SOBOLEV-TYPE LOWER BOUNDS ON $||\nabla\psi||^2$ FOR ARBITRARY REGIONS IN TWO-DIMENSIONAL EUCLIDEAN SPACE*

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Abstract. This note reports the derivation of lower bounds of the Sobolev type on $||\nabla\psi||^2 \equiv \int_R (\partial\psi/\partial x_1)^2 + (\partial\psi/\partial x_2)^2)dx_1\, dx_2$ for generic real scalar $\psi = \psi(x_1, x_2)$ of function class C° piecewise C^2 which vanish over the boundary of the (bounded or unbounded) region R in Euclidean 2-space.

1. Introduction. It has been shown [1] that for all continuous real scalar functions $\phi = \phi(x_1, x_2, x_3)$ with piecewise continuous second-derivatives we have the Sobolev inequality

$$\int |\nabla\phi|^2\, d^3x \geq 3\left(\frac{\pi}{2}\right)^{4/3}\left[\int \phi^6\, d^3x\right]^{1/3} \tag{1}$$

satisfied if ϕ is such that the integral on the right side of (1) is finite. The proof of (1) was given in [1] for unbounded Euclidean 3-space, but it is obvious that this Sobolev inequality is also valid if the domain of definition for ϕ and for the 3-dimensional integrations in (1) is any prescribed (bounded or unbounded) region, provided that ϕ is required to vanish over the boundary of the region.[1] It is shown in the present note that useful lower bounds of the Sobolev type can also be established on

$$||\nabla\psi||^2 \equiv \int_R |\nabla\psi|^2\, d^2x \equiv \int_R ((\partial\psi/\partial x_1)^2 + (\partial\psi/\partial x_2)^2)\, dx_1\, dx_2 \tag{2}$$

for generic real scalar $\psi = \psi(x_1, x_2)$ of function class C° piecewise C^2 which vanish over the boundary of the (bounded or unbounded) region R in Euclidean 2-space.

2. Primary result. Let us consider an unbounded cylindrical region in 3-space that intersects the $x_1 - x_2$ plane in the 2-dimensional region R and has a boundary surface generator parallel to the x_3 − axis. Then for $\phi \equiv \psi \exp(-\lambda\, |x_3|)$ with $\psi = \psi(x_1, x_2)$ and λ a disposable positive constant, we have $\phi = 0$ on the boundary of the cylindrical region if $\psi = 0$ on the boundary of R. If we introduce the notation

$$N^{(\nu)} \equiv \int_R |\psi|^\nu\, d^2x, \qquad \nu = 1, 2, 3, \cdots, \tag{3}$$

the Sobolev inequality (1) applies to $\phi = \psi \exp(-\lambda\, |x_3|)$ through the unbounded cylindrical region and yields

$$\lambda^{-1}\, ||\nabla\psi||^2 + \lambda N^{(2)} \geq 3\left(\frac{\pi}{2}\right)^{4/3} [N^{(6)}/3\lambda]^{1/3} \tag{4}$$

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[1] To prove this, one simply makes an extension of the domain of definition of ϕ to all 3-space with $\phi \equiv 0$ outside the region and applies the original result for unbounded Euclidean 3-space.

or equivalently

$$\lambda^{-2/3} \,||\nabla\psi||^2 + \lambda^{4/3} N^{(2)} \geq \left(\frac{\sqrt{3}\,\pi}{2}\right)^{4/3} [N^{(6)}]^{1/3}. \tag{5}$$

The left side of (5) is minimized by putting $\lambda = ||\nabla\psi||/[2N^{(2)}]^{1/2}$, and thus we obtain[2]

$$||\nabla\psi||^2 \geq \frac{\pi^2}{2\sqrt{3}} [N^{(6)}/N^{(2)}]^{1/2} \tag{6}$$

for $\psi = \psi(x_1, x_2)$ with the specified properties.

It is of interest to compare the primary Sobolev-type lower bound on $||\nabla\psi||^2$ given by (6) with the linear-theoretic result for a bounded region R of finite area $A \equiv \int_R d^2x$, namely

$$||\nabla\psi||^2 \geq \frac{\pi\alpha_0^2}{A} N^{(2)} \tag{7}$$

where $\alpha_0 = .7655\pi$ is the first zero of the zero-order Bessel function, $J_0(\alpha_0) = 0$. Because the smallest ground-state eigenvalue is obtained for fixed area A if R is a circle of radius $(A/\pi)^{1/2}$, the numerical coefficient on the right side of (7) follows from the Helmholtz equation eigenvalue problem associated with $\min_R [\min_\psi \{||\nabla\psi||^2/N^{(2)}\}]$ for $\psi = \psi(x_1, x_2)$ of function class C^0 piecewise C^2 in R and zero on the boundary of R (see, for example [2]). Our Sobolev-type result (6) is sharper than (7) for ψ and A such that $[N^{(6)}]^{1/2} > (2\sqrt{3}\,\alpha_0^2/\pi A)[N^{(2)}]^{3/2}$; moreover, (6) applies for unbounded R (i.e., $A = \infty$) if ψ is such that the three integrals in (6) exist as finite quantities.

