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SOBOLEV TYPE SPACES ASSOCIATED WITH THE q-RUBIN'S OPERATOR

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In this paper, we introduce and study some q-Sobolev type spaces by using the harmonic analysis associated with the q-Rubin operator. In particular, embedding theorems for these spaces are established. Next, we introduce the q-Rubin potential spaces and study some of its properties.

1. Introduction

In classical analysis, Sobolev spaces are vector spaces whose elements are functions defined on domains in n-dimensional Euclidean space $\mathbb{R}^n$ and whose partial derivatives satisfy certain integrability conditions. Their uses and the study of their properties were facilitated by the theory of distributions and Fourier analysis. For instance, the Sobolev space $W^s(\mathbb{R}), \quad s \in \mathbb{R}$, is defined by the use of the classical Fourier transform as the set of all tempered distributions u with classical Fourier transform $\mathcal{F}(u)$ satisfying

$$(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}(u) \in L^2(\mathbb{R}).$$

Generalization of the Sobolev spaces have been studied by replacing the classical Fourier transform by a generalized one. As far as we know, in the literature, except [7], there is no paper concerning generalizations of Sobolev spaces in the

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context of q-differential-difference operators. This paper is an attempt to fill this gap by studying the generalized Sobolev spaces associated with the q-Rubin's operator. The main tools in this study are some elements of the q-Rubin-Fourier harmonic analysis. Next, we introduce and study the q-Rubin potential spaces.

The present paper is organized as follows: In Section 2, we present some preliminary results and notations that will be useful in the sequel. In Section 3, we establish some results associated with q-Rubin Fourier analysis and we state some useful result about q-tempered distributions. Section 4 is devoted to introduce and study the Sobolev type spaces associated with q-Rubin operator by using some elements of harmonic analysis associated with q-Rubin operator. Some embedding theorem are established. Finally, in Section 5, we introduce the q-Rubin potential spaces and study some of their properties.

2. Notations and preliminaries

Throughout this paper, we assume $0 < q < 1$ and we refer to the general references [3] and [5] for the definitions and properties of the basic hypergeometric functions. In what follows, we will fix some notations and recall some preliminary results.

2.1. Basic symbols.

We put $\mathbb{R}_q = \{\pm q^n \ : \ n \in \mathbb{Z}\}$, $\widetilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}$ and
$\mathbb{R}_{q,+} = \{q^n \ : \ n \in \mathbb{Z}\}$.
For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; \ \ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \ \ n = 1,2,\ldots; \ \ (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

2.2. Operators and elementary special functions.

A q-analogue of the classical exponential function is given by (see [8, 9])

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2), \tag{1}$$

where

$$\cos(x;q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\frac{x^{2n}}{[2n]_q!}, \ \sin(x;q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\frac{x^{2n+1}}{[2n+1]_q!}. \tag{2}$$

These three functions are entire on $\mathbb{C}$ and when q tends to 1, they tend to the corresponding classical ones pointwise and uniformly on compacts.
Note that we have for all $x \in \mathbb{R}_q$ (see [8])

$$|\cos(x;q^2)| \leq \frac{1}{(q;q)_\infty}, \quad |\sin(x;q^2)| \leq \frac{1}{(q;q)_\infty}$$

and

$$|\, e(ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{3}$$

The q^2-analogue differential operators is defined as (see [8, 9])

$$\partial_q(f)(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & if \quad z \neq 0 \\ \lim\limits_{x\to 0} \partial_q(f)(x) \qquad (\text{in } \mathbb{R}_q) & if \quad z = 0. \end{cases}$$

A repeated application of the q^2-analogue differential operator n times is denoted by:

$$\partial_q^0 f = f, \quad \partial_q^{n+1} f = \partial_q(\partial_q^n f).$$

The following lemma lists some useful computational properties of ∂_q and reflects its sensitivity to parity of its argument.

Lemma 2.1.
1) $\partial_q \sin(x;q^2) = \cos(x;q^2)$, $\partial_q \cos(x;q^2) = -\sin(x;q^2)$ *and* $\partial_q e(x;q^2) = e(x;q^2)$.
2) For all function f *on* $\mathbb{R}_q$, $\partial_q f(z) = \dfrac{f_e(q^{-1}z) - f_e(z)}{(1-q)z} + \dfrac{f_o(z) - f_o(qz)}{(1-q)z}$.
3) For two functions f *and* g *on* $\mathbb{R}_q$, *we have*

$$\begin{aligned} \partial_q(fg)(z) = {} & q^{-1}(\partial_q f_o)(q^{-1}z)g_o(q^{-1}z) + q^{-1}f_o(z)(\partial_q g_o)(q^{-1}z) \\ & + (\partial_q f_o)(z)g_e(z) + qf_o(qz)(\partial_q g_e)(qz) + f_e(z)(\partial_q g_o)(z) \\ & + q(\partial_q f_e)(qz)g_o(qz) + (\partial_q f_e)(z)g_e(q^{-1}z) + f_e(z)(\partial_q g_e)(z). \end{aligned} \tag{4}$$

Here, for a function f *defined on* $\mathbb{R}_q$, f_e *and* f_o *are respectively, its even and odd parts.*

The q-Jackson integrals are defined by (see [4])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} q^n f(aq^n), \quad \int_a^b f(x)d_qx = (1-q)\sum_{n=0}^{\infty} q^n \left(bf(bq^n) - af(aq^n)\right)$$

and

$$\int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n + (1-q)\sum_{n=-\infty}^{\infty} f(-q^n)q^n,$$

provided the sums converge absolutely.

The following simple result, giving q-analogues of the integration by parts theorem, can be verified by direct calculation.

Lemma 2.2.
1) For $a > 0$, if $\int_{-a}^{a}(\partial_q f)(x)g(x)d_qx$ exists, then

$$\int_{-a}^{a}(\partial_q f)(x)g(x)d_qx = 2\left[f_e(q^{-1}a)g_o(a) + f_o(a)g_e(q^{-1}a)\right] - \int_{-a}^{a} f(x)(\partial_q g)(x)d_qx. \tag{5}$$

2) If $\int_{-\infty}^{\infty}(\partial_q f)(x)g(x)d_qx$ exists,

$$\int_{-\infty}^{\infty}(\partial_q f)(x)g(x)d_qx = -\int_{-\infty}^{\infty} f(x)(\partial_q g)(x)d_qx. \tag{6}$$

3. Elements of q-harmonic analysis related to the operator ∂_q

In what follows, we need the following sets and spaces:
• $C_q^p(\mathbb{R}_q)$ the space of functions f p times q-differentiable on $\widetilde{\mathbb{R}}_q$ such that for all $0 \le n \le p$, $\partial_q^p f$ is continuous on $\widetilde{\mathbb{R}}_q$.
• $\mathcal{D}_q(\mathbb{R}_q)$ the space of functions defined on $\mathbb{R}_q$ with compact supports.
• $\mathcal{S}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$ satisfying, for all m,n non-negative integers,

$$P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q} \mid x^m \partial_q^n f(x) \mid < \infty$$

and

$$\lim_{x\to 0} \partial_q^n f(x) \quad (\text{in} \quad \mathbb{R}_q) \qquad \text{exists.}$$

• $\mathcal{S}'_q(\mathbb{R}_q)$ the space of tempered distributions on $\mathbb{R}_q$. It is the topological dual of $\mathcal{S}_q(\mathbb{R})$.
• $L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p d_qx\right)^{\frac{1}{p}} < \infty \right\}.$
• $L_q^\infty(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty \right\}.$

4. Elements of harmonic analysis associated with the q-Rubin operator

In [9], R. L. Rubin defined the q-Rubin-Fourier transform as

$$\mathcal{F}_q(f)(x) = K\int_{-\infty}^{\infty} f(t)e(-itx;q^2)d_qt,$$

where $K = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}\left(\frac{1}{2}\right)}$ and $\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}$ is the q-Gamma function. Note that letting $q \uparrow 1$ subject to the condition

$$\frac{Log(1-q)}{Log(q)} \in 2\mathbb{Z}, \tag{7}$$

gives, at least formally, the classical Fourier transform (see [6]). In the remainder of this paper, we assume that the condition (7) holds.

