Filomat 27:7 (2013), 1183–1196
DOI 10.2298/FIL1307183F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Soft relations applied to semigroups

Feng Feng[a], Muhammad Irfan Ali[b], Muhammad Shabir[c]

[a]*Department of Applied Mathematics, Xi'an University of Posts and Telecommunications, Xi'an 710121, China*
[b]*Department of Mathematics, Islamabad Model College for Boys F-7/3, Islamabad, Pakistan*
[c]*Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan*

Abstract. Binary relations, in particular, equivalence relations play an important role in both mathematics and information sciences. The concept of soft sets was initiated by Molodtsov as a general mathematical framework for dealing with uncertainty. The present paper establishes a possible connection between binary relations and soft sets. The concept of soft binary relations is introduced and some related properties are investigated. It is shown that any fuzzy relation may be considered as a soft binary relation. Moreover, we discuss the application of soft binary relations in semigroup theory. We consider soft congruence relations over semigroups and show that all soft congruence relations over a semigroup with a fixed parameter set form a lattice. Finally, the notion of soft homomorphisms is presented and isomorphism theorems for soft semigroups are established based on soft congruence relations.

1. Introduction

In the real-world, there are many problems where uncertainty is a part of the data associated with them. This type of problems are mostly related to economics, engineering, medicine, environment and social sciences. It is very difficult to address these problems by using classical mathematical tools, because such tools are designed for certain situations. There are three major theories dealing with uncertainty viz. theory of probability, theory of fuzzy sets and interval mathematics. But these theories have their own difficulties. There are other mathematical tools available which deal with uncertainty, such as intuitionistic fuzzy sets, vague sets, and rough sets but these theories also have difficulties as mentioned by Maji et al. [24]. It has been pointed out in [24, 26] that a reason for these difficulties is the inadequacy of parametrization tools. In order to overcome these difficulties, Molodtsov [26] introduced the concept of soft set, which is free from the difficulties affecting the above mentioned theories. Theory of soft sets has been successfully applied to decision making under uncertainty [7, 11, 12, 25]. Soft set theory is also closely related to many other soft computing models including rough sets and fuzzy sets. Ali et al. [2] discussed the fuzzy sets and fuzzy soft sets induced by soft sets. Feng et al. [13, 14] combined soft sets with rough sets and fuzzy sets, obtaining three types of hybrid models: rough soft sets, soft rough sets, and soft-rough fuzzy sets. Application of soft set theory in algebraic structures such as groups was initiated by Aktaş and Çağman

2010 *Mathematics Subject Classification*. Primary 03E72; Secondary 91B06, 90B50, 06D72
Keywords. Soft set, Soft binary relation, Soft homomorphism, Semigroup, Soft congruence relation, Similarity relation
Received: 11 October 2011; Accepted: 29 July 2012
Communicated by Dragan S. Djordjević
This work was partially supported by the National Natural Science Foundation of China (Program No. 11301415), the Natural Science Basic Research Plan in Shaanxi Province of China (Program No. 2013JQ1020) and the Scientific Research Program Funded by Shaanxi Provincial Education Department of China (Program No. 2013JK1098).
Email addresses: fengnix@hotmail.com (Feng Feng), mirfanali13@yahoo.com (Muhammad Irfan Ali)

[5]. Jun and his colleagues applied it to BCK/BCI algebras [17, 18]. Feng et al. [10] applied soft set theory to the study of semirings and initiated the notion of soft semirings. Following Maji et al. [24], Ali, Feng and their colleagues defined some new operations in soft set theory [1]. A general formula for introducing binary operations for soft sets is available in [13]. Qin and Hong [29] have studied lattices of soft sets with respect to the new operations in [1]. Algebraic structures of soft sets associated with these new operations were further explored in [1].

It is well known that binary relations, especially equivalence relations, play important role in mathematics, computer science, artificial intelligence, decision making and classification. In particular, basic building blocks of Pawlak's rough set model [28] are equivalence classes which originates from equivalence relations [14]. The theory of rough sets is a powerful mathematical approach to deal with inexact, uncertain or vague knowledge. It has been successfully applied to various fields of artificial intelligence such as pattern recognition, machine learning, and automated knowledge acquisition. Using the primitive notion of a pair of upper and lower approximation operators, knowledge hidden in information systems may be discovered and expressed in the form of decision rules [13, 14, 28].

