SOBOLEV SPACES WITH ZERO BOUNDARY VALUES ON METRIC SPACES

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Abstract. We generalize the definition of the first order Sobolev spaces with zero boundary values to an arbitrary metric space endowed with a Borel regular measure. We show that many classical results extend to the metric setting. These include completeness, lattice properties and removable sets.

1. Introduction

The purpose of this note is to introduce the first order Sobolev spaces with zero boundary values on any metric space equipped with a Borel regular measure. The motivation for this generalization is twofold. First we are interested in developing the calculus of variations in the general setup and, to this end, it is crucial that we can compare the boundary values of the Sobolev functions. On the other hand, we would like to present a general theory which covers applications to manifolds, groups, vector fields, graphs and fractal sets in the Euclidean space, see for example [CDG1-2], [FGW], [GN], [HS], [VSC] and references therein.

To begin with, we need a notion of the Sobolev space with no restrictions to the boundary values. There are several different approaches available. We use the Lipschitz type characterization due to Hajłasz [H]. This definition has been employed for example in [FLW], [HM], [HKi], [HaK1], [HaK2], [HeK], [Ka], [KM], and [Vo].

1991 *Mathematics Subject Classification.* 46E35, 31B15, 28A80.
Key words and phrases. Capacity, Sobolev spaces.

Typeset by $\mathcal{AMS}$-TEX

There are two natural routes available to define the first order Sobolev spaces with zero boundary values. Recall that in the classical Euclidean case with the Lebesgue measure the space is defined as the completion of compactly supported smooth functions in the Sobolev norm. The first candidate is based on the completion of Lipschitz continuous functions which belong to the global Sobolev space and vanish in the complement of a given set; another possibility is to require that the function can be extended to the global Sobolev space and that the trace of the extension vanishes in the complement. In the classical Euclidean case of an open set both of these definitions are equivalent, but the approximation by Lipschitz functions works basically on open sets only. For an arbitrary set in a metric space the latter definition is more general and also more natural. Hence we are inclined to take it as our starting point. In order to define the trace of a Sobolev function we need the notion of capacity in the metric setup. The rudiments were established in [KM]. The Sobolev spaces of Hajłasz coincide with the classical Sobolev spaces on the extension domains but not in general. No such restriction is needed here and hence the Hajłasz spaces seem to fit perfectly well for functions with zero boundary values.

In particular, we define the Sobolev spaces with zero boundary values on an arbitrary subset of the Euclidean space. There has been previous attempts to define the Sobolev spaces with zero boundary values on nonopen sets in the Euclidean space. One possible approach is based on the representation of Sobolev functions as Bessel potentials, see [AH, Chapter 10]. Resulting spaces are characterized by a theorem of Havin [Ha] and Bagby [B], which we take as a definition in the metric space setting. These results are closely related to questions of the approximation of Sobolev functions with zero boundary values by compactly supported functions.

As our examples show, the approximation by continuous functions fails on general sets. We study the approximation on open sets and give sufficient conditions which guarantee that a Sobolev function can be approximated by Lipschitz continuous functions vanishing outside an open set. These conditions are based on Hardy type inequalities and, contrary to our definition of the Sobolev space with zero boundary values, they depend only on the values of the function in the given set.

2. Sobolev space

In this section we recall the definition due to Hajłasz of the first order Sobolev spaces on an arbitrary metric space. The details can be found in [H]. Let (X, d) be a metric space and let μ be a non-negative Borel regular outer measure on X. In the following, we keep the triple (X, d, μ) fixed, and for short, we denote it by X. $L^p(X)$, $1 < p < \infty$, is the Banach space of all $\mu-$a.e. defined $\mu-$measurable functions $u\colon X \to [-\infty, \infty]$ for which the norm

$$\|u\|_{L^p(X)} = \Big(\int_X |u|^p \, d\mu \Big)^{1/p}$$

is finite. Suppose that $u\colon X \to [-\infty, \infty]$ is $\mu-$measurable. We denote by $D(u)$ the set of all $\mu-$measurable functions $g\colon X \to [0, \infty]$ such that

$$|u(x) - u(y)| \le d(x, y)\big(g(x) + g(y)\big)$$

for every $x, y \in X \setminus N$, $x \ne y$, with $\mu(N) = 0$. The space $L^{1,p}(X)$ consists of all $\mu-$measurable functions u with $D(u) \cap L^p(X) \ne \emptyset$; the space $L^{1,p}(X)$ is endowed with the seminorm

$$\|u\|_{L^{1,p}(X)} = \inf \big\{ \|g\|_{L^p(X)} \colon g \in D(u) \cap L^p(X) \big\}. \tag{2.1}$$

An application of the uniform convexity of $L^p(X)$ implies that there is a unique minimizer of (2.1). The *Sobolev space* is $M^{1,p}(X) = L^p(X) \cap L^{1,p}(X)$ equipped with the norm

$$\|u\|_{M^{1,p}(X)} = \big(\|u\|^p_{L^p(X)} + \|u\|^p_{L^{1,p}(X)}\big)^{1/p}. \tag{2.2}$$

With this norm $M^{1,p}(X)$ is a Banach space.