It was shown in [9] that the q-Rubin-Fourier transform $\mathcal{F}_q$ verifies the following properties:
1) If $f(u),\ uf(u) \in L_q^1(\mathbb{R}_q)$, then

$$\partial_q\left(\mathcal{F}_q(f)\right)(x) = \mathcal{F}_q(-iuf(u))(x).$$

2) If $f,\ \partial_q f \in L_q^1(\mathbb{R}_q)$, then

$$\mathcal{F}_q(\partial_q f)(x) = ix\mathcal{F}_q(f)(x). \tag{8}$$

In the following theorem, we give some useful result.

Theorem 4.1. *For $f \in L_q^1(\mathbb{R}_q)$, we have*
i) $\mathcal{F}_q(f)$ is continuous on $\widetilde{\mathbb{R}}_q$.
ii) $\mathcal{F}_q(f)$ is bounded on $\mathbb{R}_q$ and we have

$$\|\mathcal{F}_q(f)\|_{\infty,q} \leq \frac{(1+q)^{\frac{1}{2}}}{\Gamma_{q^2}\left(\frac{1}{2}\right)(q;q)_\infty}\|f\|_{1,q}. \tag{9}$$

iii) We have the following reciprocity theorem

$$\forall t \in \mathbb{R}_q, \quad f(t) = K\int_{-\infty}^{\infty} \mathcal{F}_q(f)(x)e(itx;q^2)d_qx. \tag{10}$$

Theorem 4.2. *The q-Rubin Fourier transform $\mathcal{F}_q$ is an isomorphism from $\mathcal{S}_q(\mathbb{R}_q)$ onto itself. Moreover, for all $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have*

$$\mathcal{F}_q^{-1}(f)(x) = \mathcal{F}_q(f)(-x) = \overline{\mathcal{F}_q(\overline{f})}(x), \quad x \in \mathbb{R}_q. \tag{11}$$

Proof. We begin by proving that $\mathcal{F}_q$ leaves $\mathcal{S}_q(\mathbb{R}_q)$ invariant.
From the definition of $\mathcal{S}_q(\mathbb{R}_q)$ and the properties of the operator ∂_q (Lemma 2.1), one can easily see that $\mathcal{S}_q(\mathbb{R}_q)$ is also the set of all functions defined on $\mathbb{R}_q$, such that for all $k,l \in \mathbb{N}$, we have

$$\sup_{x\in\mathbb{R}_q}\left|\partial_q^k\left(x^l f(x)\right)\right| < \infty \quad \text{and} \quad \lim_{x\to 0}\partial_q^k f(x) \quad \text{exists.}$$

Now, let $f \in \mathcal{S}_q(\mathbb{R}_q)$ and $k,l \in \mathbb{N}$. On the one hand, from the properties of the operator ∂_q, we have for all $n \in \mathbb{N}$, $\partial_q^n f \in \mathcal{S}_q(\mathbb{R}_q) \subset L_q^1(\mathbb{R}_q)$.
On the other hand, from the relation (8), we have

$$\begin{aligned}\lambda^l \mathcal{F}(f)(\lambda) &= (-i)^l \mathcal{F}_q(\partial_q^l f)(\lambda)\\ &= (-i)^l K \int_{-\infty}^{\infty} \partial_q^l f(x) e(-i\lambda x; q^2) d_q x.\end{aligned}$$

So, using the relation (3), we obtain for all $\lambda \in \mathbb{R}_q$,

$$\begin{aligned}|\partial_q^k(\lambda^l \mathcal{F}_q(f)(\lambda))| &= \left|(-i)^l \frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty} \partial_q^l f(x) \partial_q^k e(-i\lambda x; q^2) d_q x\right|\\ &\leq \frac{2c_{\alpha,q}}{(q;q)_\infty}\int_{-\infty}^{\infty} |\partial_q^l f(x)| d_q x < \infty.\end{aligned}$$

This together with the Lebesgue theorem prove that $\mathcal{F}_q(f)$ belongs to $\mathcal{S}_q(\mathbb{R}_q)$.
By Theorem 4.1, we deduce that $\mathcal{F}_q$ is an isomorphism of $\mathcal{S}_q(\mathbb{R}_q)$ onto itself and for $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have $(\mathcal{F}_q)^{-1}(f)(x) = \mathcal{F}_q(f)(-x), \quad x \in \mathbb{R}_q$. □

In [8], we find the following Plancherel theorem.

Theorem 4.3. *$\mathcal{F}_q$ is an isomorphism from $L_q^2(\mathbb{R}_q)$ onto itself. For $f \in L_q^2(\mathbb{R}_q)$,*

$$\|\mathcal{F}_q(f)\|_{2,q} = \|f\|_{2,q} \tag{12}$$

and

$$\forall t \in \mathbb{R}_q, \quad f(t) = K\int_{-\infty}^{\infty} \mathcal{F}_q(f)(x) e(itx; q^2) d_q x. \tag{13}$$

The q-translation operator $\tau_{q,x}$, $x \in \widetilde{\mathbb{R}}_q$ is defined (see [8]) on $L_q^1(\mathbb{R}_q)$ by

$$\tau_{q,x}(f)(y) = K\int_{-\infty}^{\infty} \mathcal{F}_q(f)(t) e(itx; q^2) e(ity; q^2) d_q t, \;\; y \in \mathbb{R}_q, \tag{14}$$

$$\tau_{q,0}(f)(y) = f(y). \tag{15}$$

It was shown in [8] that the q-translation operator can be also defined on $L_q^2(\mathbb{R}_q)$ and we have the following result.

Proposition 4.4. *For all $f \in L^2_q(\mathbb{R}_q)$, we have $\tau_{q,x}f \in L^2_q(\mathbb{R}_q)$ and*

$$\|\tau_{q,x}f\|_{2,q} = \frac{2}{(q;q)_\infty} \leq \|f\|_{2,q}, \ \ x \in \widetilde{\mathbb{R}}_q.$$

Furthermore, it verifies the following properties.

