f\|_{p,\Omega})^{p(T(x))}\left(\frac{|\nabla f(T(x))|}{\|\nabla f\|_{p,\Omega}}\right)^{p(T(x))} JT(x)\,dx\\ &\le \int_\Omega \left(\frac{|\nabla f(x)|}{\|\nabla f\|_{p,\Omega}}\right)^{p(x)} dx \le 1,\end{aligned}$$

i.e. $\|\nabla g\|_{q,G} \le \lambda$. Since λ was an arbitrary number satisfying (4.3) we conclude that $\|\nabla g\|_{q,G} \le L^{1+n}\|\nabla f\|_{p,\Omega}$. In a similar way we obtain the estimate $\|g\|_{q,G} \le L^n\|f\|_{p,\Omega}$. ■

THEOREM 4.1. *Let* Ω *have a Lipschitz boundary. Then there exists a function* $q : \mathbb{R}^n \to [1,\infty)$ *and a bounded linear extension operator*

$$\mathcal{E} : W^{1,p(x)}(\Omega) \to W^{1,q(x)}(\mathbb{R}^n)$$

such that $q(x) = p(x)$, $x\in\Omega$, $\sup_{\mathbb{R}^n} q = \sup_\Omega p$, $\inf_{\mathbb{R}^n} q = \inf_\Omega p$, *and*

$$\|\mathcal{E}f\|_{1,q,\mathbb{R}^n} \le c\|f\|_{1,p,\Omega}, \quad f\in W^{1,p(x)}(\Omega).$$

The extension $\mathcal{E}f$ *has compact support contained in* $\{x\in\mathbb{R}^n : \operatorname{dist}(x,\Omega) \le \beta\}$ *for some positive number* β. *If, moreover,* $p \in C^{0,1}(\overline{\Omega})$, *then* $q \in C^{0,1}(\mathbb{R}^n)$.

Proof. Let $\{V_j\}_{j=1}^k$ be the covering of the boundary $\partial\Omega$ which corresponds to the local description of $\partial\Omega$. More precisely, for each $j = 1,\ldots,k$, there is a local coordinate system (x',x_n) such that

$$V_j = \{(x',x_n) : |x_i| < \delta,\ i = 1,\ldots,n-1,\ a_j(x') - \beta < x_n < a_j(x') + \beta\},$$
$$V_j\cap\Omega = \{x\in V_j : a_j(x') < x_n < a_j(x') + \beta\}$$

and

$$\{x\in\overline{V}_j : x_n < a(x')\}\cap\overline{\Omega} = \emptyset, \tag{4.4}$$

where β, δ are some fixed positive numbers and $a_j \in C^{0,1}((-\delta,\delta)^{n-1})$ are the functions describing the boundary. Define the mappings

$$T_j : G = (-\delta,\delta)^{n-1} \times (-\beta,\beta) \to \mathbb{R}^n, \quad j = 1,\ldots,n,$$

by

$$T_j(x', x_n) = (x', x_n + a_j(x')).$$

Then the T_j are bi-Lipschitz mappings. Let $V_0 \subset \Omega$ be an open set such that $\overline{V}_0 \subset \Omega$ and $\overline{\Omega} \subset \bigcup_{j=0}^k V_j$. Let $\{\varphi_j\}$ be a partition of unity subordinate to $\{V_j\}$, i.e. $\varphi_j \in C_0^\infty(V_j)$, $0 \le \varphi_j \le 1$ and $\sum_{j=0}^k \varphi_j = 1$ on $\overline{\Omega}$.

Let $f \in W^{1,p(x)}(\Omega)$. We define the functions f_j by

$$f_j(x) = f(x)\varphi_j(x), \quad x \in \Omega,\ j = 0,\ldots,k.$$

Then $f_j \in W^{1,p(x)}(V_j \cap \Omega)$ and

$$\|f_j\|_{1,p,V_j\cap\Omega} \le c_1 \|f\|_{1,p,\Omega} \tag{4.5}$$

where c_1 depends on p and on $\{\varphi_j\}$. We set $G_+ = (-\delta,\delta)^{n-1} \times (0,\beta)$ and define the functions g_j by

$$g_j(x) = \begin{cases} f_j(T_j(x)), & x \in G_+, \\ 0, & x \in \mathbb{R}^n_+ \setminus G_+. \end{cases}$$

Let $j = 1,\ldots,k$. Set $r_j = p \circ T_j$. We can use Lemma 4.1 to extend g_j to $Eg_j \in W^{1,Er_j(x)}(\mathbb{R}^n)$ so that

$$\|Eg_j\|_{1,Er_j,\mathbb{R}^n} \le 2\|g_j\|_{1,r_j,G_+}. \tag{4.6}$$

It follows from the construction of E that $\operatorname{supp} Eg_j \subset G$.

We define the functions q_j, $j = 1,\ldots,k$, by

$$q_j(x) = \begin{cases} p(x), & x \in \Omega, \\ Er_j(T_j^{-1}(x)), & x \in V_j \setminus \Omega, \end{cases}$$

and extend them on $\mathbb{R}^n$ preserving their upper and lower bounds.

Now, we define the function q by

$$q(x) = \min_{1\le j\le k} q_j(x), \quad x \in \mathbb{R}^n,$$

and the function $\mathcal{E}f$ by

$$\mathcal{E}f(x) = f_0(x) + \sum_{j=1}^k Eg_j(T_j^{-1}(x)), \quad x \in \mathbb{R}^n,$$

where f_0 and $Eg_j \circ T_j^{-1}$ are extended by zero to the whole $\mathbb{R}^n$.