If $X = \mathbf{R}^n$ with the Euclidean metric and the Lebesgue measure, then $M^{1,p}(\mathbf{R}^n) = W^{1,p}(\mathbf{R}^n)$, $1 < p < \infty$. Moreover, the norms are comparable. Here $W^{1,p}(\mathbf{R}^n)$ is the classical Sobolev space, that is, the space of functions in $L^p(\mathbf{R}^n)$ whose first distributional derivatives belong to $L^p(\mathbf{R}^n)$ with the norm

$$\|u\|_{W^{1,p}(\mathbf{R}^n)} = \big(\|u\|^p_{L^p(\mathbf{R}^n)} + \|Du\|^p_{L^p(\mathbf{R}^n)}\big)^{1/p}.$$

3. Capacity

There is a natural capacity in the Sobolev space. For $1 < p < \infty$, the *Sobolev $p-$capacity* of the set $E \subset X$ is the number

$$\mathrm{C}_p(E) = \inf\big\{\|u\|^p_{M^{1,p}(X)} : u \in \mathcal{A}(E)\big\},$$

where

$$\mathcal{A}(E) = \big\{u \in M^{1,p}(X) : u \geq 1 \text{ on a neighbourhood of } E\big\}.$$

If $\mathcal{A}(E) = \emptyset$, we set $\mathrm{C}_p(E) = \infty$. Functions belonging to $\mathcal{A}(E)$ are called *admissible* functions for E. Since the Sobolev norm decreases under truncation we may restrict ourselves to those admissible functions u for which $0 \leq u \leq 1$. For the basic properties of the Sobolev capacity we refer to [KM].

A property holds *$p-$quasieverywhere* ($p-$q.e.), if it holds except of a set of $p-$capacity zero. A function $u\colon X \to [-\infty, \infty]$ is *$p-$quasicontinuous* in X if for every

$\varepsilon > 0$ there is a set E such that $\mathrm{C}_p(E) < \varepsilon$ and the restriction of u to $X \setminus E$ is continuous. Since C_p is an outer capacity, we may assume that E is open.

By [Ha, Theorem 5] we see that

$$\mathrm{Lip}^{1,p}(X) = \{u \in M^{1,p}(X) \colon u \text{ is Lipschitz in } X\}$$

is a dense subspace of $M^{1,p}(X)$. Hence for each $u \in M^{1,p}(X)$ there exist sequences of functions $u_i \in \mathrm{Lip}^{1,p}(X)$ and $g_i \in D(u_i - u)$, $i = 1, 2, \dots$, such that $u_i \to u$ and $g_i \to 0$ in $L^p(X)$ as $i \to \infty$. It was proved in [KM, Theorem 3.7] that a subsequence of u_i converges uniformly outside a set of arbitrary small capacity. Thus each Sobolev function has a $p-$quasicontinuous representative [KM, Corollary 3.9]:

3.1. Theorem. *For every $u \in M^{1,p}(X)$ there is a $p-$quasicontinuous function $\widetilde{u} \in M^{1,p}(X)$ such that $\widetilde{u} = u$ $\mu-$a.e. in X.*

The quasicontinuous representative given by Theorem 3.1 is essentially unique. In the classical case this was first proved by Deny and Lions [DL]. The proof of the following Theorem can be found in [Ki].

3.2. Theorem. *Suppose that u and v are $p-$quasicontinuous on an open set $O \subset X$. If $u = v$ $\mu-$a.e. in O, then $u = v$ p-q.e. in O.*

3.3. Remark. Observe that if u and v are $p-$quasicontinuous and $u \leq v$ $\mu-$a.e. in an open set O, then $\max(u-v, 0) = 0$ $\mu-$a.e. in O and $\max(u-v,0)$ is $p-$quasicontinuous. Then Theorem 3.2 implies $\max(u-v,0) = 0$ p-q.e. in O and consequently $u \leq v$ p-q.e. in O.

The previous theorem enables us to define the trace of a Sobolev function to an arbitrary set. If $u \in M^{1,p}(X)$ and $E \subset X$, then the *trace* of u to E is the restriction to E of any $p-$quasicontinuous representative of u. Of course, this definition is useful only if $\mathrm{C}_p(E) > 0$.

The next theorem gives a characterization of the capacity in terms of quasicontinuous functions. For $E \subset X$ and $1 < p < \infty$, we denote

$$\widetilde{\mathrm{C}}_p(E) = \inf\big\{\|u\|^p_{M^{1,p}(X)} \colon u \in \mathcal{QA}(E)\big\},$$

where

$$\mathcal{QA}(E) = \big\{u \in M^{1,p}(X) \colon u \text{ is } p-\text{quasicontinuous and } u \geq 1 \ p\text{-q.e. in } E\big\}.$$

Here we use the convention that $\widetilde{\mathrm{C}}_p(E) = \infty$ if $\mathcal{QA}(E) = \emptyset$.

3.4. Theorem. *Let $1 < p < \infty$. Then $\widetilde{\mathrm{C}}_p(E) = \mathrm{C}_p(E)$ for every $E \subset X$.*

Proof. The inequality $\widetilde{\mathrm{C}}_p(E) \geq \mathrm{C}_p(E)$ follows from the uniqueness of the quasicontinuous representatives (Theorem 3.2). Indeed, if $u \in M^{1,p}(X)$ and $u \geq 1$ on an open neighbourhood of O of E, then the quasicontinuous representative $\widetilde{u}$ of u satifies $\widetilde{u} \geq 1$ p-q.e. on O (see Remark 3.3) and hence $\widetilde{u} \geq 1$ p-q.e. on E.