Proposition 4.5. *For $f, g \in L^1_q(\mathbb{R}_q)$, we have*
i) $\tau_{q,x}(f)(y) = \tau_{q,y}(f)(x), \ \ x, \ y \in \mathbb{R}_q$.
ii) $\int_{-\infty}^{\infty} \tau_{q,x}(f)(-y)g(y)d_qy = \int_{-\infty}^{\infty} f(y)\tau_{q,x}(g)(-y)d_qy, \ \ x \in \widetilde{\mathbb{R}}_q$.
iii)

$$\mathcal{F}_q(\tau_{q,x}f)(\lambda) = e(i\lambda x; q^2)\mathcal{F}_q(f)(\lambda), \ \ x \in \widetilde{\mathbb{R}}_q. \tag{16}$$

iv) $\partial_q \tau_{q,x} f = \tau_{q,x}\partial_q f, \ \ x \in \widetilde{\mathbb{R}}_q$.

By using the q-translation operator, we define the generalized convolution product $f *_q g$ of two functions $f, g \in \mathcal{S}_q(\mathbb{R}_q)$ as follows:

$$f *_q g = K \int_{-\infty}^{\infty} \tau_{q,x} f(y) g(y) d_q y$$

Proposition 4.6. *For $f, g \in \mathcal{S}_q(\mathbb{R}_q)$ we have*

$$\mathcal{F}_q(f *_q g) = \mathcal{F}_q(f)\mathcal{F}_q(g).$$

We finish this section by stating some useful results about the q-tempered space.

Definition 4.7. The q-Rubin transform of a q-distribution u in $\mathcal{S}'_q(\mathbb{R}_q)$ is defined by

$$\langle \mathcal{F}_q(u), \varphi\rangle = \langle u, \mathcal{F}_q(\varphi)\rangle \ \ u \in \mathcal{S}'{}_q(\mathbb{R}_q), \ \varphi \in \mathcal{S}_q(\mathbb{R}_q).$$

Proposition 4.8. *The q-Rubin transform $\mathcal{F}_q$ is a topological isomorphism from $\mathcal{S}'{}_q(\mathbb{R}_q)$ onto itself.*

Proof. The result is a consequence of Theorem 2. □

For $u \in \mathcal{S}'{}_q(\mathbb{R}_q)$, we define the distribution $\partial_q u$, by

$$\langle \partial_q u, \psi\rangle = -\langle u, \partial_q \psi\rangle, \ \ \psi \in \mathcal{S}_q(\mathbb{R}_q).$$

These distributions satisfy the following properties

$$\forall p \in \mathbb{N}, u \in \mathcal{S}'{}_q(\mathbb{R}_q), \quad \mathcal{F}_q(\partial_q^p u) = (-iy)^p \mathcal{F}_q(u). \tag{17}$$

5. q-Sobolev spaces

In this Section, we establish the main properties of the Sobolev spaces associated with the q-Rubin operator.

Definition 5.1. For $s \in \mathbb{R}$, we define the Sobolev space $\mathcal{W}_q^s(\mathbb{R}_q)$ as

$$\mathcal{W}_q^s(\mathbb{R}_q) = \left\{ u \in \mathcal{S}_q'(\mathbb{R}_q) : (1+|\xi|^2)^{\frac{s}{2}} \mathcal{F}_q(u) \in L_q^2(\mathbb{R}_q) \right\}.$$

We provide $\mathcal{W}_q^s(\mathbb{R}_q)$ with the scalar product

$$\langle u, v \rangle_s = \int_{-\infty}^{\infty} (1+|\xi|^2)^s \mathcal{F}_q(u)(\xi) \overline{\mathcal{F}_q(v)(\xi)} d_q\xi$$

and the norm

$$\|u\|_{\mathcal{W}_q^s(\mathbb{R}_q)} := \left(\int_{-\infty}^{\infty} (1+|\xi|^2)^s |\mathcal{F}_q(u)(\xi)|^2 d_q\xi \right)^{\frac{1}{2}}. \tag{18}$$

Remark 5.2. Let $u \in \mathcal{W}_q^s(\mathbb{R}_q)$. Then, using the relations (11) and (18), and the change of variables $\xi = -t$, we obtain

$$\int_{-\infty}^{\infty} (1+|\xi|^2)^s |\mathcal{F}_q(u)(\xi)|^2 d_q\xi = \int_{-\infty}^{\infty} (1+|t|^2)^s |\mathcal{F}_q(\overline{u})(t)|^2 d_qt.$$

Then, $\overline{u} \in \mathcal{W}_q^s(\mathbb{R}_q)$ and $\|\overline{u}\|_{\mathcal{W}_q^s(\mathbb{R}_q)} = \|u\|_{\mathcal{W}_q^s(\mathbb{R}_q)}$.

Proposition 5.3. *i) For all $s \in \mathbb{R}$, we have*

$$\mathcal{S}_q(\mathbb{R}_q) \subset \mathcal{W}_q^s(\mathbb{R}_q).$$

ii) We have

$$\mathcal{W}_q^0(\mathbb{R}_q) = L_q^2(\mathbb{R}_q).$$

iii) For all s_1, s_2 in $\mathbb{R}$, such that $s_1 \geq s_2$, the space $\mathcal{W}_q^{s_1}(\mathbb{R}_q)$ is continuously contained in $\mathcal{W}_q^{s_2}(\mathbb{R}_q)$.

Proof. i) and ii) are immediately from the definition of the generalized Sobolev space.
iii) Let $s_1, s_2 \in \mathbb{R}$ such that $s_1 > s_2$ and $u \in \mathcal{W}_q^{s_1}(\mathbb{R}_q)$.
Then,

$$\forall \xi \in \mathbb{R}_q, \quad (1+|\xi|^2)^{s_2} \leq (1+|\xi|^2)^{s_1}$$

and

$$\int_{-\infty}^{\infty} \left|(1+|\xi|^2)^{s_2} \mathcal{F}_q(u)(\xi)\right|^2 d_q\xi \leq \int_{-\infty}^{\infty} \left|(1+|\xi|^2)^{s_1} \mathcal{F}_q(u)(\xi)\right|^2 d_q\xi < \infty.$$

So, $u \in \mathcal{W}_q^{s_2}(\mathbb{R}_q)$ and $\|u\|_{\mathcal{W}_q^{s_2}(\mathbb{R}_q)} \leq \|u\|_{\mathcal{W}_q^{s_1}(\mathbb{R}_q)}$.
Then, the space $\mathcal{W}_q^{s_1}(\mathbb{R}_q)$ is continuously contained in $\mathcal{W}_q^{s_2}(\mathbb{R}_q)$.