Clearly, $\mathcal{E}f(x) = f(x)$ for $x \in \Omega$. Applying Lemma 4.2 for Eg_j and T_j^{-1} and using the estimates (4.5), (4.6) and (2.2) we obtain

$$\begin{aligned} \|\mathcal{E}f\|_{1,q,\mathbb{R}^n} &\le \|\mathcal{E}f\|_{1,q,\Omega} + \sum_{j=1}^k \|\mathcal{E}f\|_{1,q,V_j\setminus\Omega} \\ &= \|f\|_{1,p,\Omega} + \sum_{j=1}^k \|Eg_j \circ T_j^{-1}\|_{1,q,V_j\setminus\Omega} \\ &\le \|f\|_{1,p,\Omega} + \sum_{j=1}^k \|Eg_j \circ T_j^{-1}\|_{1,q_j,V_j\setminus\Omega}(|V_j\setminus\Omega| + 1) \\ &\le c\|f\|_{1,p,\Omega}, \end{aligned}$$

where $c > 0$ is a constant which depends on n, p and on the parameters of description of the boundary $\partial\Omega$.

If $p \in C^{0,1}(\overline{\Omega})$ then $r_j \in C^{0,1}(G_+)$, $q_j \in C^{0,1}(V_j \cup \Omega)$ (cf. (4.4)) and q_j can be extended to a Lipschitz function on $\mathbb{R}^n$. Thus also $q \in C^{0,1}(\mathbb{R}^n)$. ■

5. Embedding theorems. Using the extension operator from Theorem 4.1 and the Sobolev inequality from Theorem 3.1 we can easily obtain the following embedding theorem.

THEOREM 5.1. *Let Ω have a Lipschitz boundary. Let $p \in C^{0,1}(\overline{\Omega})$ and let q be such that $1 \le p(x) \le q < n$ for all $x \in \Omega$. Then there exists a constant $c > 0$ such that*

$$\|f\|_{p^*,\Omega} \le c\|f\|_{1,p,\Omega}$$

for all $f \in W^{1,p(x)}(\Omega)$.

If $p = n$ then the classical Sobolev inequality $\|f\|_{q,\Omega} \le c(q)\|\nabla f\|_{n,\Omega}$ and the embedding theorem hold for every $q \in [0,\infty)$ while the constant $c(q)$ is not uniformly bounded. It is therefore natural to introduce an appropriate weight in $L^{p(x)}$ if p is not bounded away from n. To prove the corresponding result we shall need the following covering lemma of Besicovitch type; the proof uses ideas from [Gu, Lemma 1.6] and [EvR, Lemma 1].

LEMMA 5.1. *Let $p \in C^{0,1}(\mathbb{R}^n)$ be such that $1 \le p(x) < n = \sup_\Omega p = \sup_{\mathbb{R}^n} p$ for all $x \in \Omega$. Let L be the Lipschitz constant for p and let κ, δ satisfy $0 < 2\kappa < \delta^{-1} < 1$. Define the function σ by $\sigma(x) = \kappa L^{-1}(n - p(x))$, $x \in \mathbb{R}^n$. Then there exists a sequence of points $x_k \in \Omega$ with the following properties:*

(i) *$\Omega \subset \bigcup_k B_k^* \subset \bigcup_k B_k$, where $B_k^* = B(x_k, \sigma(x_k))$, $B_k = B(x_k, \delta\sigma(x_k))$;*
(ii) *$\lim_{k\to\infty} p(x_k) = n$;*
(iii) *$p(x) < n$ for all $x \in \overline{B}_k$;*

(iv) $\operatorname{diam}(\bigcup_k B_k) \le \operatorname{diam}\Omega + 2(n-1)\delta\kappa L^{-1} \le \operatorname{diam}\Omega + (n-1)L^{-1}$;
(v) *there exists a number* $\theta = \theta(n, L, \kappa, \delta)$ *such that* $\sum_k \chi_{B_k} \le \theta$.

Proof. According to the assumptions, for all $x \in \Omega$, $y \in \overline{B(x, \sigma(x))}$ we have

$$(5.1)\qquad 1-\kappa \le \frac{n-p(y)}{n-p(x)} \le 1+\kappa,$$

$$(5.2)\qquad p(y) \le (1-\kappa)p(x) + \kappa n < n.$$

The balls $B(x, \sigma(x)/5)$, $x \in \Omega$, cover the bounded set Ω and the radii $\sigma(x)$ are bounded. By the so-called $5r$-covering lemma (see [Ma, Thm. 2.1]) there exist $x_k \in \Omega$ such that the balls $B_k(x_k, \sigma(x_k)/5)$ are pairwise disjoint and $\Omega \subset \bigcup_k B(x_k, \sigma(x_k))$. We claim that $\{x_k\}$ is the required sequence.

The properties (i) and (ii) are obvious.

If $x \in \overline{B}$, then $|x - x_k| < \delta\kappa L^{-1}(n - p(x_k))$ and $p(x) \le p(x_k) + |p(x) - p(x_k)| < p(x_k) + \delta\kappa(n - p(x_k)) < p(x_k) + \frac{1}{2}(n - p(x_k)) < n$. Thus (iii) holds.

The property (iv) follows from the estimate $\delta\sigma(x) \le \delta(n-1)\kappa L^{-1}$.

