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SOBOLEV'S METHOD FOR HAMMERSTEIN INTEGRAL EQUATIONS

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Abstract- Sobolev's initial value method is used to solve the Hammerstain equation of the second kind.
Keywords- Hammerstain integral equation, initial value problem, Sobolev's initial value method.

1. INTRODUCTION

Sobolev, [4], proposed a numerical method for the solution of linear Fredholm integral equations. He recommended that the solution of the initial value problem for the resolvent,

$$k_y(t, x, y) = k(t, y, y)\, k(y, x, y),$$
$$k(t,x,0) = f(t, x), \tag{1}$$
$$0 \le t, x, y \le Y,$$

can be established using Euler's method (see [1]). In [2], Kagiwada and Kalaba have extended Solobev's method to the case of Urysohn integral equations of the second kind.

In this paper, we apply the same method to the nonlinear integral equation of the second kind, namely

$$\phi(t) = f(t) + \int_0^y p(t, x)\, q(x, \phi(x))\, dx,\ 0 \le t,\ y \le Y, \tag{2}$$

where Y is sufficiently small. Eq. (2) is known as the Hammerstein equation of the second kind (see [3]).

To show the dependence of the solution ϕ upon the upper limit of integration y, as well as upon t, we rewrite Eq. (2) in the form

$$\phi(t, y) = f(t) + \int_0^y p(t, x)\, q(x, \phi(x, y))\, dx,\ 0 \le t,\ y \le Y. \tag{3}$$

2. METHOD OF SOLUTION

Assume that the functions ϕ, p, and q are differentiable. Differentiating both sides of Eq. (1.3) with respect to y, we obtain

$$\phi_y(t, y) = p(t, y)\, q(y, \phi(y, y)) + \int_0^y p(t, x)\, q_\phi(x, \phi(x, y))\, \phi_y(x, y)\, dx, \tag{4}$$

$0 \le t,\ y \le Y$, where the subscripts denotes partial differentiation.

Eq. (4) represents a linear Fredholm integral equation of the second kind for the unknown function $\phi_y(t, y)$ with kernel $p(t, x)\, q_\phi(x, \phi(x, y))$. Assume that the linear Fredholm integral equation of the second kind

$$\psi(t, y) = F(t, y) + \int_0^y p(t, x)\, q_\phi(x, \phi(x, y))\, \psi_y(x, y)\, dx,\ 0 \le t,\ y \le Y, \tag{5}$$

possesses a unique solution, where ϕ is a solution of Eq. (3) and F is arbitrary.

Then the solution of Eq. (4) can be expressed in terms of the resolvent kernel in the form,

$$\phi_y(t, y) = F(t, y) + \int_0^y F(x, y)\, k(t, x, y)\, dx,\ 0 \le t,\ y \le Y, \qquad (6)$$

where $F(u, v) = p(u, v)\, q(v, \phi(v, v))$ is the forcing function and $k(t, x, y) = p(t, x)\, q(x, \phi(x, y))$ is the resolvent kernel given by

$$k(t, x, y) = p(t, x)\, q_\phi(x, \phi(x, y)) + \int_0^y p(t, \tilde{x})\, q_\phi(\bar{x}, \phi(\tilde{x}, y))\, k(\tilde{x}, x, y)\, d\tilde{x}, \qquad (7)$$

$0 \le t, x, y \le Y$. Let us denote the right-hand side of Eq. (6) by H,

$$H(t, y) = F(t, y) + \int_0^y F(x, y)\, k(t, x, y)\, dx,\ 0 \le t,\ y \le Y, \qquad (8)$$

then Eq. (6) takes the form

$$\phi_y(t, y) = H(t, y),\ \ 0 \le t, y \le Y. \qquad (9)$$

Eq. (9) is a partial integro-differential equation for the unknown function $\phi(t, y)$.

Now, we are going to obtain a differential equation for the resolvent kernel k. For this, differentiating Eq. (7) with respect to y, making use of (9), leads to

$$\begin{aligned} k_y(t,x,y) = {} & p(t,x)\, q_{\phi\phi}(x,\phi(x,y))\, H(x,y) + \\ & + p(t,y)\, q_{\phi\phi}(y,\phi(y,y))\, k(y,x,y) + \\ & + \int_0^y p(t,\tilde{x})\, q_{\phi\phi}(\tilde{x},\phi(\tilde{x},y))\, H(\tilde{x},y)\, k(\tilde{x},x,y)\, d\tilde{x} + \\ & + \int_0^y p(t,\tilde{x})\, q_\phi(\tilde{x},\phi(\tilde{x},y))\, k_y(\tilde{x},x,y)\, d\tilde{x}. \end{aligned} \qquad (10)$$