□

Proposition 5.4. *The space $\mathcal{W}_q^s(\mathbb{R}_q)$ provided with the norm $\|.\|_{\mathcal{W}_q^s(\mathbb{R}_q)}$ is a Banach space.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{W}_q^s(\mathbb{R}_q)$. Then, from the definition of the norm $\|.\|_{\mathcal{W}_q^s(\mathbb{R}_q)}$, it is easy to see that $(\mathcal{F}_q(u_n))_n$ is a Cauchy sequence in $L^2(\mathbb{R}_q,(1+|\xi|^2)^s d_q\xi)$.
But $L^2(\mathbb{R}_q,(1+|\xi|^2)^s d_q\xi)$ is complete, then there exists a function u in $L^2(\mathbb{R}_q,(1+|\xi|^2)^s d_q\xi)$ such that

$$\lim_{n\to+\infty}\|\mathcal{F}_q(u_n)-u\|_{L^2(\mathbb{R}_q,(1+|\xi|^2)^s d_q\xi)}=0. \tag{19}$$

Then $u\in\mathcal{S}_q'(\mathbb{R}_q)$ and from Proposition 4.8, we obtain

$$v=(\mathcal{F}_q)^{-1}(u)\in\mathcal{S}_q'(\mathbb{R}_q).$$

So, $\mathcal{F}_q(v)=u\in L^2(\mathbb{R}_q,(1+|\xi|^2)^s d_q\xi)$, which proves that $v\in\mathcal{W}_q^s(\mathbb{R}_q)$.
Furthermore, using the relation (19), we get:

$$\lim_{n\to+\infty}\|u_n-v\|_{\mathcal{W}_q^s(\mathbb{R}_q)}=\lim_{n\to+\infty}\|\mathcal{F}_q(u_n)-u\|_{L^2(\mathbb{R}_q,(1+|\xi|^2)^s d_q\xi)}=0.$$

Hence, $\mathcal{W}_q^s(\mathbb{R}_q)$ is complete. □

Lemma 5.5 (Convexity)**.** *Let $s_1,s_2\in\mathbb{R}$, such that $s_1<s_2$ and $s=(1-t)s_1+ts_2$, $t\in]0,1[$. Then we have*

$$\forall\, u\in\mathcal{W}_q^{s_2}(\mathbb{R}_q),\quad \|u\|_{\mathcal{W}_q^s(\mathbb{R}_q)}\le\|u\|_{\mathcal{W}_q^{s_1}(\mathbb{R}_q)}^{1-t}\times\|u\|_{\mathcal{W}_q^{s_2}(\mathbb{R}_q)}^{t}$$

Proof. Let $s_1,s_2\in\mathbb{R}$, such that $s_1<s_2$ and $s=(1-t)s_1+ts_2$, $t\in]0,1[$.
Let $u\in\mathcal{W}_q^{s_2}(\mathbb{R}_q)$. Then,

$$\begin{aligned}\|u\|^2_{\mathcal{W}_q^s(\mathbb{R}_q)}&=\int_{-\infty}^{\infty}\left|(1+|\xi|^2)^s\mathcal{F}_q(u)(\xi)\right|^2 d_q\xi\\&=\int_{-\infty}^{\infty}\left|(1+|\xi|^2)^{s_1(1-t)}\mathcal{F}_q(u)(\xi)\right|^{2(1-t)}\left|(1+|\xi|^2)^{s_2t}\mathcal{F}_q(u)(\xi)\right|^{2t}d_q\xi.\end{aligned}$$

Then, using the Hölder's inequality, we get

$$\begin{aligned}\|u\|^2_{\mathcal{W}_q^s(\mathbb{R}_q)}&\le\left[\int_{-\infty}^{\infty}|(1+|\xi|^2)^{s_1}\mathcal{F}_q(u)(\xi)|^2 d_q\xi\right]^{1-t}\\&\quad\times\left[\int_{-\infty}^{\infty}|(1+|\xi|^2)^{s_2}\mathcal{F}_q(u)(\xi)|^2 d_q\xi\right]^{t}\\&\le\|u\|^{2(1-t)}_{\mathcal{W}_q^{s_1}(\mathbb{R}_q)}\times\|u\|^{2t}_{\mathcal{W}_q^{s_2}(\mathbb{R}_q)}.\end{aligned}$$

□

Proposition 5.6. *Let s_1, s, s_2 be three real numbers, satisfying $s_1 < s < s_2$. Then, for all $\varepsilon > 0$, there exists a nonnegative constant C_ε such that for all $u \in \mathcal{W}_q^s(\mathbb{R}_q)$, we have*

$$\|u\|_{\mathcal{W}_q^s(\mathbb{R}_q)} \leq C_\varepsilon \|u\|_{\mathcal{W}_q^{s_1}(\mathbb{R}_q)} + \varepsilon \|u\|_{\mathcal{W}_q^{s_2}(\mathbb{R}_q)}.$$

Proof. Let $s_1, s, s_2 \in \mathbb{R}$, $s_1 < s_2$ and $s \in]s_1, s_2[$. Then there exists $t \in]0,1[$ such that $s = (1-t)s_1 + ts_2$.

From the previous lemma and using $\left(\varepsilon^{\frac{-t}{1-t}}\right)^{1-t} . \varepsilon^t = 1$, we get for $u \in \mathcal{W}_q^s(\mathbb{R}_q)$,

$$\begin{aligned} \|u\|_{\mathcal{W}_q^s(\mathbb{R}_q)} &\leq \|u\|^{1-t}_{\mathcal{W}_q^{s_1}(\mathbb{R}_q)} \|u\|^t_{\mathcal{W}_q^{s_2}(\mathbb{R}_q)} \\ &= \left(\varepsilon^{-\frac{t}{1-t}} \|u\|_{\mathcal{W}_q^{s_1}(\mathbb{R}_q)}\right)^{1-t} \left(\varepsilon \|u\|_{\mathcal{W}_q^{s_2}(\mathbb{R}_q)}\right)^t. \end{aligned}$$

So from the fact,

$$\forall a, b > 0, \ a^t b^{1-t} \leq a + b,$$

we obtain

$$\|u\|_{\mathcal{W}_q^s(\mathbb{R}_q)} \leq \varepsilon^{-\frac{t}{1-t}} \|u\|_{\mathcal{W}_q^{s_1}(\mathbb{R}_q)} + \varepsilon \|u\|_{\mathcal{W}_q^{s_2}(\mathbb{R}_q)}.$$

This completes the proof by taking $C_\varepsilon = \varepsilon^{-\frac{t}{1-t}} = \varepsilon^{\frac{s-s_1}{s-s_2}}$. □

A characterization of $\mathcal{W}_q^s(\mathbb{R}_q)$, for $s = m$, a positive integer, is given below.

Proposition 5.7. *Let $m \in \mathbb{N}$. Then*

$$\mathcal{W}_q^m(\mathbb{R}_q) = \left\{ u \in \mathcal{S}_q'(\mathbb{R}_q) : \mathcal{F}_q(\partial_q^j u) \in L_q^2(\mathbb{R}_q), \ 0 \leq j \leq m \right\}.$$

Proof. Let $u \in \mathcal{W}_q^m(\mathbb{R}_q)$. Then, using the formula (17), we obtain

$$\mathcal{F}_q(\partial_q^j u) = (-i\lambda)^j \mathcal{F}_q(u), \ 0 \leq j \leq m \tag{20}$$

and

$$\begin{aligned} \forall\, 0 \leq j \leq m, \ \int_{-\infty}^{\infty} |\mathcal{F}_q(\partial_q^j u)(\xi)|^2 d_q\xi &= \int_{-\infty}^{\infty} |(-i\xi)^j \mathcal{F}_q(u)(\xi)|^2 d_q\xi \\ &\leq \int_{-\infty}^{\infty} (1 + |\xi|^2)^{\frac{j}{2}} |\mathcal{F}_q(u)(\xi)|^2 d_q\xi \\ &\leq \int_{-\infty}^{\infty} (1 + |\xi|^2)^{\frac{m}{2}} |\mathcal{F}_q(u)(\xi)|^2 d_q\xi \\ &< \infty. \end{aligned}$$