For the reverse inequality, let $v \in \mathcal{QA}(E)$. By truncation, we may assume that $0 \leq v \leq 1$. Fix ε, $0 < \varepsilon < 1$, and choose an open set V with $\mathrm{C}_p(V) < \varepsilon$ so that $v = 1$ on $E \setminus V$ and that $v|_{X \setminus V}$ is continuous. By topology, there is an open set $U \subset X$ such that $\big\{x \in X \colon v(x) > 1-\varepsilon\big\} \setminus V = U \setminus V$. Observe, that $E \setminus V \subset U \setminus V$. Then choose $u \in \mathcal{A}(V)$ such that $\|u\|_{M^{1,p}(X)} < \varepsilon$ and that $0 \leq u \leq 1$. We define $w = v/(1-\varepsilon) + u$. Then $w \geq 1$ $\mu-$a.e. in $(U \setminus V) \cup V = U \cup V$, which is an open neighbourhood of E and hence $w \in \mathcal{A}(E)$. Thus

$$\mathrm{C}_p(E)^{1/p} \leq \|w\|_{M^{1,p}(X)} \leq \frac{1}{1-\varepsilon}\|v\|_{M^{1,p}(X)} + \|u\|_{M^{1,p}(X)} \leq \frac{1}{1-\varepsilon}\|v\|_{M^{1,p}(X)} + \varepsilon.$$

Since $\varepsilon > 0$ and $v \in \mathcal{QA}(E)$ were arbitrary we arrive at the desired inequality $\mathrm{C}_p(E) \leq \widetilde{\mathrm{C}}_p(E)$. This completes the proof. □

The following result is a sharpening of [KM, Theorem 3.7].

3.5. Lemma. *Suppose that (u_i) is a sequence of $p-$quasicontinuous functions $u_i \in M^{1,p}(X)$, $i = 1, 2, \dots,$ such that $u_i \to u$ in $M^{1,p}(X)$, where u is $p-$quasicontinuous. Then there is a subsequence of (u_i) which converges to u p-q.e. in X.*

Proof. There is a subsequence of (u_i), which we denote again by (u_i), such that

$$\sum_{i=1}^{\infty} 2^{ip} \|u_i - u\|^p_{M^{1,p}(X)} < \infty.$$

For $i = 1, 2, \dots,$ denote $E_i = \big\{x \in X \colon |u_i(x) - u(x)| > 2^{-i}\big\}$ and $F_j = \bigcup_{i=j}^{\infty} E_i$. Clearly $2^i|u_i - u| \in \mathcal{QA}(E_i)$ and hence using Theorem 3.4 we obtain $\mathrm{C}_p(E_i) \le 2^{ip}\|u_i - u\|^p_{M^{1,p}(X)}$ and, by subadditivity, we obtain

$$\mathrm{C}_p(F_j) \le \sum_{i=j}^{\infty} \mathrm{C}_p(E_i) \le \sum_{i=j}^{\infty} 2^{ip}\|u_i - u\|^p_{M^{1,p}(X)}.$$

Hence $\mathrm{C}_p\Big(\bigcap_{j=1}^{\infty} F_j\Big) \le \lim_{j\to\infty} \mathrm{C}_p(F_j) = 0$ and $u_i \to u$ pointwise in $X \setminus \bigcap_{j=1}^{\infty} F_j$. The claim follows. $\square$

4. Sobolev space with zero boundary values

Let E be a subset of X. We say that u belongs to the *Sobolev space with zero boundary values*, and denote $u \in M^{1,p}_0(E)$, if there is a $p-$quasicontinuous function $\widetilde{u} \in M^{1,p}(X)$ such that $\widetilde{u} = u$ $\mu-$a.e. in E and $\widetilde{u} = 0$ p-q.e. in $X \setminus E$. In other words, u belongs to $M^{1,p}_0(E)$ if there is $\widetilde{u} \in M^{1,p}(X)$ as above such that the trace of $\widetilde{u}$ vanishes p-q.e. in $X \setminus E$. The space $M^{1,p}_0(E)$ is endowed with the norm

$$\|u\|_{M^{1,p}_0(E)} = \|\widetilde{u}\|_{M^{1,p}(X)}.$$

Since $\mathrm{C}_p(E) = 0$ implies that $\mu(E) = 0$ for every $E \subset X$, it follows that the norm does not depend on the choice of the quasicontinuous representative.

Observe, that even though $\widetilde{u}$ vanishes q.e. in $X \setminus E$, the functions $g \in D(u)$ need not be zero in $X \setminus E$.

Obviously $M_0^{1,p}(E)$ is a linear space. It is also complete:

4.1. Theorem. *$M_0^{1,p}(E)$ is a Banach space.*

Proof. Suppose that (u_i) is a Cauchy sequence in $M_0^{1,p}(E)$. Then for every u_i, $i = 1, 2, \dots$, there is a $p-$quasicontinuous function $\widetilde{u}_i \in M^{1,p}(X)$ such that $\widetilde{u}_i = u_i$ $\mu-$a.e. in E and $\widetilde{u}_i = 0$ p-q.e. in $X \setminus E$. Since $M^{1,p}(X)$ is complete [H, Theorem 3], there is $u \in M^{1,p}(X)$ such that $\widetilde{u}_i \to u$ in $M^{1,p}(X)$ as $i \to \infty$. Let $\widetilde{u}$ be a $p-$quasicontinuous representative of u given by Theorem 3.1. By Lemma 3.5 there is a subsequence of $(\widetilde{u}_i)$ such that $\widetilde{u}_i \to \widetilde{u}$ p-q.e. in X as $i \to \infty$. This shows that $\widetilde{u} = 0$ p-q.e. in $X \setminus E$ and consequently $u \in M_0^{1,p}(E)$. The theorem follows. $\square$

4.2. Remarks. (1) Clearly $M_0^{1,p}(E) \subset M^{1,p}(E)$ if $E \subset X$ is $\mu-$measurable and $M_0^{1,p}(X) = M^{1,p}(X)$.

(2) If $E_1 \subset E_2 \subset X$, then $M_0^{1,p}(E_1) \subset M_0^{1,p}(E_2)$.

(3) If $u \in M^{1,p}(X)$ is a continuous function which vanishes on $X \setminus E$, then $u \in M_0^{1,p}(E)$. In particular, if $\mu(E) < \infty$, then every Lipschitz continuous function vanishing on $X \setminus E$ belongs to $M_0^{1,p}(E)$.