To prove (v) we assume that $x \in B(x_k, \sigma(x_k)) \cap B(x_m, \sigma(x_m))$. Then $B(x_k, \sigma(x_k)/5) \subset B(x, 6\sigma(x_k)/5)$ and from (5.1) we have

$$\frac{1-\kappa}{1+\kappa} \le \frac{\sigma(x_k)}{\sigma(x_m)} \le \frac{1+\kappa}{1-\kappa}.$$

Since the balls $B(x_k, \sigma(x_k)/5)$ are pairwise disjoint, we conclude that

$$\theta \le \sup\left\{\left[6\frac{\sigma(x_k)}{\sigma(x_m)}\right]^n : B(x_k, \sigma(x_k)) \cap B(x_m, \sigma(x_m)) \ne \emptyset\right\} \le \left[6\frac{1+\kappa}{1-\kappa}\right]^n. \ \blacksquare$$

THEOREM 5.2. *Let Ω have a Lipschitz boundary. Let $p \in C^{0,1}(\overline{\Omega})$ be such that*

$$(5.3)\qquad 1 \le p(x) < n = \sup_\Omega p, \qquad x \in \Omega.$$

Let $b > 4 - 1/n$ and

$$(5.4)\qquad w(x) = \min\{(n-p(x))^{bp^*(x)}, 1\}, \qquad M(x,t) = t^{p^*(x)}w(x), \qquad x \in \Omega,\ t \ge 0.$$

Then there exists a constant $c > 0$ such that

$$(5.5)\qquad \|f\|_{M,\Omega} \le c\|f\|_{1,p,\Omega}$$

for all $f \in W^{1,p(x)}(\Omega)$.

Proof. According to Theorem 4.1, there exists a bounded linear extension operator $\mathcal{E} : W^{1,p(x)}(\Omega) \to W^{1,\widetilde{p}(x)}(\mathbb{R}^n)$ where $\widetilde{p} \in C^{0,1}(\mathbb{R}^n)$ is an extension of p on $\mathbb{R}^n$, with the same Lipschitz constant L and such that $\inf_\Omega p = \inf_{\mathbb{R}^n} \widetilde{p}$, $\sup_\Omega p = \sup_{\mathbb{R}^n} \widetilde{p}$. We shall denote the function $\widetilde{p}$ again by p.

Let κ and δ satisfy

$$(5.6)\qquad \delta > 1, \quad 0 < \kappa < \delta^{-1}\min\left\{\frac{b-4+1/n}{b+4-1/n}, \frac{1}{(n-1)(2n+1)}\right\}.$$

There exists a sequence of points x_k and a sequence of functions $\varphi_k \in C_0^\infty(B_k)$ such that

$$\Omega \subset \bigcup_k B_k^* \subset \bigcup_k B_k, \qquad B_k^* = B(x_k, \sigma_k), \qquad B_k = B(x_k, \delta\sigma_k),$$

$$(5.7)\qquad \sigma_k = \kappa L^{-1}(n - p_k), \quad p_k = p(x_k) < n, \quad p_k \to n \quad \text{as } k \to \infty,$$

$$(5.8)\qquad \sum_k \chi_{B_k} \le \theta = \theta(n, L, \kappa, \delta) < \infty,$$

$0 \le \varphi_k \le 1$ on $\mathbb{R}^n$, $\sum_k \varphi_k = 1$ on Ω, and $|\nabla\varphi_k| \le c_0\sigma_k^{-1}$, where $c_0 > 0$ is a constant dependent on δ. To show this we set $F = \{x \in \mathbb{R}^n : p(x) = n\}$ and apply Lemma 5.1 for the domain $\widetilde{\Omega} = \{x \in \mathbb{R}^n \setminus F : \operatorname{dist}(x, \Omega) < 3(n-1)L^{-1}\}$ to obtain the corresponding sequences of points $x_k \in \widetilde{\Omega}$ and balls B_k. There exist functions $\psi_k \in C_0^\infty(B_k)$, $k \in \mathbb{N}$, such that $\psi_k(x) = 1$ for $x \in B_k^*$, $|\nabla\psi_k(x)| \le c_0\sigma_k$ for $x \in \mathbb{R}^n$, and a function $\psi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\psi_0(x) = 1$ if $\operatorname{dist}(x, \Omega) > 2(n-1)L^{-1}$ and $\psi_0(x) = 0$ if $\operatorname{dist}(x, \Omega) < (n-1)L^{-1}$. Then $\psi = \sum_k \psi_k \in C^\infty(\mathbb{R}^n \setminus F)$ and $\psi \ge 1$ on $\mathbb{R}^n \setminus F$. We set $\varphi_k = \psi_k\psi^{-1}$ and consider only those k for which $B_k \cap \Omega \ne \emptyset$.

For $x \in B_k$ we have

$$(5.9)\qquad r_k := \max\{1, (1+\kappa\delta)p_k - \kappa\delta n\} \le p(x) \le q_k := (1-\kappa\delta)p_k + \kappa\delta n,$$

which implies

$$(5.10)\qquad 1 - \kappa\delta = \frac{n-q_k}{n-p_k} \le \frac{n-p(x)}{n-p_k} \le \frac{n-r_k}{n-p_k} = 1 + \kappa\delta.$$

Let $f \in W^{1,p(x)}(\Omega)$ be such that $\|f\|_{1,p,\Omega} \le 1$. Then $g = \mathcal{E}f \in W^{1,p(x)}(\mathbb{R}^n)$ satisfies $\|g\|_{1,p,\mathbb{R}^n} \le A$ where A is the norm of the extension operator $\mathcal{E}$.

As in the proof of Lemma 3.1 we can assume that g is absolutely continuous on almost all closed segments in B_k parallel to coordinate axes and that for a.a. x the classical derivatives $\partial g(x)/\partial x_i$, $i = 1, \ldots, n$, exist and coincide with the corresponding generalised derivatives. Set $g_k = g\varphi_k$ and let γ be defined by (3.4). Then $g_k \in W^{1,p(x)}(B_k)$ and for $i = 1, \ldots, n$ and for a.a. $x \in B_k$ we have

$$|D_i(|g_k(x)|^{\gamma(x)})| \le |D_i\gamma(x)| \cdot |g_k(x)|^{\gamma(x)}|\log|g_k(x)|| + \gamma(x)|g_k(x)|^{\gamma(x)-1}|\nabla g_k(x)|$$

and

$$(5.11)\quad \int_{(B_k)_i} \sup_{x_i} |g_k(x)|^{\gamma(x)}\,dx_i' \le \frac{n(n-1)L}{(n-q_k)^2}\int_{B_k} |g_k(x)|^{\gamma(x)}\big|\log|g_k(x)|\big|\,dx + \frac{nq_k}{n-q_k}\int_{B_k}|g_k(x)|^{\gamma(x)-1}|\nabla g_k(x)|\,dx.$$