So,

$$\mathcal{F}_q(\partial_q^j u) \in L_q^2(\mathbb{R}_q), \ 0 \leq j \leq m.$$

Hence,

$$\mathcal{W}_q^m(\mathbb{R}_q) \subset \left\{u \in \mathcal{S}_q^{'}(\mathbb{R}_q) : \mathcal{F}_q(\partial_q^j u) \in L_q^2(\mathbb{R}_q),\ 0 \le j \le m\right\}.$$

Conversely, assume that

$$\mathcal{F}_q(\partial_q^j u) \in L_q^2(\mathbb{R}_q), 0 \le j \le m.$$

It is easy to see that there exists a positive constant C such that

$$(1+|\xi|^2)^{\frac{m}{2}} \le C \sum_{j=0}^{m} |\xi|^j.$$

Then using the formula (20), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} |(1+|\xi|^2)^{\frac{m}{2}} \mathcal{F}_q(u)(\xi)|^2 d_q\xi &\le C \sum_{k=0}^{m} \int_{-\infty}^{\infty} |(-i\xi)^j \mathcal{F}_q(u)(\xi)|^2 d_q\xi \\ &= C \sum_{k=0}^{m} \int_{-\infty}^{\infty} |\mathcal{F}_q(\partial_q^j u)(\xi)|^2 d_q\xi < \infty.\end{aligned}$$

Hence

$$u \in \mathcal{W}_q^m(\mathbb{R}_q).$$

Finally, we obtain

$$\left\{u \in \mathcal{S}_q^{'}(\mathbb{R}_q) : \mathcal{F}_q(\partial_q^j u) \in L_q^2(\mathbb{R}_q),\ 0 \le j \le m\right\} \subset \mathcal{W}_q^m(\mathbb{R}_q).$$

This leads to the result. □

Using the q-Plancherel theorem, we obtain the following result.

Corollary 5.8. *For $m \in \mathbb{N}$, we have*

$$\mathcal{W}_q^m(\mathbb{R}_q) = \left\{f \in L_q^2(\mathbb{R}_q) : \partial_q^j f \in L_q^2(\mathbb{R}_q)\ j = 0, \dots, m.\right\}$$

Proposition 5.9. *Let $s \in \mathbb{R}_q$ and $p \in \mathbb{N}$ such that $s > \frac{1}{2} + p$. Then, we have $\mathcal{W}_q^s(\mathbb{R}_q) \subset C_q^p(\mathbb{R}_q)$.*

Proof. Let $s \in \mathbb{R}$ such that $s > \frac{1}{2} + p$ and $u \in \mathcal{W}_q^s(\mathbb{R}_q)$. Then, for $0 \le n \le p$, we have

$$\int_{-\infty}^{\infty} |\lambda^n \mathcal{F}_q(u)(\lambda)| d_q\lambda = \int_{-\infty}^{\infty} |\lambda^n (1+|\lambda|^2)^{-\frac{s}{2}} (1+|\lambda|^2)^{\frac{s}{2}} \mathcal{F}_q(u)(\lambda)| d_q\lambda.$$

Using the Cauchy-Schwarz inequality, we deduce that

$$\int_{-\infty}^{\infty} |\lambda^n \mathcal{F}_q(u)(\lambda)| d_q\lambda \leq \left(\int_{-\infty}^{\infty} \left(\lambda^n (1+|\lambda|^2)^{-\frac{s}{2}} \right)^2 d_q\lambda \right)^{\frac{1}{2}}$$

$$\times \left(\int_{-\infty}^{\infty} \left[(1+|\lambda|^2)^{\frac{s}{2}} \left| \mathcal{F}_q(u)(\lambda) \right| \right]^2 d_q\lambda \right)^{\frac{1}{2}}. \quad (21)$$

Since $s > \frac{1}{2} + p$ and $u \in \mathcal{W}_q^s(\mathbb{R}_q)$, then for all $0 \leq n \leq p$, we have

$$C_{q,n} = \left(\int_{-\infty}^{\infty} \left(\lambda^n (1+|\lambda|^2)^{-\frac{s}{2}} \right)^2 d_q\lambda \right)^{\frac{1}{2}} < \infty$$

and

$$\int_{-\infty}^{\infty} |\lambda^n \mathcal{F}_q(u)(\lambda)| d_q\lambda < \infty.$$

So,

$$\lambda^n \mathcal{F}_q(u)(\lambda) \in L_q^1(\mathbb{R}_q) \quad \text{for all} \quad 0 \leq n \leq p.$$

In particular $\mathcal{F}_q(u) \in L_q^1(\mathbb{R}_q)$. Then, from (13), we have

$$u(x) = K \int_{-\infty}^{\infty} \mathcal{F}_q(u)(\lambda) e(ix\lambda; q^2) d_q\lambda, \ \ x \in \mathbb{R}_q. \quad (22)$$

The q-derivation under the q-integral sign gives

$$\forall\, 0 \leq n \leq p, \quad \forall x \in \mathbb{R}_q, \quad \partial_q^n u(x) = K \int_{-\infty}^{\infty} (i\lambda)^n \mathcal{F}_q(u)(\lambda) e(ix\lambda; q^2) d_q\lambda. \quad (23)$$

Then since $\lambda^n \mathcal{F}_q(u) \in L_q^1(\mathbb{R}_q)$, the inequality (3), the Lebesgue theorem and Theorem 4.1 show that $\partial_q^n u$ is continuous on $\widetilde{\mathbb{R}}_q$ for all $0 \leq n \leq p$.
So $u \in C_q^P(\mathbb{R}_q)$. This shows that $\mathcal{W}_q^s(\mathbb{R}_q) \subset C_q^p(\mathbb{R}_q)$, which completes the proof. □

Theorem 5.10. *For all $s \in (0,1)$, we have*

$$\mathcal{W}_q^s(\mathbb{R}_q) = \left\{ f \in L_q^2(\mathbb{R}_q) : \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{|(f - \tau_{q,x} f)(\xi)|^2}{|x|^{1+2s}} d_q x \right) d_q\xi < \infty \right\},$$

where $\tau_{q,x}$ is the q-translation operator defined by (14).