(4) If Ω is an open set in $\mathbf{R}^n$ and μ is the Lebesgue measure, then $M_0^{1,p}(\Omega) = W_0^{1,p}(\Omega)$, see [AH, Theorem 9.1.3]. Moreover, the norms of these spaces are equivalent. Originally this characterization of Sobolev spaces with zero boundary values is due to Havin [Ha] and Bagby [B].

(5) The *Bessel potential spaces* in $\mathbf{R}^n$ are defined as $\mathcal{L}^{1,p}(\mathbf{R}^n) = \{G_1 * g \colon g \in$

$L^p(\mathbf{R}^n)\}$, where

$$G_1(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{e^{ix\cdot\xi}}{(1+|\xi|^2)^{1/2}}\, d\xi$$

is the *Bessel kernel* of order 1. The norm on $\mathcal{L}^{1,p}(\mathbf{R}^n)$ is given by $\|u\|_{\mathcal{L}^{1,p}(\mathbf{R}^n)} = \|g\|_{L^p(\mathbf{R}^n)}$. For an arbitary set $E \subset \mathbf{R}^n$, let $\mathcal{L}^{1,p}_0(E)$ be the completion of the functions in $\mathcal{L}^{1,p}(\mathbf{R}^n)$ with compact support contained in E.

A theorem of Calderón [St, Theorem V.3] implies that $W^{1,p}(\mathbf{R}^n) = \mathcal{L}^{1,p}(\mathbf{R}^n)$, $1 < p < \infty$, and that the norms are equivalent. This means that $\mathcal{L}^{1,p}_0(E)$ is the completion of the $p-$quasicontinuous functions in $W^{1,p}(\mathbf{R}^n)$ whose support is a compact subset of E. A special case of Netrusov's theorem [AH, Theorem 10.1.1] asserts that $u \in \mathcal{L}^{1,p}_0(E)$ if and only if $u \in M^{1,p}_0(E)$. Observe that $\mathcal{L}^{1,p}_0(E) \subset M^{1,p}_0(E)$ by Lemma 3.5, but the reverse inclusion is deeper.

In applications it is useful to know that $M^{1,p}_0(E)$ posesses the following lattice properties. The straightforward proof is omitted.

4.3. Theorem. *Suppose that $u, v \in M^{1,p}_0(E)$. Then the following claims are true.*

(1) *If $\lambda \geq 0$, then $\min(u,\lambda) \in M^{1,p}_0(E)$ and $\|\min(u,\lambda)\|_{M^{1,p}_0(E)} \leq \|u\|_{M^{1,p}_0(E)}$.*

(2) *If $\lambda \leq 0$, then $\max(u,\lambda) \in M^{1,p}_0(E)$ and $\|\max(u,\lambda)\|_{M^{1,p}_0(E)} \leq \|u\|_{M^{1,p}_0(E)}$.*

(3) *$|u| \in M^{1,p}_0(E)$ with $\||u|\|_{M^{1,p}_0(E)} \leq \|u\|_{M^{1,p}_0(E)}$.*

(4) *$\max(u,v) \in M^{1,p}_0(E)$.*

(5) *$\min(u,v) \in M^{1,p}_0(E)$.*

The following two theorems generalize classically known facts to the metric setting.

4.4. Theorem. *Suppose that E is $\mu-$measurable and that $u \in M^{1,p}_0(E)$ and $v \in M^{1,p}(E)$. If $|v| \leq u$ $\mu-$a.e. in E, then $v \in M^{1,p}_0(E)$.*

Proof. Let w be the zero extension of v to $X \setminus E$ and let $\widetilde{u} \in M^{1,p}(X)$ be a $p-$quasicontinuous function such that $\widetilde{u} = u$ $\mu-$a.e. in E and that $\widetilde{u} = 0$ p-q.e. in $X \setminus E$. Suppose that $g_1 \in D(\widetilde{u}) \cap L^p(X)$ and $g_2 \in D(v) \cap L^p(E)$. Then it is easy to see that

$$g_3(x) = \begin{cases} \max\big(g_1(x), g_2(x)\big), & x \in E, \\ g_1(x), & x \in X \setminus E. \end{cases}$$

belongs to $D(w) \cap L^p(X)$ and hence $w \in M^{1,p}(X)$. Let $\widetilde{w} \in M^{1,p}(X)$ be a $p-$quasicontinuous function such that $\widetilde{w} = w$ $\mu-$a.e. in X given by Theorem 3.1. Then $|\widetilde{w}| \le \widetilde{u}$ $\mu-$a.e. in X. Remark 3.3 yields $|\widetilde{w}| \le \widetilde{u}$ p-q.e. in X and consequently $\widetilde{w} = 0$ p-q.e. in $X \setminus E$. This shows that $v \in M_0^{1,p}(E)$. □

4.5. Theorem. *Suppose that $u \in M_0^{1,p}(E)$ and $v \in M^{1,p}(X)$ are bounded functions. Then $uv \in M_0^{1,p}(E)$.*

The proof of Theorem 4.5 is easy.

The next theorem shows that the sets of capacity zero are removable in the Sobolev spaces with zero boundary values.

4.6. Theorem. *If $N \subset E$ such that $\mathrm{C}_p(N) = 0$, then $M_0^{1,p}(E) = M_0^{1,p}(E \setminus N)$.*

Proof. Suppose that $\mathrm{C}_p(N) = 0$. Clearly $M_0^{1,p}(E \setminus N) \subset M_0^{1,p}(E)$. If $u \in M_0^{1,p}(E)$, then there is a $p-$quasicontinuous $\widetilde{u} \in M^{1,p}(X)$ such that $\widetilde{u} = u$ $\mu-$a.e. in E and $\widetilde{u} = 0$ p-q.e. in $X \setminus E$. Since $\mathrm{C}_p(N) = 0$, we see that $\widetilde{u} = 0$ p-q.e. in $X \setminus (E \setminus N)$. This implies that $u|_{E \setminus N} \in M_0^{1,p}(E \setminus N)$. Moreover, we have

$$\big\| u|_{E \setminus N} \big\|_{M_0^{1,p}(E \setminus N)} = \|u\|_{M_0^{1,p}(E)}.$$

□

4.7. Remarks. (1) In particular $M_0^{1,p}(\operatorname{int} E) = M_0^{1,p}(\overline{E})$ if $\mathrm{C}_p(\partial E) = 0$.