Using the same arguments as in the proof of Lemma 3.1 we obtain

$$(5.12)\quad \int_{\{|g_k(x)|\le 1\}} |g_k(x)|^{\gamma(x)}\big|\log|g_k(x)|\big|\,dx \le c_p e^{-1}\|\,|g_k|^{\gamma-1}\|_{p',B_k}\,\|1\|_{p,\Omega^*},$$

and

$$(5.13)\quad \int_{\{|g_k(x)|>1\}} |g_k(x)|^{\gamma(x)}\big|\log|g_k(x)|\big|\,dx \le c_p(e\varepsilon)^{-1}\|\,|g_k|^{\gamma-1}\|_{p',B_k}\,\|\,|g_k|^{1+\varepsilon}\|_{p,B_k},$$

where

$$(5.14)\quad \varepsilon = \frac12\left(b\,\frac{1-\kappa\delta}{1+\kappa\delta} - 4 + \frac1n\right) > 0.$$

From (5.9), (5.6) and (5.14) we derive that $\varepsilon \le r_k^*/q_k - 1$ and so $(1+\varepsilon)p(x) \le r_k^*$ for $x \in B_k$.

The classical Sobolev inequality

$$\|u\|_{r^*,\mathbb{R}^n} \le c(r)\|\nabla u\|_{r,\mathbb{R}^n},\qquad u \in W^{1,r}(\mathbb{R}^n),\ \operatorname{supp} u \text{ compact},$$

holds with the constant

$$(5.15)\quad c(r) = \left(\frac{(n-1)!}{\omega_n}\right)^{1/n}\frac{n^{-1/r}(r-1)^{1-1/r}}{[\Gamma(n/r)\Gamma(1+n-n/r)]^{1/n}}\cdot\frac{1}{(n-r)^{1-1/r}},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ (cf. [Ta]). Using Lemma 2.1, (2.2) and the Hölder inequality (2.3) we obtain

$$\begin{aligned}(5.16)\quad \|\,|g_k|^{1+\varepsilon}\|_{p,B_k} &= \|\,g_k\|_{p(1+\varepsilon),B_k}^{1+\varepsilon} \le (|B_k|+1)^{1+\varepsilon}\|g_k\|_{r_k^*,B_k}^{1+\varepsilon}\\ &\le (|B_k|+1)^{1+\varepsilon}c(r_k)^{1+\varepsilon}\|\nabla g_k\|_{r_k,B_k}^{1+\varepsilon}\\ &\le (|B_k|+1)^{1+\varepsilon}c(r_k)^{1+\varepsilon}(\|\nabla g\|_{r_k,B_k} + c_0\sigma_k^{-1}\|g\|_{r_k,B_k})^{1+\varepsilon}\\ &\le (|B_k|+1)^{2(1+\varepsilon)}c(r_k)^{1+\varepsilon}(\|\nabla g\|_{p,\mathbb{R}^n} + c_0\sigma_k^{-1}\|g\|_{p,\mathbb{R}^n})^{1+\varepsilon}\\ &\le [A^{1/2}(|B_k|+1)]^{2(1+\varepsilon)}c(r_k)^{1+\varepsilon}\max\{1,(c_0\sigma_k^{-1})^{1+\varepsilon}\}.\end{aligned}$$

Similarly,

$$\begin{aligned}(5.17)\quad \int_{B_k}|g_k(x)|^{\gamma(x)-1}|\nabla g_k(x)|\,dx &\le c_p\|\,|g_k|^{\gamma-1}\|_{p',B_k}\|\nabla g_k\|_{p,B_k}\\ &\le Ac_p\|\,|g_k|^{\gamma-1}\|_{p',B_k}\max\{1,c_0\sigma_k^{-1}\}.\end{aligned}$$

Moreover, (5.7), (5.10) and (5.15) yield

$$(5.18)\quad \sigma_k = \kappa L^{-1}(1+\kappa\delta)^{-1}(n-r_k),\qquad c(r_k) \le \widetilde{c}(n-r_k)^{1/n-1},$$

where $\widetilde{c} > 0$ depends only on n.

From (5.11)–(5.18) we conclude that there is a constant $c > 0$ which depends on $|\Omega|$, p, n, δ, κ, b such that

$$\int_{(B_k)_i}\sup_{x_i}|g_k(x)|^{\gamma(x)}\,dx_i' \le c(n-r_k)^{-2-(1+\varepsilon)(2-1/n)}\|\,|g_k|^{\gamma-1}\|_{p,B_k}.$$

As in (3.11) we obtain

$$\int_{B_k}|g_k(x)|^{p^*}\,dx \le [c(n-r_k)^{-4+1/n-\varepsilon(2-1/n)}\|\,|g_k|^{\gamma-1}\|_{p,B_k}]^{n'}.$$

If $p_k > (1+\kappa\delta n)(1+\kappa\delta)^{-1}$, then (5.6), (5.9) imply that p satisfies the assumptions of Lemma 3.1 on B_k and we proceed as in (3.12)–(3.14) to obtain

$$(5.19)\quad \|g_k\|_{p^*,B_k} \le [c(n-r_k)^{-4+1/n-\varepsilon(2-1/n)}]^{a_k},$$

where

$$(5.20)\quad a_k = \frac{r_k'-n'}{q_k'-n'} = \frac{(1+\kappa\delta)(q_k-1)}{(1-\kappa\delta)(r_k-1)}.$$