Proof. Since $0 < s < 1$, then

$$\forall \xi \in \mathbb{R}, \ \max(1, |\xi|^s) \leq (1+|\xi|^2)^{\frac{s}{2}} \leq 1 + |\xi|^s.$$

So,

$$\mathcal{W}_q^s(\mathbb{R}_q) = \left\{ f \in L_q^2(\mathbb{R}_q) : |\xi|^s \mathcal{F}_q(f) \in L_q^2(\mathbb{R}_q) \right\}.$$

Put,

$$I_{q,s} = \int_{-\infty}^{\infty} \frac{|1 - e(it;q^2)|^2}{|t|^{1+2s}} d_q t.$$

Since $s \in (0,1)$, then the relation (3) and the fact that

$$0 < \frac{|1 - e(it;q^2)|^2}{|t|^{1+2s}} \underset{t \to 0}{\sim} \frac{1}{|t|^{2s-1}}$$

imply that

$$0 < I_{q,s} < \infty.$$

Using the change of variables $t = \xi x$, we get

$$I_{q,s} = |\xi|^{-2s} \int_{-\infty}^{\infty} \frac{|1 - e(ix\xi;q^2)|^2}{|x|^{1+2s}} d_q x. \tag{24}$$

Now, let $f \in L_q^2(\mathbb{R}_q)$, then by the relation (24), we get for all $\xi \in \mathbb{R}_q$,

$$|\xi|^{2s} |\mathcal{F}_q(f)(\xi)|^2 = \frac{1}{I_{q,s}} \int_{-\infty}^{\infty} \frac{|\mathcal{F}_q(f)(\xi) - \mathcal{F}_q(f)(\xi) e(ix\xi;q^2)|^2}{|x|^{1+2s}} d_q x.$$

Then, form the relation (16), we deduce that

$$|\xi|^{2s} |\mathcal{F}_q(f)(\xi)|^2 = \frac{1}{I_{q,s}} \int_{-\infty}^{\infty} \frac{|\mathcal{F}_q(f - \tau_{q,x} f)(\xi)|^2}{|x|^{1+2s}} d_q x.$$

So, by q-integration, we obtain

$$\int_{-\infty}^{\infty} |\xi|^{2s} |\mathcal{F}_q(f)(\xi)|^2 d_q\xi = \frac{1}{I_{q,s}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\mathcal{F}_q(f - \tau_{q,x} f)(\xi)|^2}{|x|^{1+2s}} d_q x d_q \xi.$$

Hence, by Fubini's theorem and Plancherel formula, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} |\xi|^{2s} |\mathcal{F}_q(f)(\xi)|^2 d_q\xi &= \frac{1}{I_{q,s}} \int_{-\infty}^{\infty} \frac{1}{|x|^{1+2s}} \left(\int_{-\infty}^{\infty} |\mathcal{F}_q(f - \tau_{q,x} f)(\xi)|^2 d_q \xi \right) d_q x \\
&= \frac{1}{I_{q,s}} \int_{-\infty}^{\infty} \frac{1}{|x|^{1+2s}} \left(\int_{-\infty}^{\infty} |(f - \tau_{q,x} f)(\xi)|^2 d_q \xi \right) d_q x \\
&= \frac{1}{I_{q,s}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{\left|(f - \tau_{q,x} f)(\xi)\right|^2}{|x|^{1+2s}} d_q x \right) d_q \xi.
\end{aligned}$$

This leads to the desired result. □

Notation. For all s $\in \mathbb{R}$, we denote by $\left(\mathcal{W}_q^s(\mathbb{R}_q)\right)'$ the topological dual of $\mathcal{W}_q^s(\mathbb{R}_q)$.

Theorem 5.11. *Let $s \in \mathbb{R}$. Then, every tempered distribution $u \in \mathcal{W}_q^s(\mathbb{R}_q)$ extends uniquely to a continuous linear form L_u on $\left(\mathcal{W}_q^{-s}(\mathbb{R}_q), \|.\|_{\mathcal{W}_q^{-s}(\mathbb{R}_q)}\right)$.*

Proof. For all $\varphi \in \mathcal{S}_q(\mathbb{R}_q)$ and $u \in \mathcal{W}_q^s(\mathbb{R}_q)$, we have

$$\begin{aligned}\langle u, \varphi\rangle &= \left\langle \mathcal{F}_q(u), \mathcal{F}_q^{-1}(\varphi)\right\rangle \\ &= \int_{-\infty}^{\infty} \mathcal{F}_q(u)(\lambda)\mathcal{F}_q(\varphi)(-\lambda)d_q\lambda \\ &= \int_{-\infty}^{\infty} \left(1+|\lambda|^2\right)^{\frac{s}{2}} \mathcal{F}_q(u)(\lambda) \left(1+|\lambda|^2\right)^{-\frac{s}{2}} \mathcal{F}_q(\varphi)(-\lambda)d_q\lambda.\end{aligned}$$

By using Cauchy-Schwarz's inequality, we obtain for all $\varphi \in \mathcal{S}_q(\mathbb{R}_q)$

$$\begin{aligned}|\langle u, \varphi\rangle| &\le \left(\int_{-\infty}^{\infty} \left(1+|\lambda|^2\right)^{s} \left|\mathcal{F}_q(u)(\lambda)\right|^2 d_q\lambda\right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{-\infty}^{\infty} \left(1+|\lambda|^2\right)^{-s} \left|\mathcal{F}_q(\varphi)(\lambda)\right|^2 d_q\lambda\right)^{\frac{1}{2}} \\ &\le \|u\|_{\mathcal{W}_q^s(\mathbb{R}_q)} \|\varphi\|_{\mathcal{W}_q^{-s}(\mathbb{R}_q)}.\end{aligned}$$

Since $\mathcal{S}_q(\mathbb{R}_q)$ is a subspace of $\mathcal{W}_q^{-s}(\mathbb{R}_q)$, we deduce by the Hahn-Banach theorem [2] that u extends uniquely to a continuous linear form L_u on $\mathcal{W}_q^{-s}(\mathbb{R}_q)$. Moreover, we have

$$\|L_u\|_{\left(\mathcal{W}_q^{-s}(\mathbb{R}_q)\right)'} \le \|u\|_{\mathcal{W}_q^s(\mathbb{R}_q)}.$$

□

Theorem 5.12. *Let $s \in \mathbb{R}$. Then the map*

$$\begin{aligned}\chi : \mathcal{W}_q^{-s}(\mathbb{R}_q) &\longrightarrow \left(\mathcal{W}_q^s(\mathbb{R}_q)\right)' \\ u &\longmapsto L_u\end{aligned}$$

is an isometric isomorphism.

Proof. The linearity of χ is a direct consequence of the uniqueness of the extension of each $u \in \mathcal{W}_q^{-s}(\mathbb{R}_q)$ in a continuous linear form

$$L_u = \chi(u) \in \left(\mathcal{W}_q^s(\mathbb{R}_q)\right)'$$

It remains to show that χ is a bijective isometry.
Let $L \in \left(\mathcal{W}_q^s(\mathbb{R}_q)\right)'$ be a continuous linear form on $\mathcal{W}_q^s(\mathbb{R}_q)$, then by the Riesz theorem [2], there exists a unique $v \in \mathcal{W}_q^s(\mathbb{R}_q)$ such that

$$\|v\|_{\mathcal{W}_q^s(\mathbb{R}_q)} = \|L\|_{\left(\mathcal{W}_q^s(\mathbb{R}_q)\right)'} \tag{25}$$

and

$$\begin{aligned} \forall \phi \in \mathcal{W}_q^s(\mathbb{R}_q), L(\phi) &= \langle \phi, \upsilon \rangle_s \\ &= \int_{-\infty}^{\infty} \left(1+|\lambda|^2\right)^s \mathcal{F}_q(\phi)(\lambda)\overline{\mathcal{F}_q(\upsilon)(\lambda)} d_q\lambda \\ &= \left\langle \left(1+|\lambda|^2\right)^s \overline{\mathcal{F}_q(\upsilon)}, \mathcal{F}_q(\phi) \right\rangle. \end{aligned} \tag{26}$$