(2) We easily infer that $M_0^{1,p}(X \setminus N) = M_0^{1,p}(X) = M^{1,p}(X)$ if and only if $\mathrm{C}_p(N) = 0$. However, the converse of Theorem 4.6 need not be true in general. For example, let $X = \mathbf{R}$ with the Lebesgue measure and the standard metric, $E = (0, 1]$ and $N = \{1\}$. Then $\mathrm{C}_p(N) > 0$, but $M_0^{1,p}(E) = M_0^{1,p}(E \setminus N) = W_0^{1,p}\big((0, 1)\big)$.

The following theorem shows that the converse of Theorem 4.6 holds for open sets.

4.8. Theorem. *Suppose that μ is finite on bounded sets and that $D \subset X$ is open. Then $M_0^{1,p}(D \setminus N) = M_0^{1,p}(D)$ if and only if $\mathrm{C}_p(N \cap D) = 0$.*

Proof. Only the necessity calls for a proof. Assume that $N \subset D$. Let $x_0 \in D$ and write

$$D_i = B(x_0, i) \cap \big\{x \in D \colon \operatorname{dist}(x, X \setminus D) > 1/i\big\}, \qquad i = 1, 2, \dots$$

Define $u_i \colon X \to \mathbf{R}$ by $u_i(x) = \max\big(0, 1 - \operatorname{dist}(x, N \cap D_i)\big)$, $i = 1, 2, \dots$ Then $u_i \in M^{1,p}(X)$ is continuous, $0 \le u_i \le 1$ and $u = 1$ in $N \cap D_i$. Define $v_i \colon D_i \to \mathbf{R}$ as $v_i(x) = \operatorname{dist}(x, X \setminus D_i)$, $i = 1, 2, \dots$ Then $v_i \in M_0^{1,p}(D_i) \subset M_0^{1,p}(D)$ and by Theorem 4.5 we have $u_i v_i \in M_0^{1,p}(D) = M_0^{1,p}(D \setminus N)$, $i = 1, 2, \dots$ Fix i. If w is $p-$quasicontinuous function such that $w = u_i v_i$ $\mu-$a.e. in $D \setminus N$, then $w = u_i v_i$ $\mu-$a.e. in D since $\mu(N) = 0$. Theorem 3.2 implies that $w = u_i v_i$ p-q.e. in D. In particular, $w = u_i v_i > 0$ p-q.e. in $N \cap D_i$. On the other hand, since $u_i v_i \in M_0^{1,p}(D \setminus N)$ we may define $w = 0$ p-q.e. in $X \setminus (D \setminus N)$. In particular, we have $w = 0$ p-q.e. in $N \setminus D_i$. This is possible only if $\mathrm{C}_p(N \setminus D_i) = 0$ for every $i = 1, 2, \dots$ and hence

$$\mathrm{C}_p(N) \le \sum_{i=1}^{\infty} \mathrm{C}_p(N \cap D_i) = 0.$$

This completes the proof. $\square$

5. Approximation by Lipschitz continuous functions

In the classical Euclidean case when μ is the Lebesgue measure and Ω is an open set in $\mathbf{R}^n$ the Sobolev space $W_0^{1,p}(\Omega)$ can be characterized as the completion of

$$\{u \in C^\infty(\mathbf{R}^n) \cap W^{1,p}(\mathbf{R}^n)\colon u = 0 \text{ in } \mathbf{R}^n \setminus \Omega\}$$

with respect to the Sobolev norm. Obviously the class $C^\infty(\mathbf{R}^n)$ can be replaced by the Lipschitz continuous functions for the first order Sobolev space.

A natural way in a metric space X would be to describe the Sobolev space with zero boundary values on $E \subset X$ as the completion of

$$\mathrm{Lip}_0^{1,p}(E) = \{u \in M^{1,p}(X)\colon u \text{ is Lipschitz in } X \text{and } u = 0 \text{ in } X \setminus E\}$$

in the norm (2.2). Since $M^{1,p}(X)$ is complete, this completion is the closure of $\mathrm{Lip}_0^{1,p}(E)$ in $M^{1,p}(X)$. We denote it by $H_0^{1,p}(E)$.

In the classical Euclidean case with the Lebesgue measure and $\Omega \subset \mathbf{R}^n$ is open we have $H_0^{1,p}(\Omega) = W_0^{1,p}(\Omega) = M_0^{1,p}(\Omega)$. However, this is not true in the general setting.

To start we recall that $H_0^{1,p}(X) = M_0^{1,p}(X)$ by [H, Theorem 5]. Moreover, since $\mathrm{Lip}_0^{1,p}(E) \subset M_0^{1,p}(E)$ and the latter space is complete we have: If $E \subset X$, then $H_0^{1,p}(E) \subset M_0^{1,p}(E)$. Simple examples show that the equality may fail in general.

We start with a trivial example: Let B be the unit ball in $\mathbf{R}^n$ and $u(x) = \mathrm{dist}(x, \partial B)$. Using the standard measure and metric in $\mathbf{R}^n$ we have that $u \in M_0^{1,p}(E)$, where $E \subset B$ is such that $B \setminus E$ is countable and dense, and $1 < p \leq n$. Clearly, u cannot be approximated in $M^{1,p}(\mathbf{R}^n) = W^{1,p}(\mathbf{R}^n)$ by continuous functions that vanish outside E, for such functions vanish identically. Thus $u \notin H_0^{1,p}(E)$.