Similarly, the inequalities

$$(5.21)\quad p_k \le \frac{1+\kappa\delta n}{1+\kappa\delta},$$

(5.6) and (5.9) imply that p satisfies the assumptions of Lemma 3.2 on B_k and we proceed as in (3.17)–(3.19) to obtain (5.19) with

$$(5.22)\quad a_k = \frac{n-q_k}{2n-q_k(n+1)}.$$

According to (5.8), for every $x \in \Omega$ at most θ members of the sequence $\{g_k(x)\}_k$ are different from zero and we can write

$$\begin{aligned}(5.23)\quad \int_\Omega |f(x)|^{p^*(x)}w(x)\,dx &= \int_\Omega |g(x)|^{p^*(x)}w(x)\,dx\\ &= \int_\Omega\Big|\sum_k g_k(x)\Big|^{p^*(x)}w(x)\,dx\\ &\le \theta^{-1}\sum_k\int_{B_k}(\theta|g_k(x)|)^{p^*(x)}w(x)\,dx.\end{aligned}$$

Since (5.21) implies $\sigma_k \ge \kappa L^{-1}(n-1)(1+\kappa\delta)^{-1}$ we conclude that only a finite number of p_k satisfy (5.21). Let $k_0 \in \mathbb{N}$ be such that $p_k >$

$(1+\kappa\delta n)(1+\kappa\delta)^{-1}$ for $k \ge k_0$. From (5.20), (5.9), (5.7), (5.6) and (5.14) we have

$$\lim_{k\to\infty} a_k = \frac{1+\kappa\delta}{1-\kappa\delta} < \frac{b}{4-1/n+\varepsilon(2-1/n)} < \frac{b}{4-1/n}. \tag{5.24}$$

Hence we can assume that

$$a_k < \frac{b}{4-1/n+\varepsilon(2-1/n)} \quad \text{for } k \ge k_0.$$

Let $k \ge k_0$. By (5.4) and (5.19),

$$\int_{B_k} (\theta|g_k(x)|)^{p^*(x)} w(x)\,dx \le \sup_{x\in B_k} [\theta(c(n-r_k)^{-4+1/n-\varepsilon(2-1/n)})^{a_k}(n-p(x))^b]^{p^*(x)}. \tag{5.25}$$

We use (5.10) to obtain

$$c^{a_k}(n-p(x))^b(n-r_k)^{a_k(-4+1/n-\varepsilon(2-1/n))} \le \max\{1, c^{b/(4-1/n)}\}[(1+\kappa\delta)(n-r_k)]^{b-a_k(4-1/n+\varepsilon(2-1/n))}. \tag{5.26}$$

Since $\lim_{k\to\infty} r_k = n$ by (5.7), (5.9), and $b - a_k(4-1/n+\varepsilon(2-1/n)) > \frac{1}{2}(b-4+1/n) > 0$ by (5.14), (5.24), we can assume k_0 is so large that

$$\theta \max\{1, c^{b/(4-1/n)}\}[(1+\kappa\delta)(n-r_k)]^{b-a_k(4-1/n+\varepsilon(2-1/n))} < s < 1 \tag{5.27}$$

for $k \ge k_0$. Taking into account that $p^*(x) \ge n(n-1)^{-1}$ for all x we conclude from (5.23), (5.25)–(5.27) that

$$\int_{\Omega} |f(x)|^{p^*(x)} w(x)\,dx \le \theta^{-1}\sum_{k=1}^{k_0-1} \sup_{x\in B_k} [\theta(c(n-r_k)^{-4+1/n-\varepsilon(2-1/n)})^{a_k}(n-p(x))^b]^{p^*(x)} + \theta^{-1}\sum_{k=k_0}^{\infty} s^{n/(n-1)} \le K < \infty \tag{5.28}$$

where a_k satisfy (5.20) and (5.22) and K is a constant independent of f. If $K \le 1$ then $\|f\|_{M,\Omega} \le 1$. If $K > 1$ then (5.28) yields

$$\int_{\Omega} (|f(x)|K^{-1/p^*(x)})^{p^*(x)} w(x)\,dx \le 1,$$

i.e., by (5.3), $\|f\|_{M,\Omega} \le K^{1-1/n}$. Hence

$$\|f\|_{M,\Omega} \le \max\{1, K^{1-1/n}\}$$

and (5.5) follows. ∎

REMARK. According to Lemma 5.1(iii), the balls B_k have a positive distance from the set $F = \{x \in \mathbb{R}^n : p(x) = n\}$. Therefore B_k possibly cross only that part of the boundary $\partial\Omega$ which contains points x such that $p(x) < n$ and it is sufficient to assume that only this part of $\partial\Omega$ satisfies the Lipschitz condition. More precisely, let $\eta > 0$ and let $G_\eta = \{x \in \mathbb{R}^n : |x-y| < \eta$ for some $y \in \partial\Omega \setminus F\}$. Let $\partial\Omega \cap \overline{G}_\eta$ have a local description by Lipschitz functions in the sense of the proof of Theorem 4.1. We can modify this proof to obtain a bounded extension operator from $W^{1,p(x)}(\Omega)$ to $W^{1,q}(\Omega \cup G_\eta)$ which is sufficient for the proof of Theorem 5.1.

In particular, if $p(x) = n$ for all $x \in \partial\Omega$ we do not need an extension of functions from $W^{1,p(x)}(\Omega)$ since all balls B_k are contained in Ω. Therefore, in this case we need no assumptions on the smoothness of the boundary $\partial\Omega$. This result is formulated in the following theorem.