The last equation is considered to be a linear Fredholm integral equation of the second kind for the unknown function K_y with kernel as in equation (6). The forcing function is given by the first three terms on the right hand side. We introduce, for simplicity, the auxiliary function A(t, x, y) to be the sum of these forcing terms, then Eq. (10) takes the form

$$k_y(t,x,y) = A(t,x,y) + \int_0^y p(t,\tilde{x})\, q_\phi(\bar{x}, \phi(\tilde{x},y))\, k_y(\tilde{x},x,y)\, d\tilde{x}, \quad 0 \le t,x,y \le Y. \qquad (11)$$

Therefore, the solution of Eq. (10) may be written in the form

$$k_y(t,x,y) = A(t,x,y) + \int_0^y k(t,\tilde{x},y)\, A(\tilde{x},x,y)\, d\tilde{x}, \quad 0 \le t,x,y \le Y. \qquad (12)$$

Now, we summarize what we have proved in this section. The solution of the Hammerstain integral equation, (3), satisfies the initial value problem consisting of Eqs. (9) and (12) and the initial conditions

$$\phi(t, 0) = f(t) \qquad (13)$$

and

$$k(t, x, 0) = p(t, x)\, q_\phi(x, \phi(x)), \qquad (14)$$

which follow from Eqs. (3) and (7), respectively.

3. VERIFICATION

In this section, we shall show that a solution of the initial value system in Eqs. (9), (12), (13) and (14) provides a solution of the Hammerstain equation (3).

Firstly, we show that k, as determined by the initial value system, satisfies Eq. (7). We define B(t, x, y) to be the right-hand side of (7), i.e.,

$$B(t,x,y) = p(t,x)\,q_\phi(x,\phi,(x,y)) + \int_0^y p(t,\tilde{x})\; q_\phi(\bar{x},\, \phi(\tilde{x},y))\; k(\tilde{x},x,y)\; d\tilde{x}, \qquad (15)$$

$0 \le t,\ x,\ y \le Y$. Differentiating both sides of the last equation with respect to y, we obtain

$$B_y(t,x,y) = A(t,x,y) + \int_0^y p(t,\tilde{x})\; q_\phi(\bar{x},\, \phi(\tilde{x},y))\; k_y(\tilde{x},x,y)\; d\tilde{x}. \qquad (16\ )$$

From Eqs. (11) and (16), we have

$$B_y(t,x,y) = A(t,x,y) + \int_0^y [p(t,\tilde{x})\; q_\phi(\tilde{x},\phi(\tilde{x},y)) \; +$$

$$+ \int_0^y p(t,\hat{x})\; q_\phi(\hat{x},\phi(\hat{x},y))\; k(\hat{x},\tilde{x},y)\; d\hat{x}]\; A(\tilde{x},x,y)\,d\tilde{x} \qquad (17)$$

$$= \int_0^y B(t,\tilde{x},y)\; A(\tilde{x},x,y)\; d\tilde{x},$$

which is a linear integro-differential equation for the function B(t, x, y). The initial condition is

$$B(t, x, 0) = p(t, x)\, q_\phi(x, \phi(x)). \qquad (18)$$

By comparing Eqs. (17) and (18) with Eqs. (12) and (14), and assuming uniqueness of the solution, we obtain

$$B(t, x, y) = k(t, x, y). \qquad (19)$$

Thus the integral equation (1.3) is satisfied by k.

Secondly, we prove that the function $\phi(t, y)$ satisfies Eq. (3). We introduce the function C to be

$$C(t,\ y) = f(t) + \int_0^y p(t,\ x)\ q(x,\ \phi(x,\ y))\ dx,\ 0 \le t,\ y \le Y. \qquad (20)$$

We shall show that $\phi \equiv C$. Differentiating (20) with respect to y and making use of Eqs. (6) and (7), we obtain

$$C_x(t,\ y) = F(t,\ y) + \int_0^y k(t,\ x,\ y)\ F(x,\ y)\ dx,\ 0 \le t,\ y \le Y. \qquad (21)$$

Furthermore, the function C satisfies the initial condition

$$C(t, 0) = f(t). \qquad (22)$$

Comparing Eqs. (21) and (22) with Eqs. (9) and (13), and assuming uniqueness of the solution, we obtain

$$C(t, y) = \phi(t, y). \qquad (23)$$

Therefore, the function $\phi(t, y)$ satisfies the integral equation (3).

Remark . *One may use Euler's method to solve the initial value system of equations* (9), (12), (13) and (14), see [1, 2].

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