3. Alternative lower bound. Excluding from consideration a trivial ψ which vanishes identically in R, the functional

$$\Phi[\psi] \equiv N^{(1)}[N^{(2)}]^{-1} \,||\nabla\psi|| \tag{8}$$

is stationary about solutions to the inhomogeneous Helmholtz equation

$$\nabla^2\psi + k^2\psi = [N^{(1)}]^{-1} \,||\nabla\psi||^2 \operatorname{sgn}(\psi) \tag{9}$$

where the positive quantity $k^2 \equiv 2[N^{(2)}]^{-1} \,||\nabla\psi||^2$. In terms of the variable

$$(\psi - \tfrac{1}{2}[N^{(1)}]^{-1}N^{(2)} \operatorname{sgn}(\psi)),$$

Eq. (9) reduces *presque partout*[3] to the homogeneous Helmholtz equation, and thus the established linear theory for proper vibrations of membranes [2] provides the solution to $\min_R [\min_\psi \{\Phi[\psi]\}]$ for bounded regions R of fixed area A. The minimum value of (8) obtains for ψ of function class C^0 piecewise C^2 in R and zero on the boundary with R a circle of radius $r_A \equiv (A/\pi)^{1/2}$ and ψ proportional to the nonnegative (nodeless) function

$$\hat{\psi} = J_0(kr) - J_0(\alpha_1) \simeq J_0(kr) + (.4026) \tag{10}$$

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[2] The somewhat sharper numerical coefficient $\pi^{3/2}/2^{1/2}3^{1/4} \cong 2.992$ is obtained in place of $\pi^2/2\sqrt{3} \cong 2.849$ in (6) if one puts $\phi = \psi\, e^{-\lambda x_3^2}$ in place of the form $\phi = \psi\, e^{-\lambda |x_3|}$ used here. One is tempted to conjecture that $\min_\psi \{||\nabla\psi||^2 [N^{(2)}/N^{(6)}]^{1/2}\}$ equals either 3 or π, but the author has not been able to solve the associated nonlinear eigenvalue problem which yields the maximum value for the numerical coefficient in (6).

[3] Along the nodal lines $\psi = 0$ the quantity $\nabla^2 \operatorname{sgn}(\psi)$ is not defined, and continuity of the solution must be evoked.

in which $kr_A = \alpha_1 \equiv 1.2197\pi$ is the first positive zero of the first-order Bessel function, $J_1(\alpha_1) = 0$. By making use of the definite integrals (for example, [3] $\int_0^1 J_0(\alpha_1 x)x\,dx = 0$ and $\int_0^1 J_0(\alpha_1 x)^2 x\,dx = \int_0^1 J_1(\alpha_1 x)^2 x\,dx = \frac{1}{2}J_2(\alpha_1)^2 = \frac{1}{2}J_0(\alpha_1)^2$), one obtains the quantities associated with (10)

$$N^{(1)}(\hat{\psi}) = \pi r_A^2\,|J_0(\alpha_1)|\,, \qquad N^{(2)}(\hat{\psi}) = 2\pi r_A^2[J_0(\alpha_1)]^2, \tag{11}$$

verifies that (10) satisfies (9) with $k = \alpha_1/r_A$, and evaluates $\Phi[\hat{\psi}] = \frac{1}{2}\sqrt{\pi}\alpha_1$. Hence, from (8) and $\Phi[\psi] \geq \Phi[\hat{\psi}]$ we get the alternative Sobolev-type lower bound

$$||\nabla\psi||^2 \geq \frac{\pi}{4}\alpha_1^2[N^{(2)}/N^{(1)}]^2. \tag{12}$$

Since the area of the region does not appear on the right side of (12), this result also applies for unbounded R if ψ is such that the three integrals in (12) exist as finite quantities. The equality sign in (12) holds only for a circle of finite radius and ψ proportional to $\hat{\psi}$ given by (10), thus for a ψ which also has its normal derivative equal to zero over the boundary: $(d\hat{\psi}/dr)|_{r=r_A} = 0$. Finally, it should be observed that (12) is sharper than (6) if $[N^{(6)}]^{1/2} < (\sqrt{3}\alpha_1^2/2\pi)[N^{(1)}]^{-2}[N^{(2)}]^{5/2} \simeq (4.07)[N^{(1)}]^{-2}[N^{(2)}]^{5/2}$, a circumstance not precluded by the general Hölder inequality for all ψ, $[N^{(6)}]^{1/2} > [N^{(1)}]^{-2}[N^{(2)}]^{5/2}$.

References

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