On the other hand, the concept of fuzzy equivalence relations, which is the fuzzy generalization of crisp equivalence relations, was introduced by Zadeh [33, 34]. Due to the great importance of the concept, many authors contributed in this direction. Murali explored the lattice of fuzzy equivalence relations in [27]. Kuroki [22] and Tan [31] discussed fuzzy congruences in semigroups. Fuzzy congruences in groups were studied by Kim and Bae [19]. Ajmal and Thomas [4] introduced the concept of t-fuzzy congruences. Lattice of fuzzy congruences in inverse semigroups was studied by Das in [8]. K.C. Gupta and R. K. Gupta introduced the concept of t-equivalence relations in [15]. Application of fuzzy congruences in the study of rough algebraic structures can be found in [21, 32]. This list of authors on fuzzy relations is by no means complete but give us a slight idea about the importance of fuzzy equivalence relations. In this paper, we initiate an extension of fuzzy binary relations based on the theory of soft sets, which will be called soft binary relations. Some interesting properties of soft equivalence and soft congruence relations are discussed. Moreover, we apply soft binary relations to the study of semigroups, and obtain that soft congruence relations over a semigroup with a fixed set of parameters form a lattice. Finally, we introduce soft homomorphisms and prove the isomorphism theorems for soft semigroups.

2. Preliminaries

We assume that the reader is familiar with the rudiments of binary relations and semigroup theory. For the terms not defined here we refer to [16]. Let us first introduce the notion of soft sets which is a newly-emerging mathematical approach to vagueness.

Let U be an initial universe of objects and E_U (simply denoted by E) the set of parameters in relation to the objects in U. By parameters we usually mean attributes, characteristics, or properties of the objects in U. Let $P(U)$ denote the power set of U.

Definition 2.1. ([26]) A pair $S = (F, A)$ is called a *soft set* over U, where $A \subseteq E$ and $F : A \to P(U)$ is a set-valued mapping.

Roughly speaking we can say that soft sets are crisp sets determined by parameters. In other words, a soft set over U is a parameterized family of subsets of the universe U. The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable in practice. We may use any suitable parametrization—with the help of words and sentences, real numbers, functions, mappings, etc. The set of all soft sets over U is denoted by $\mathfrak{S}(U)$.

Definition 2.2. ([13]) Let (F, A) and (G, B) be soft sets over U. Then (G, B) is called a *soft subset* of (F, A) if $B \subseteq A$ and $G(b) \subseteq F(b)$ for all $b \in B$.

Two soft sets (F, A) and (G, B) over U are said to be *soft equal* if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.3. Let S be a semigroup and let (F, A) be a soft set over S. Then (F, A) is called a *soft semigroup* over S if $F(\alpha)$ is a subsemigroup of S for all $\alpha \in A$ with $F(\alpha) \neq \emptyset$.

Definition 2.4. A soft set (F, A) over a semigroup S is called a *soft ideal* (resp. *soft left ideal, soft right ideal*) over S, if $F(\alpha)$ is an ideal (resp. left ideal, right ideal) of S for all $\alpha \in A$ with $F(\alpha) \neq \emptyset$.

Recall that the product of two soft sets over a universe U with a binary operation $*$ is defined by Molodstov [26] as follows.

Definition 2.5. Let (F, A) and (G, B) be two soft sets over U, then the operation $*$ for soft sets is defined as $(F, A) * (G, B) = (H, A \times B)$, where $H(a, b) = F(a) * G(b), a \in A, b \in B$, and $A \times B$ is the Cartesian product of the sets A and B.

If there does not arise any ambiguity then we can simply write $(F, A)(G, B)$ instead of $(F, A) * (G, B)$ and $F(a) G(b)$ for $F(a) * G(b)$. Note that Molodstov's idea on soft products was further developed by Maji et al. [24] as follows.

Definition 2.6. Let (F, A) and (G, B) be any two soft sets over U.

(1) The $\wedge$-*product* (called AND-product) of (F, A) and (G, B) is defined as the soft set $(H, C) = (F, A) \wedge (G, B)$, where $C = A \times B$, and $H(a, b) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$.
(2) The $\vee$-*product* (called OR-product) of (F, A) and (G, B) is defined as the soft set $(H, C) = (F, A) \vee (G, B)$, where $C = A \times B$, and $H(a, b) = F(a) \cup G(b)$ for all $(a, b) \in A \times B$.

Definition 2.7. ([1]) Let (F, A) and (G, B) be any two soft sets over U.