THEOREM 5.3. *Let $p \in C^{0,1}(\overline{\Omega})$ be such that*

$$1 \le p(x) < n, \quad x \in \Omega, \quad p(y) = n, \quad y \in \partial\Omega.$$

Let b, w and M be as in Theorem 5.2. Then there exists a constant $c > 0$ such that (5.5) holds for all $f \in W^{1,p(x)}(\Omega)$.

If $p(x) \equiv p > n$ then the Sobolev space $W^{1,p}$ is embedded into a space of continuous and Hölder-continuous functions. An analogous result for $p \in C^{0,1}(\overline{\Omega})$ is given in the following two assertions. It is natural that the degree of Hölder-continuity of functions from $W^{1,p(x)}(\Omega)$ depends on $x \in \Omega$ and that the behaviour of these functions for $p(x)$ close to n is compensated with an appropriate weight.

THEOREM 5.4. *Let p be such that*

$$p(x) > n, \quad x \in \Omega,$$

and

$$\sup_{|y-x|<\sigma} \left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \le \frac{a}{|\log\sigma|}, \quad 0 < \sigma < \min\{1, \operatorname{dist}(x,\partial\Omega)\}, \tag{5.29}$$

where $a > 0$ is independent of x and σ. Define the function $\lambda : \Omega\times(0,\infty) \to (0,\infty)$ by $\lambda(x,t) = t^{1-n/p(x)}$ and the seminorm

$$|f|_{\lambda,\Omega} = \sup_{\substack{x,y\in\Omega \\ x\ne y}} \frac{|f(x)-f(y)|}{\lambda(x,|x-y|)}.$$

Then there exists a constant $c > 0$ such that

$$|f|_{\lambda,\Omega} \le c\,\|\nabla f\|_{p,\Omega},$$

for all $f \in W^{1,p(x)}(\Omega) \cap C^1(\Omega)$.

Proof. Let $f \in C^1(\Omega)$ be such that $\|\nabla f\|_{p,\Omega} \le 1$ and let $x \in \Omega$. Let $0 < r_0 < \min\{1, \operatorname{dist}(x, \partial\Omega)\}$ and set $p_0 = \inf_{B(x,r_0)} p$. It follows from (5.29) that p is continuous in Ω. Thus it is bounded in $B(x,r_0)$ and there exists $r \in (0, r_0]$ such that $|B(x,r)| \le 1$ and

$$\|\nabla f\|_{p,B(x,r)} \le 1 - n/p_0. \tag{5.30}$$

Let $y \in B(x, r/2)$, $y \ne x$, and set $\sigma = |x-y|$. For every $z \in B(x,\sigma)$ we have

$$|f(y) - f(z)| = \left|\int_0^1 \frac{d}{dt} f(y + t(z-y))\,dt\right| \le \sigma \int_0^1 |\nabla f(y + t(y-z))|\,dt.$$

Hence

$$\begin{aligned}\Big|f(y) - |B(y,\sigma)|^{-1} \int_{B(y,\sigma)} f(z)\,dz\Big| &\le |B(y,\sigma)|^{-1} \int_{B(y,\sigma)} |f(y) - f(z)|\,dz \\ &\le \omega_n^{-1}\sigma^{1-n} \int_{B(y,\sigma)} \int_0^1 |\nabla f(y + t(y-z))|\,dt\,dz \\ &= \omega_n^{-1}\sigma^{1-n} \int_0^1 t^{-n} \int_{B(y,\sigma t)} |\nabla f(z)|\,dz\,dt \\ &\le \omega_n^{-1}\sigma^{1-n} c_p \|\nabla f\|_{p,B(y,\sigma)} \int_0^1 t^{-n} \|1\|_{p',B(y,\sigma t)}\,dt.\end{aligned}$$

Since $|B(y,\sigma t)| \le |B(x,r)| \le 1$, we have

$$\|1\|_{p',B(y,\sigma t)} \le \sup_{z \in B(y,\sigma t)} |B(y,\sigma t)|^{1/p'(z)} \le |B(y,\sigma t)|^{1-1/p_\sigma}, \tag{5.31}$$

where $p_\sigma = \inf_{B(x,2\sigma)} p$. Thus

$$\Big|f(y) - |B(y,\sigma)|^{-1} \int_{B(y,\sigma)} f(z)\,dz\Big| \le \omega_n^{-1/p_\sigma} c_p \|\nabla f\|_{B(y,\sigma)} \sigma^{1-n/p_\sigma} \int_0^1 t^{-n/p_\sigma}\,dt \tag{5.32}$$

and using (5.30) and the estimate $\int_0^1 t^{-n/p_\sigma}\,dt = p_\sigma/(p_\sigma - n) \le p_0/(p_0 - n)$ we obtain

$$\Big|f(y) - |B(y,\sigma)|^{-1} \int_{B(y,\sigma)} f(z)\,dz\Big| \le c(n,p)\sigma^{1-n/p_\sigma}$$

where $c(n,p) = c_p \max\{1, \omega^{-1}\}$. Similarly,

$$\begin{aligned}\Big|f(x) - |B(y,\sigma)|^{-1} \int_{B(y,\sigma)} f(z)\,dz\Big| &\le |B(y,\sigma)|^{-1} \int_{B(y,\sigma)} |f(x) - f(z)|\,dz \\ &\le 2^n |B(x,2\sigma)|^{-1} \int_{B(x,2\sigma)} |f(x) - f(z)|\,dz \\ &\le 2^{n+1} c(n,p) \sigma^{1-n/p_\sigma}.\end{aligned} \tag{5.33}$$

From (5.32), (5.33) we obtain

$$\frac{|f(x) - f(y)|}{\sigma^{1-n/p(x)}} \le c\sigma^{n(1/p(x) - 1/p_\sigma)},$$

where c is a constant independent of f and σ. It remains to observe that the assumption (5.29) implies $\sigma^{1/p(x) - 1/p_\sigma} \le e^a$. ∎

REMARK. Every function $p \in C^{0,1}(\overline{\Omega})$ such that $p(x) \ge 1$ for $x \in \Omega$ also satisfies (5.29). Indeed, if $x, y \in \Omega$, $|x-y| \le \sigma \le 1$, we have

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| = \frac{|p(x) - p(y)|}{p(x)p(y)} \le L\sigma \le \frac{Le^{-1}}{|\log \sigma|}$$

where L is the Lipschitz constant for p.