In particular, for all $\varphi \in \mathcal{S}_q(\mathbb{R}_q)$, we get

$$\begin{aligned} L(\varphi) &= \left\langle \left(1+|\lambda|^2\right)^s \overline{\mathcal{F}_q(\upsilon)}, \mathcal{F}_q(\varphi) \right\rangle \\ &= \left\langle \mathcal{F}_q\left(\left(1+|\lambda|^2\right)^s \overline{\mathcal{F}_q(\upsilon)}\right), \varphi \right\rangle \\ &= \langle u, \varphi \rangle, \end{aligned}$$

where

$$u = \mathcal{F}_q\left(\left(1+|\lambda|^2\right)^s \overline{\mathcal{F}_q(\upsilon)}\right).$$

Using (11), we get

$$\left(1+|\lambda|^2\right)^{\frac{-s}{2}} \mathcal{F}_q(\overline{u}) = \left(1+|\lambda|^2\right)^{\frac{s}{2}} \mathcal{F}_q(\upsilon).$$

So, since $v \in \mathcal{W}_q^s(\mathbb{R}_q)$, we deduce that $\overline{u} \in \mathcal{W}_q^{-s}(\mathbb{R}_q)$ and by using the relation (15), we get

$$\|\overline{u}\|_{\mathcal{W}_q^{-s}(\mathbb{R}_q)} = \|v\|_{\mathcal{W}_q^s(\mathbb{R}_q)} = \|L\|_{\left(\mathcal{W}_q^s(\mathbb{R}_q)\right)'}.$$

Hence, from Remark 5.2, we obtain $u \in \mathcal{W}_q^{-s}(\mathbb{R}_q)$ and

$$\|u\|_{\mathcal{W}_q^{-s}(\mathbb{R}_q)} = \|\overline{u}\|_{\mathcal{W}_q^{-s}(\mathbb{R}_q)} = \|L\|_{\left(\mathcal{W}_q^s(\mathbb{R}_q)\right)'}.$$

This proves that χ is effectively an isometric isomorphism from $\mathcal{W}_q^{-s}(\mathbb{R}_q)$ onto $\left(\mathcal{W}_q^s(\mathbb{R}_q)\right)'$. Its inverse is given by

$$\chi^{-1}(L) = \mathcal{F}_q\left(\left(1+|\lambda|^2\right)^s \overline{\mathcal{F}_q(\upsilon)}\right),$$

where υ is the unique q-tempered distribution in $\mathcal{W}_q^s(\mathbb{R}_q)$ satisfying the relation (15) and (26). □

6. Generalized Potential spaces

Definition 6.1. For $s \in \mathbb{R}$, we define the generalized q-potential of order s, as follows

$$\mathcal{P}_q^s(u) = (\mathcal{F}_q)^{-1}\left[(\lambda^2+1)^{-s/2}\mathcal{F}_q(u)(\lambda)\right], \quad u \in \mathcal{S}'_q(\mathbb{R}_q).$$

Lemma 6.2. *Let $f \in \mathcal{S}'_q(\mathbb{R}_q)$. Then*

$$\mathcal{P}_q^s\mathcal{P}_q^t(f) = \mathcal{P}_q^{s+t}(f), \quad s,t \in \mathbb{R}$$

and

$$\mathcal{P}_q^0(f) = f.$$

Proof. By definition,

$$(\mathcal{P}_q^t f)(x) = (\mathcal{F}_q)^{-1}\left[(\lambda^2+1)^{-t/2}\mathcal{F}_q(f)(\lambda)\right](x).$$

Then,

$$\begin{aligned}\mathcal{P}_q^s\mathcal{P}_q^t f(x) &= (\mathcal{F}_q)^{-1}\left[(\lambda^2+1)^{-s/2}(\lambda^2+1)^{-t/2}\mathcal{F}_q(f)(\lambda)\right](x)\\ &= (\mathcal{F}_q)^{-1}\left[(\lambda^2+1)^{-(s+t)/2}\mathcal{F}_q(f)(\lambda)\right](x)\\ &= (\mathcal{P}_q^{s+t})f(x)\end{aligned}$$

On the other hand, $\mathcal{P}_q^0 f(x) = (\mathcal{F}_q)^{-1}(\mathcal{F}_q(f))(x) = f(x)$. □

Remark 6.3. From the lemma, it is clear that for all $s \in \mathbb{R}$, $\mathcal{P}_q^s$ is bijective on $\mathcal{S}'_q(\mathbb{R}_q)$ and $(\mathcal{P}_q^s)^{-1} = \mathcal{P}_q^{-s}$.

Definition 6.4. For $s \in \mathbb{R}$, we define the generalized potential space as

$$\mathfrak{C}_q^s(\mathbb{R}_q) := \{\phi \in \mathcal{S}'_q(\mathbb{R}_q) : \mathcal{P}_q^{-s}(\phi) \in L_q^2(\mathbb{R}_q)\}.$$

The norm on $\mathfrak{C}_q^s(\mathbb{R}_q)$ is given by

$$\|\phi\|_{\mathfrak{C}_q^s(\mathbb{R}_q)} = \|\mathcal{P}_q^{-s}(\phi)\|_{2,q}.$$

Lemma 6.5. *The generalized q-potential $\mathcal{P}_q^t$ is an isometry of $\mathfrak{C}_q^s(\mathbb{R}_q)$ onto $\mathfrak{C}_q^{s+t}(\mathbb{R}_q)$, satisfying*

$$\|\mathcal{P}_q^t\phi\|_{\mathfrak{C}_q^{s+t}(\mathbb{R}_q)} = \|\phi\|_{\mathfrak{C}_q^s(\mathbb{R}_q)}, \quad \phi \in \mathfrak{C}_q^s(\mathbb{R}_q).$$

Proof. Let $\phi \in \mathfrak{C}_q^s(\mathbb{R}_q)$. By Definition 6.1 and Lemma 6.2, we have

$$\|\mathcal{P}_q^t \phi\|_{\mathfrak{C}_q^{s+t}(\mathbb{R}_q)} = \|\mathcal{P}_q^{-s-t}\mathcal{P}_q^t \phi\|_{2,q} = \|\mathcal{P}_q^{-s}\phi\|_{2,q} = \|\phi\|_{\mathfrak{C}_q^s(\mathbb{R}_q)}.$$

Now, let $f \in \mathfrak{C}_q^{s+t}(\mathbb{R}_q)$. Then, $\mathcal{P}_q^{-t} f \in \mathfrak{C}_q^s(\mathbb{R}_q)$ and $\mathcal{P}_q^t \mathcal{P}_q^{-t} f = f$. This achieves the proof. □

Proposition 6.6. *For $s \in \mathbb{R}$, $\mathfrak{C}_q^s(\mathbb{R}_q)$ is a Banach space.*

Proof. Let $(\phi_n)_n$ be a Cauchy sequence in $\mathfrak{C}_q^s(\mathbb{R}_q)$. By the definition of $\mathfrak{C}_q^s(\mathbb{R}_q)$ the sequence $\{\mathcal{P}_q^{-s}\phi_n\}$ is a Cauchy sequence in $L_q^2(\mathbb{R}_q)$. As $L_q^2(\mathbb{R}_q)$ is complete, it follows that there exists a function f in $L_q^2(\mathbb{R}_q)$ such that $\{\mathcal{P}_q^{-s}\phi_n\}$ converges to f in $L_q^2(\mathbb{R}_q)$. Thus, it is easy to see that $(\phi_n)_n$ converges to $\phi = \mathcal{P}_q^s(f)$ in $\mathfrak{C}_q^s(\mathbb{R}_q)$. □