(1) The *extended union* of (F, A) and (G, B) is defined as the the soft set $(H, C) = (F, A) \cup_{\mathcal{E}} (G, B)$ where $C = A \cup B$ and for all $c \in C$,

$$H(c) = \begin{cases} F(c), & \text{if } c \in A \backslash B, \\ G(c), & \text{if } c \in B \backslash A, \\ F(c) \cup G(c), & \text{if } c \in A \cap B. \end{cases}$$

(2) The *extended intersection* of (F, A) and (G, B) is defined as the the soft set $(H, C) = (F, A) \cap_{\mathcal{E}} (G, B)$ where $C = A \cup B$ and for all $c \in C$,

$$H(c) = \begin{cases} F(c), & \text{if } c \in A \backslash B, \\ G(c), & \text{if } c \in B \backslash A, \\ F(c) \cap G(c), & \text{if } c \in A \cap B. \end{cases}$$

(3) The *restricted intersection* of (F, A) and (G, B) is defined as the soft set $(H, C) = (F, A) \cap_{\mathcal{R}} (G, B)$, where $C = A \cap B \neq \emptyset$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$.
(4) The *restricted union* of (F, A) and (G, B) is defined as the soft set $(H, C) = (F, A) \cup_{\mathcal{R}} (G, B)$, where $C = A \cap B \neq \emptyset$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$.

Aktaş and Çağman [5] defined the concept of Cartesian products for soft groups. Here we introduce a similar notion as follows.

Definition 2.8. Let (F, A) and (G, B) be two soft sets over a semigroup S. Their *Cartesian product* is defined as $(F, A) \times (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \times G(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 2.9. Let (F, A) and (G, B) be two soft sets over a semigroup $(S, *)$ with $C = A \cap B \neq \emptyset$. Their *restricted product* is defined as $(F, A) \widehat{*} (G, B) = (H, C)$, where $H(c) = F(c) * G(c)$ for all $c \in C$.

3. Soft binary relations

In this section we define the notion of soft binary relations over a set X. Some basic concepts, operations, characterizations and related properties with regard to soft binary relations are proposed here.

Definition 3.1. Let (σ, A) be a soft set over $X\times X$, then (σ, A) is called a *soft binary relation* over X.

In fact (σ, A) is a parameterized collection of binary relations over X. That is, we have a binary relation $\sigma(\alpha)$ on X for each parameter $\alpha \in A$. In what follows, we shall denote the collection of all soft binary relations over X by $\mathfrak{S}_{Br}(X)$.

Definition 3.2. Let (σ, A) and (ρ, B) be two soft binary relations over X. Then their *composition* is defined as the soft set $(\delta, C) = (\sigma, A) \circ (\rho, B)$, where $C = A \times B$ and $\delta(\alpha, \beta) = \sigma(\alpha) \diamond \rho(\beta)$ for all $(\alpha, \beta) \in C$.

Note that $\sigma(\alpha) \diamond \rho(\beta)$ denotes the ordinary composition of binary relations on X. Specifically, we have

$$\sigma(\alpha) \diamond \rho(\beta) = \{(x, y) \in X \times X : \exists z \in X, (x, z) \in \sigma(\alpha) \text{ and } (z, y) \in \rho(\beta)\}.$$

Definition 3.3. Let (σ, A) and (ρ, B) be two soft binary relations over X such that $A \cap B \neq \emptyset$. Then their *restricted composition* is defined as the soft set $(\delta, C) = (\sigma, A) \widehat{\circ} (\rho, B)$, where $C = A \cap B$ and $\delta(\gamma) = \sigma(\gamma) \diamond \rho(\gamma)$ for all $\gamma \in C$.

It is well known that the composition of binary relations on a set is associative; hence we can show that if $A \cap B \cap C \neq \emptyset$, then for soft binary relations (σ, A), (ρ, B) and (δ, C) we have

$$(\sigma, A) \widehat{\circ} [(\rho, B) \widehat{\circ} (\delta, C)] = [(\sigma, A) \widehat{\circ} (\rho, B)] \widehat{\circ} (\delta, C).$$

Definition 3.4. A soft binary relation (σ, A) over a set X is called a *soft reflexive relation* over X if $\sigma(\alpha)$ is a reflexive relation on X for all $\alpha \in A$ with $\sigma(\alpha) \neq \emptyset$.

If (σ, A) is a soft binary relation over a set X, then the *converse soft binary relation* of (σ, A) is also a soft binary relations over X, denoted by $(\sigma, A)^{-1}$. Concretely, $(\sigma, A)^{-1} = (\rho, A)$ is a soft set over X, where $\rho(\alpha) = \sigma^{-1}(\alpha)$ and $\sigma^{-1}(\alpha) = \{(x, y) \in X \times X : (y, x) \in \sigma(\alpha)\}$, i.e. the converse of $\sigma(\alpha)$ for all $\alpha \in A$.

Definition 3.5. A soft binary relation (σ, A) over X is called a *soft symmetric relation* over X if $(\sigma, A)^{-1} = \left(\sigma^{-1}, A\right)$.