THEOREM 5.5. *Let $p \in C^{0,1}(\overline{\Omega})$ be such that $p(x) > n$ for all $x \in \Omega$. Then there exists a constant $c > 0$ such that every function $f \in C^1(\Omega) \cap W^{1,p(x)}(\Omega)$ with $\operatorname{supp} f \cap \overline{\{x \in \partial\Omega : p(x) > n\}} = \emptyset$ satisfies the estimate*

$$\sup_{x \in \Omega} |f(x)| w(x) \le c \|f\|_{1,p,\Omega} \tag{5.34}$$

with the weight function given by

$$w(x) = \min\{p(x) - n, 1\}. \tag{5.35}$$

Proof. If $\inf_\Omega p = p_0 > n$ then, by (2.2), the space $L^{p(x)}(\Omega)$ is embedded in $L^{p_0}(\Omega)$, the weight function w satisfies $0 < \min\{p_0 - n, 1\} \le w(x) \le 1$, $x \in \Omega$, and thus we can use the classical embedding theorem.

Hence we can assume $\inf_\Omega p = n$. As in the proof of Theorem 3.1 we set $\widetilde{p}(x) = \inf\{p(y) + L|x-y| : y \in \Omega\}$ for $x \in \mathbb{R}^n$. Then $\widetilde{p} \in C^{0,1}(\mathbb{R}^n)$ is an extension of p with the same Lipschitz constant. Moreover, for all $x \in \mathbb{R}^n \setminus \overline{\Omega}$ we have $\widetilde{p}(x) \ge n + L\operatorname{dist}(x, \Omega) > n$ and we truncate $\widetilde{p}$ from above by $q = \sup_\Omega p$. For simplicity we denote the extended function again by p. Set $F = \{x \in \partial\Omega : p(x) = n\}$ and $G = \partial\Omega \setminus F$.

Let $f \in C^1(\Omega) \cap W^{1,p(x)}(\Omega)$ be such that $\operatorname{supp} f \cap \overline{G} = \emptyset$. We extend the function f by zero to $\mathbb{R}^n \setminus F$. Let $x \in \Omega$. Fix κ, $0 < \kappa <$

$\min\{\omega_n^{-1/n}, 1\}L(q-n)^{-1}$, put $\sigma = (p(x)-n)\kappa L^{-1}$ and $B = B(x,\sigma)$. Then $\sigma \le 1$, $|B| \le 1$ and for all $y \in \overline{B}$ we have

$$(5.36) \qquad 1-\kappa \le \frac{p(y)-n}{p(x)-n} \le 1+\kappa, \quad p(y) > n$$

(cf. (5.1), (5.2)). Hence $\overline{B} \cap F = \emptyset$ and $f \in C^1(\overline{B})$. Using polar coordinates and the Hölder inequality we obtain the estimate

$$(5.37) \qquad |f(x)| \le |B|^{-1}\int_B |f(y)|\,dy + c(n)\int_B \frac{|\nabla f(y)|}{|x-y|^{n-1}}\,dy$$
$$\le c_p|B|^{-1}\|1\|_{p',B}\,\|f\|_{p,B} + c(n)c_p\|g\|_{p',B}\,\|\nabla f\|_{p,B},$$

where $g(y) = |x-y|^{1-n}$.

Set $p_0 = \inf_B p$. Then $\|1\|_{p',B} \le |B|^{1-1/p_0}$ (cf. (5.31)) and thus

$$(5.38) \qquad |B|^{-1}\|1\|_{p',B} \le (\omega_n\sigma^n)^{-1/p_0} \le c_1(p(x)-n)^{-n/p_0},$$

where c_1 depends on n, p and κ. Using (5.36) we obtain

$$\int_B g(y)^{p'(y)}\,dy \le \int_B |x-y|^{(1-n)p_0'}\,dy = n\omega_n\int_0^\sigma t^{(1-n)(p_0'-1)}\,dt$$
$$\le c_2(p(x)-n)^{(1-n)/(p_0-1)} \le c_2\max\{(p(x)-n)^{-1}, 1\},$$

where c_2 depends on n, p and κ. We can assume that $c_2 \ge 1$. The convexity of the modular $\varrho_{p'}$ then yields

$$(5.39) \qquad \|g\|_{p',B} \le c_2(p(x)-n)^{-1+1/q}.$$

From (5.35), (5.37)–(5.39) we obtain

$$|f(x)|w(x) \le c[(p(x)-n)^{1-n/p_0} + (p(x)-n)^{1/q}]\|f\|_{1,p,\Omega},$$

which yields (5.34). ∎

REMARKS. 1. There are two reasons for the limiting assumption on f in Theorem 5.5. First, the balls $B(x,\sigma)$, in general, intersect the complement of Ω and thus we need f well extendable outside Ω. The second reason is more essential. As we mentioned in the Introduction, we do not have an analogue of the classical result on density of smooth functions in $W^{1,p(x)}(\Omega)$. If we did, we could simply assume that $f \in W^{1,p(x)}(\Omega)$ in both Theorems 5.5 and 5.6.