On the other hand, we may construct $X \subset \mathbf{R}$ as follows. Let $b_1 > a_1 > b_2 > a_2 > \cdots > 0$ such that $2b_i \le a_{i-1}$, $\lim_{i\to\infty} b_i = 0$ and that $\int_A |x|^{-p}\,dx < \infty$, where $A = \bigcup_{i=1}^{\infty} (a_i, b_i)$. This is clearly possible by choosing the intervals (a_i, b_i) short and sparse enough. Let $X = (-1, 0] \cup A$. The metric is the one induced by the Euclidean metric and the measure is the Lebesgue measure on the real line restricted to X. Then the characteristic function u of $E = (-1, 0]$ is p-quasicontinuous in X and hence belongs to $M_0^{1,p}(E)$, but it does not belong to $H_0^{1,p}(E)$: it cannot be approximated by continuous functions that vanish in $X \setminus E$, for $\mathrm{C}_p(\{0\}) > 0$. (Note that such a continuous function must vanish at 0 and recall Lemma 3.5).

Since continuous functions vanish on a closed set, we have that $\mathrm{Lip}_0^{1,p}(E) = \mathrm{Lip}_0^{1,p}(\operatorname{int} E)$, and hence $H_0^{1,p}(E) = H_0^{1,p}(\operatorname{int} E)$. Furthermore, if $H_0^{1,p}(E) = M_0^{1,p}(E)$, then $M_0^{1,p}(E) = M_0^{1,p}(\operatorname{int} E)$. The claim follows since

$$M_0^{1,p}(E) = H_0^{1,p}(E) = H_0^{1,p}(\operatorname{int} E) \subset M_0^{1,p}(\operatorname{int} E) \subset M_0^{1,p}(E).$$

This supports our choice to restrict the study of the $H = M$ question to open sets. Next we prove a sufficient condition.

5.1. Theorem. *Let D be an open subset of X and suppose that $u \in M^{1,p}(D)$, $1 < p < \infty$. If*

$$\frac{u(x)}{\operatorname{dist}(x, X \setminus D)} \in L^p(D),$$

then $u \in H_0^{1,p}(D)$.

Proof. Let $g \in D(u) \cap L^p(D)$ and define

$$\overline{g}(x) = \begin{cases} \max\big(g(x), |u(x)|/\operatorname{dist}(x, X \setminus D)\big), & x \in D, \\ 0, & x \in X \setminus D. \end{cases}$$

Let $\overline{u}$ be the zero extension of u to $X \setminus D$. We claim that $\overline{g} \in D(\overline{u}) \cap L^p(X)$. By the assumptions $\overline{g} \in L^p(X)$, and hence we need to show that

$$|\overline{u}(x) - \overline{u}(y)| \le d(x,y)\big(\overline{g}(x) + \overline{g}(y)\big) \tag{5.2}$$

for every $x, y \in X \setminus N$ with $\mu(N) = 0$. Now for $\mu-$a.e. $x, y \in D$ or $x, y \in X \setminus D$ this is clear. For $\mu-$a.e. $x \in D$ and $y \in X \setminus D$ we obtain

$$|\overline{u}(x) - \overline{u}(y)| = |u(x)| \le d(x,y)\frac{|u(x)|}{\operatorname{dist}(x, X \setminus D)} \le d(x,y)\big(\overline{g}(x) + \overline{g}(y)\big)\,.$$

Hence $\overline{g} \in D(\overline{u}) \cap L^p(X)$ and consequently $\overline{u} \in M^{1,p}(X)$.

Write

$$F_i = \big\{x \in D \setminus N \colon |\overline{u}(x)| \le i,\, \overline{g}(x) \le i\big\} \cup (X \setminus D) \tag{5.3}$$

for $i = 1, 2, \dots$ By (5.2) we see that $\overline{u}|_{F_i}$ is $2i-$Lipschitz continuous and we extend it to a $2i-$Lipschitz continuous function on X using the McShane extension

$$\overline{u}_i(x) = \inf\big\{\overline{u}(y) + 2i\, d(x,y) \colon y \in F_i\big\}\,.$$

Finally, we truncate $\overline{u}_i$ at the level i and set $u_i(x) = \min\big(\max(\overline{u}_i(x), -i), i\big)$. The function u_i has the following properties:

(1) u_i is $2i-$Lipschitz in X.

(2) $u_i = \overline{u}$ in F_i and, in particular, $u_i = 0$ in $X \setminus D$.

(3) $|u_i| \le i$ in X.

Moreover, by (5.3) we see that

$$i^p \mu(X \setminus F_i) \le i^p \mu\big(\{x \in X \colon |\overline{u}(x)| > i\}\big) + i^p \mu\big(\{x \in X \colon \overline{g}(x) > i\}\big) \to 0, \tag{5.4}$$

as $i \to \infty$. Here we used the fact that $\overline{u}, \overline{g} \in L^p(X)$.

Next we show that $u_i \in M^{1,p}(X)$. To this end, we define

$$g_i(x) = \begin{cases} \overline{g}(x), & x \in F_i, \\ 2i, & x \in X \setminus F_i. \end{cases}$$

Then

$$|u_i(x) - u_i(y)| \le d(x,y)\big(g_i(x) + g_i(y)\big) \tag{5.5}$$

when $x, y \in X \setminus N$. Indeed, if $x, y \in F_i$, then (5.5) is clearly true. For $y \in X \setminus F_i$ we have

$$|u_i(x) - u_i(y)| \le \begin{cases} 2i\, d(x,y) \le d(x,y)\big(g_i(x) + g_i(y)\big), & \text{if } x \in X \setminus F_i \\ 2i\, d(x,y) \le d(x,y)\big(\overline{g}(x) + 2i\big), & \text{if } x \in F_i\,. \end{cases}$$

Hence (5.5) holds and thus $g_i \in D(u_i)$.