Proposition 6.7. *For $s \in \mathbb{R}$, $\mathcal{S}_q(\mathbb{R}_q)$ is dense in $\mathfrak{C}_q^s(\mathbb{R}_q)$.*

Proof. Let $f \in \mathfrak{C}_q^s(\mathbb{R}_q)$. Then, $\mathcal{P}_q^{-s} f \in L_q^2(\mathbb{R}_q)$. Since $\mathcal{S}_q(\mathbb{R}_q)$ is dense in $L_q^2(\mathbb{R}_q)$, there exists a sequence $(\phi_j)_j$ in $\mathcal{S}_q(\mathbb{R}_q)$ such that

$$\phi_j \to \mathcal{P}_q^{-s} f \text{ in } L_q^2(\mathbb{R}_q). \tag{27}$$

From Theorem 4.3, we deduce that,

$$\mathcal{F}_q(\phi_j)(\lambda) \in \mathcal{S}_q(\mathbb{R}_q)$$

and then

$$(\lambda^2+1)^{-s/2}\mathcal{F}_q(\phi_j)(\lambda) \in \mathcal{S}_q(\mathbb{R}_q).$$

Now, define

$$g_j = \mathcal{P}_q^s \phi_j = (\mathcal{F}_q)^{-1}\left[(\lambda^2+1)^{-s/2}\mathcal{F}_q(\phi_j)(\lambda)\right] \quad j \in \mathbb{N}.$$

So, Theorem 4.3 leads to

$$g_j = (\mathcal{F}_q)^{-1}\left[(\lambda^2+1)^{-s/2}\mathcal{F}_q(\phi_j)(\lambda)\right] \in \mathcal{S}_q(\mathbb{R}_q), \quad j \in \mathbb{N}.$$

Hence, by (27), we obtain

$$\begin{aligned}\|f - g_j\|_{\mathfrak{C}_q^s(\mathbb{R}_q)} &= \left(\int_{-\infty}^{\infty} |\mathcal{P}_q^{-s} f(x) - \mathcal{P}_q^{-s} g_j(x)|^2 d_q x\right)^{1/2}\\ &= \left(\int_{-\infty}^{\infty} |\mathcal{P}_q^{-s} f(x) - \phi_j(x)|^2 d_q x\right)^{1/2} \to 0, \quad \text{as } j \to \infty.\end{aligned}$$

□

Proposition 6.8. *For $s>1$, $\mathcal{P}_q^{-s}$ maps $L_q^2(\mathbb{R}_q)$ into $L_q^2(\mathbb{R}_q)$. More precisely there exists $g\in L_q^2(\mathbb{R}_q)\cap L_q^\infty(\mathbb{R}_q)$ such that for all $f\in L_q^2(\mathbb{R}_q)$, we have*

$$\mathcal{P}_q^{-s}(f)=f*_q g$$

and there exists a positive constant C such that

$$\|\mathcal{P}_q^{-s}f\|_{2,q}\leq C\|f\|_{2,q}.$$

Proof. As $s>1$, the function $\lambda\mapsto(1+\lambda^2)^{-\frac{s}{2}}$ belongs to $L_q^2(\mathbb{R}_q)\cap L_q^\infty(\mathbb{R}_q)$. Then, using the inversion theorem for the q-Rubin-Fourier transform, we deduce that there exists a function $g\in L_q^2(\mathbb{R}_q)$, such that

$$(1+\lambda^2)^{-\frac{s}{2}}=\mathcal{F}_q(g)(\lambda).$$

But $\mathcal{F}_q(g)\in L_q^\infty(\mathbb{R}_q)$, then for all $f\in L_q^2(\mathbb{R}_q)$, $\mathcal{F}_q(g)\mathcal{F}_q(f)\in L_q^2(\mathbb{R}_q)$.
So, for all $f\in L_q^2(\mathbb{R}_q)$, we have $g*_q f\in L_q^2(\mathbb{R}_q)$ and

$$\mathcal{F}_q(g*_q f)(\lambda)=\mathcal{F}_q(g)(\lambda)\mathcal{F}_q(f)(\lambda)=(1+\lambda^2)^{-\frac{s}{2}}\mathcal{F}_q(f)(\lambda).$$

On the other hand, we have

$$\mathcal{F}_q(\mathcal{P}_q^{-s}f)(\lambda)=(1+\lambda^2)^{-\frac{s}{2}}\mathcal{F}_q(f)(\lambda).$$

We conclude by using Proposition 4.8 that

$$\mathcal{P}_q^{-s}f=f*_q g.$$

Finally, applying the Plancherel formula, we obtain

$$\begin{aligned}\|\mathcal{P}_q^{-s}f\|_{2,q}&=\|\mathcal{F}_q(\mathcal{P}_q^{-s}f)\|_{2,q}=\|\mathcal{F}_q(g*_q f)\|_{2,q}=\|\mathcal{F}_q(g)\mathcal{F}_q(f)\|_{2,q}\\&\leq\|\mathcal{F}_q(g)\|_{\infty,q}\|\mathcal{F}_q(f)\|_{2,q}=\|\mathcal{F}_q(g)\|_{\infty,q}\|f\|_{2,q}.\end{aligned}$$

This completes the proof of the proposition. □

Proposition 6.9. *Let $s,t\in\mathbb{R}$, such that $t>1+s$. Then, we have*

$$\mathfrak{C}_q^s(\mathbb{R}_q)\subset\mathfrak{C}_q^t(\mathbb{R}_q).$$

Moreover, there exits a positive constant C, such that for all $u\in\mathfrak{C}_q^s(\mathbb{R}_q)$

$$\|u\|_{\mathfrak{C}_q^t(\mathbb{R}_q)}\leq C\|u\|_{\mathfrak{C}_q^s(\mathbb{R}_q)}.$$

Proof. Let $u \in \mathfrak{C}^s_q(\mathbb{R}_q)$. Then, we have $\mathcal{P}^{-s}_q(u) = f \in L^2_q(\mathbb{R}_q)$. From Lemma 6.5 and Proposition 6.8, we can write

$$\mathcal{P}^{-t}_q(u) = \mathcal{P}^{-t+s}_q\left(\mathcal{P}^{-s}_q(u)\right) = \mathcal{P}^{-t+s}_q(f) = f *_q g \in L^2_q(\mathbb{R}_q),$$

where g is such that

$$(1+\lambda^2)^{-\frac{t-s}{2}} = \mathcal{F}_q(g).$$

So $u \in \mathfrak{C}^t_q(\mathbb{R}_q)$. Furthermore, we have

$$\|u\|_{\mathfrak{C}^t_q(\mathbb{R}_q)} = \|f *_q g\|_{2,q} \leq C\|f\|_{2,q} = C\|u\|_{\mathfrak{C}^s_q(\mathbb{R}_q)}.$$

□

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