2. We can see from the proof of Theorem 5.5 that it is sufficient if p is Lipschitz-continuous on the set Ω_δ for some $\delta > 0$, where $\Omega_\delta = \{x \in \Omega : p(x) < n+\delta\}$, and if $\inf_{\Omega\setminus\Omega_\delta} p(x) = p_0 > n$. Indeed, we set $\widetilde{p}(x) = \min\{p(x), p_0, \inf_{\Omega_\delta\setminus\Omega_{\delta/2}} p\}$, $x \in \Omega$. Then $\widetilde{p} \in C^{0,1}(\overline{\Omega})$, $\widetilde{p}(x) = p(x)$ for $x \in \Omega_{\delta/2}$, and we use the embedding of $W^{1,p(x)}(\Omega)$ in $W^{1,\widetilde{p}(x)}(\Omega)$.

3. If $p(x) = n$ for all $x \in \partial\Omega$ then the balls $B(x,\sigma)$ in the proof of Theorem 5.5 are contained in Ω and we do not need an extension operator. Thus we have the following counterpart of Theorem 5.3.

THEOREM 5.6. *Let $p \in C^{0,1}(\overline{\Omega}_\delta)$ and $\inf_{\Omega\setminus\Omega_\delta} p > n$, where $\delta > 0$ and $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \delta\}$. Let w be given by* (5.35). *Then* (5.34) *holds for every $f \in C^1(\Omega) \cap W^{1,p(x)}(\Omega)$.*

6. Non-embedding examples. Our aim was to present the Sobolev inequality and embedding theorems under the assumption that the exponent function p is Lipschitz. The following examples show that some of the other assumptions cannot be improved within this frame.

EXAMPLE 1 (the Sobolev conjugate function p^* in Theorems 3.1 and 5.1). Let Ω be a non-empty open set in $\mathbb{R}^n$. Let $p \in C^{0,1}(\overline{\Omega})$ and $q \in C(\Omega)$ be such that $1 \le p(x) < n$ and $1 \le q(x) < \infty$ for $x \in \Omega$. Let $q(x_0) > p^*(x_0)$ for some $x_0 \in \Omega$. Then

$$W_0^{1,p(x)}(\Omega) \setminus L^{q(x)}(\Omega) \ne \emptyset.$$

Indeed, since

$$\frac{1}{q(x_0)} < \frac{1}{p^*(x_0)} = \frac{1}{p(x_0)} - \frac{1}{n},$$

there exist numbers s, t and a ball $B \subset \Omega$ centred at x_0 such that

$$\frac{1}{q(x)} \le \frac{1}{t} < \frac{1}{s} - \frac{1}{n} \le \frac{1}{p(x_0)} - \frac{1}{n}, \qquad x \in B.$$

Since $t > s^*$, there exists a function $f \in W_0^{1,s}(B) \setminus L^t(B)$. It suffices to realize that $W_0^{1,s}(B) \subset W_0^{1,p(x)}(\Omega)$ and $L^{q(x)}(B) \subset L^t(B)$.

EXAMPLE 2 (the Hölder-continuity exponent λ in Theorem 5.4). Let Ω be a non-empty open set in $\mathbb{R}^n$ and let $p \in C(\Omega)$ satisfy $p(x) > n$ for $x \in \Omega$. Let $\mu : \Omega \times (0,\infty) \to (0,\infty)$ be such that $\mu(x_0,t) = t^\sigma$ for some $x_0 \in \Omega$ and $\sigma > 1 - n/p(x_0)$. Then there exists a sequence of functions $f_k \in C^1(\Omega)$ such that $\{\|\nabla f_k\|_{p,\Omega}\}$ is bounded and

$$\lim_{k\to\infty} |f_k|_{\mu,\Omega} = \infty.$$

To prove the assertion let us consider q, $p(x_0) < q < s := n/(1-\sigma)$. Since p is continuous, there exists a ball $B = B(x_0,r) \subset \Omega$ such that $p(x) < q$ for $x \in B$. Define the function $f(x) = \max\{(r/2)^{1-n/q} - |x-x_0|^{1-n/q}, 0\}$, $x \in \mathbb{R}^n$. Then $f \in W^{1,q}(\Omega)$ and $f(x) = 0$ if $|x-x_0| \ge r/2$. Using the standard mollification method we define $f_k(x) = k^n f * \varphi(kx)$, $k = 1, 2, \ldots$, where $\varphi \in C_0^\infty(B(0,r/2))$, $\int\varphi = 1$. Then $f_k \in C^\infty(B)$ and

$$(6.1) \qquad f_k \to f \quad \text{in } W^{1,q}(\Omega).$$

Let $y_k \in B(x_0, r/2)$ be such that $\lim_{k\to\infty} y_k = x_0$. Since $q > n$, there is a bounded embedding of $W^{1,q}(B)$ in $C(\overline{B})$ and (6.1) implies that $\{f_k\}$ contains a subsequence (we shall denote it again by $\{f_k\}$) such that

$$\sup_{x\in B} |f_k(x) - f(x)| \le c\|f_k - f\|_{1,q,B} < \tfrac{1}{4}|x_0 - y_k|^{1-n/q}.$$

Then

$$|f_k(x_0) - f_k(y_k)| \ge |f(x_0) - f(y_k)| - |f_k(x_0) - f(x_0)| - |f_k(y_k) - f(y_k)| \ge \tfrac{1}{2}|x_0 - y_k|^{1-n/q}$$

and

$$|f_k|_{\mu,\Omega} \ge \tfrac{1}{2}|x_0 - y_k|^{n(1/q-1/s)} = \infty.$$

Using (2.2) and (6.1) we obtain

$$\|\nabla f_k\|_{p,\Omega} = \|\nabla f_k\|_{p,B} \le (|B|+1)\|\nabla f_k\|_{q,B} \le (|B|+1)(\|\nabla f\|_{q,\Omega} + 1)$$

for sufficiently large k and so the sequence $\{\|\nabla f_k\|_{p,\Omega}\}$ is bounded.

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Received October 25, 1999
Revised version September 18, 2000 (4418)