Since

$$\int_X g_i^p \, d\mu \le \int_{F_i} \overline{g}^p \, d\mu + (2i)^p \mu(X \setminus F_i) < \infty$$

and

$$\int_X |u_i|^p \, d\mu \le \int_{F_i} |\overline{u}|^p \, d\mu + i^p \mu(X \setminus F_i) < \infty,$$

it follows that $u_i \in M^{1,p}(X)$, and hence $u_i \in \operatorname{Lip}_0^{1,p}(D)$.

It remains to prove that $u_i \to \overline{u}$ in $M^{1,p}(X)$. First observe that (5.4) yields

$$\begin{aligned} \big\|\overline{u} - u_i\big\|_{L^p(X)}^p &= \int_{X\setminus F_i} |\overline{u} - u_i|^p \, d\mu \\ &\le 2^{p-1}\Big(\int_{X\setminus F_i} |\overline{u}|^p \, d\mu + i^p \mu(X \setminus F_i)\Big) \to 0, \end{aligned}$$

as $i \to \infty$. Letting

$$f_i(x) = \begin{cases} \overline{g}(x) + 3i, & x \in X \setminus F_i, \\ 0, & x \in F_i, \end{cases}$$

we have that $f_i \in D(\overline{u} - u_i) \cap L^p(X)$; in fact, the only nontrivial case is $x \in F_i$ and $y \in X \setminus F_i$, but then

$$\begin{aligned} \big|(\overline{u} - u_i)(x) - (\overline{u} - u_i)(y)\big| &\le d(x,y)\big(\overline{g}(x) + \overline{g}(y) + 2i\big) \\ &\le d(x,y)\big(\overline{g}(y) + 3i\big). \end{aligned}$$

¿From (5.4) we obtain

$$\|\overline{u} - u_i\|^p_{L^{1,p}(X)} \le \int_X f_i^p \, d\mu \le 2^{p-1}\Big(\int_{X\setminus F_i} \overline{g}^p \, d\mu + (3i)^p \mu(X \setminus F_i)\Big) \to 0,$$

as $i \to \infty$. Thus $\overline{u} \in H_0^{1,p}(D)$ and the proof is complete. □

Next we give a condition for the open set D such that the assumptions of Theorem 5.1 hold for every $u \in M_0^{1,p}(D)$. To this end, we recall that a locally finite Borel measure μ is *doubling* if there is a constant $c > 0$ such that

$$\mu\big(B(x, 2r)\big) \le c\, \mu\big(B(x, r)\big)$$

for every $x \in X$ and $r > 0$. A nonempty set $E \subset X$ is *uniformly $\mu-$thick*, if there are constants $c > 0$ and $0 < r_0 \le 1$ such that

$$\mu\big(B(x, r) \cap E\big) \ge c\, \mu\big(B(x, r)\big)$$

for every $x \in E$, and $0 < r < r_0$. We use c to denote various constants which may differ even on the same line.

We have the following Hardy type inequality.

5.6. Theorem. *Let $1 < p < \infty$ and suppose that μ is doubling. Let $D \subset X$ be an open set such that $X \setminus D$ is uniformly $\mu-$thick. Then there is a constant $c > 0$ such that*

$$\int_D \Big(\frac{|u(x)|}{\operatorname{dist}(x, X \setminus D)}\Big)^p \, d\mu(x) \le c\, \|u\|^p_{M_0^{1,p}(D)}$$

holds for every $u \in M_0^{1,p}(D)$. The constant c is independent of u.

Proof. Let $\widetilde{u} \in M^{1,p}(X)$ be p-quasicontinuous such that $u = \widetilde{u}$ $\mu-$a.e. in D and $\widetilde{u} = 0$ p-q.e. in $X \setminus D$. Let $g \in D(\widetilde{u}) \cap L^p(X)$. We define

$$D_0 = \big\{x \in D \colon \operatorname{dist}(x, X \setminus D) < r_0\big\}.$$

Fix $x \in D_0$ and choose $x_0 \in X \setminus D$ such that $r_x = \operatorname{dist}(x, X \setminus D) = d(x, x_0)$. Then the uniform $\mu-$thickness and the doubling conditions yield

$$\begin{aligned}
\frac{1}{\mu\big(B(x_0,r_x)\setminus D\big)} &\int_{B(x_0,r_x)\setminus D} g(y)\,d\mu(y) \\
&\le \frac{c}{\mu\big(B(x_0,r_x)\big)} \int_{B(x_0,r_x)} g(y)\,d\mu(y) \\
&\le \frac{c}{\mu\big(B(x,2r_x)\big)} \int_{B(x,2r_x)} g(y)\,d\mu(y) \le c\,Mg(x)
\end{aligned}$$

for $\mu-$a.e. $x \in D_0$; here

$$Mg(x) = \sup_{r>0} \frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} g(y)\,d\mu(y)$$

is the *Hardy–Littlewood maximal function* of g. For $\mu-$a.e. $x \in D_0$ there is $y \in B(x_0, r_x) \setminus D$ such that

$$\begin{aligned}
|u(x)| &\le d(x,y)\Big(g(x) + \frac{1}{\mu\big(B(x_0,r_x)\setminus D\big)} \int_{B(x_0,r_x)\setminus D} g(y)\,d\mu(y)\Big) \\
&\le cr_x\big(g(x) + Mg(x)\big) \le c\operatorname{dist}(x, X\setminus D) Mg(x).
\end{aligned}$$

Since μ is doubling, the Hardy–Littlewood maximal operator is bounded from $L^p(X)$ into itself, see [CW], and hence

$$\int_{D_0} \Big(\frac{|u(x)|}{\operatorname{dist}(x, X\setminus D)}\Big)^p d\mu(x) \le c\int_X (Mg(x))^p\,d\mu(x) \le c\int_X g(x)^p\,d\mu(x).$$

On the other hand

$$\int_{D\setminus D_0} \Big(\frac{|u(x)|}{\operatorname{dist}(x, X\setminus D)}\Big)^p d\mu(x) \le r_0^{-p} \int_D |u(x)|^p\,d\mu(x).$$

We conclude that

$$\int_{D} \Big(\frac{|u(x)|}{\operatorname{dist}(x, X\setminus D)}\Big)^p d\mu(x) \le c\Big(\int_X |\widetilde{u}(x)|^p\,d\mu(x) + \int_X g(x)^p\,d\mu(x)\Big).$$

The claim follows by taking the infimum over all $g \in D(\widetilde{u}) \cap L^p(X)$. $\square$

Theorem 5.6 together with Theorem 5.1 yield

5.7. Corollary. *Let $1 < p < \infty$ and assume that μ is doubling. Suppose that $D \subset X$ is an open set such that $X \setminus D$ is uniformly $\mu-$thick. Then $M_0^{1,p}(D) = H_0^{1,p}(D)$.*

Also we obtain the following compactness result.

5.8. Corollary. *Let $1 < p < \infty$ and assume that μ is doubling. Suppose that $D \subset X$ is an open set such that $X \setminus D$ is uniformly $\mu-$thick. Let $u_j \in M_0^{1,p}(D)$ be a sequence that is bounded in $M_0^{1,p}(D)$. If $u_j \to u$ μ-a.e., then $u \in M_0^{1,p}(D)$.*

Since $H_0^{1,p}(D) = M_0^{1,p}(D)$ in the situation of Corollary 5.8 we see that then the following property (CH) is satisfied for sets D whose complement is μ-thick:

(CH) Let $u_j \in H_0^{1,p}(D)$ be a sequence that is bounded in $H_0^{1,p}(D)$. If $u_j \to u$ μ-a.e., then $u \in H_0^{1,p}(D)$.

If $M^{1,p}(X)$ is reflexive, then by Mazur's lemma closed convex sets are weakly closed. Hence every open subset D of X satisfies property (CH). However, in general we do not know whether the space $M^{1,p}(X)$ is reflexive or not.

5.9. Theorem. *Suppose that X is a proper space, i.e. bounded closed sets are compact. If D is an open subset of X that satisfies* (CH)*, then $H_0^{1,p}(D) = M_0^{1,p}(D)$.*

Proof. Since $H_0^{1,p}(D) \subset M_0^{1,p}(D)$, it suffices to prove the reverse inclusion. For this, let $u \in M_0^{1,p}(D)$ be a quasicontinuous function from $M^{1,p}(X)$ such that $u = 0$ p-q.e. on $X \setminus D$. Using (CH) we easily infer that by truncating and considering the positive and negative parts separately we may assume that u is bounded and non-negative. Fix $x_0 \in D$. If we write

$$\eta_j(x) = \begin{cases} 1 & \text{if } d(x_0, x) \le j-1 \\ j - d(x_0, x) & \text{if } j-1 < d(x_0, x) < j \\ 0 & \text{if } d(x_0, x) \ge j \end{cases}$$

for $j = 1, 2, \dots$, and $v_j = u\eta_j$, then by (CH) it clearly suffices to show that $v_j \in H_0^{1,p}(D)$, because $v_j \to u$ μ-a.e. in X, and $\|v_j\|_{M^{1,p}(X)} \le 2\|u\|_{M^{1,p}(X)}$. Note that

$$\begin{aligned} |v_j(x) - v_j(y)| &\le |u(x) - u(y)| + u(x)\big|\eta_j(x) - \eta_j(y)\big| \\ &\le d(x, y)\big(g(x) + g(y) + u(x)\big), \end{aligned}$$

and hence $v_j \in M^{1,p}(X)$.

Next fix j and let $v = v_j$. Since v vanishes outside a bounded set we find a bounded open subset U of D such that $v = 0$ p-q.e. in $X \setminus U$. Now we may choose a sequence $w_k \in M^{1,p}(X)$ of quasicontinuous functions such that $0 \le w_k \le 1$, $w_k = 1$ on an open set G_k, with $\|w_k\|_{M^{1,p}(X)} \to 0$, and moreover so that the restrictions $v|_{X\setminus G_k}$ are continuous, and $v = 0$ in $X \setminus (U \cup G_k)$. The functions

$$\varphi_k = (1 - w_k) \max\big(v - \frac{1}{k}, 0\big)$$

form a bounded sequence in $M^{1,p}(X)$ and (passing to a subsequence) $\varphi_k \to v$ μ-a.e. Moreover, the continuity of $v|_{X\setminus G_k}$ implies that

$$\overline{\{\varphi_k \ne 0\}} \subset \{v \ge \frac{1}{k}\} \setminus G_k \subset U\,.$$

Hence $\overline{\{\varphi_k \ne 0\}}$ is a compact subset of D, whence $\varphi_k \in H_0^{1,p}(D)$ by Theorem 5.1. Consequently, (CH) yields $v \in H_0^{1,p}(D)$, and the proof is complete. □

5.10. Remark. Suppose that X is a proper space. If $M^{1,p}(X)$ is reflexive, then $H_0^{1,p}(D) = M_0^{1,p}(D)$ whenever D is an open subset of X.

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