Lemma 3.6. *Let (σ, A) be a soft binary relation over a set X. Then (σ, A) is soft symmetric if and only if $\sigma(\alpha) \neq \emptyset$ is a symmetric relation on X for all $\alpha \in A$.*

Proof. Straightforward. □

Definition 3.7. Let (σ, A) be a soft binary relation over a set X. Then (σ, A) is called a *soft transitive relation* over X if $(\sigma, A) \widehat{\circ} (\sigma, A) \subseteq (\sigma, A)$.

Lemma 3.8. *Let (σ, A) be a soft binary relation over a set X. Then (σ, A) is a soft transitive relation over X if and only if $\sigma(\alpha) \neq \emptyset$ is a transitive relation on X for all $\alpha \in A$.*

Proof. Let (σ, A) be a soft transitive relation over X. For any $\alpha \in A$ such that $\sigma(\alpha) \neq \emptyset$, if $(a, b) \in \sigma(\alpha)$ and $(b, c) \in \sigma(\alpha)$ then $(a, c) \in \sigma(\alpha) \diamond \sigma(\alpha) \subseteq \sigma(\alpha)$. This just shows that $\sigma(\alpha)$ is transitive for all $\alpha \in A$.

Conversely, assume that $\sigma(\alpha) \neq \emptyset$ is a transitive relation on X for all $\alpha \in A$. Let $(\sigma, A) \widehat{\circ} (\sigma, A) = (\rho, A)$. By definition, we have $\rho(\alpha) = \sigma(\alpha) \diamond \sigma(\alpha)$ for all $\alpha \in A$. If $\sigma(\alpha) \neq \emptyset$, then we have $\rho(\alpha) \subseteq \sigma(\alpha)$ since by hypothesis $\sigma(\alpha) \neq \emptyset$ is a transitive relation on X. On the other hand, if $\sigma(\alpha) = \emptyset$, then clearly we deduce that $\rho(\alpha) = \sigma(\alpha) = \emptyset$. Therefore, we conclude that $(\rho, A) = (\sigma, A) \widehat{\circ} (\sigma, A) \subseteq (\sigma, A)$. □

A soft binary relation (σ, A) over a set X is called a *soft equivalence relation* over X if it is soft reflexive, soft symmetric and soft transitive. In what follows, the collection of all soft equivalence relations over a set X will be denoted by $\mathfrak{S}_{Eq}(X)$.

As an immediate consequence of Definition 3.4, Lemma 3.6 and Lemma 3.8, we have the following:

Corollary 3.9. *A soft binary relation (σ, A) over a set X is a soft equivalence relation over X if and only if $\sigma(\alpha) \neq \emptyset$ is an equivalence relation on X for all $\alpha \in A$.*

It is well known that each equivalence relation on a set divides the set into disjoint equivalence classes and a partition of the set provides us an equivalence relation on the set. Therefore a soft equivalence relation over X provides us a parameterized collection of partitions of X. Let $[x]_{\sigma(\alpha)}$ denotes the equivalence class containing $x \in X$ determined by $\sigma(\alpha)$ for $\alpha \in A$. Then it is clear that $y \in [x]_{\sigma(\alpha)}$ if and only if $(x, y) \in \sigma(\alpha)$.

For soft binary relations, we can introduce some concepts which are similar to what have been defined for soft sets as mentioned above.

Definition 3.10. Let (σ, A) and (ρ, B) be two soft binary relations over a set X. Then

(1) $(\sigma, A) \subseteq (\rho, B) \Leftrightarrow A \subseteq B$ and $\sigma(\alpha) \subseteq \rho(\alpha)$ for all $\alpha \in A$.
(2) $(\sigma, A) \cap_{\mathcal{R}} (\rho, B) = (\delta, C)$, where $C = A \cap B \neq \emptyset$ and $\delta(\gamma) = \sigma(\gamma) \cap \rho(\gamma)$ for all $\gamma \in C$.
(3) $(\sigma, A) \cup_{\mathcal{R}} (\rho, B) = (\delta, C)$, where $C = A \cap B \neq \emptyset$ and $\delta(\gamma) = \sigma(\gamma) \cup \rho(\gamma)$ for all $\gamma \in C$.
(4) $(\sigma, A) \wedge (\rho, B) = (\delta, C)$, where $C = A \times B$ and $\delta(\alpha, \beta) = \sigma(\alpha) \cap \rho(\beta)$ for all $(\alpha, \beta) \in C$.
(5) $(\sigma, A) \vee (\rho, B) = (\delta, C)$, where $C = A \times B$ and $\delta(\alpha, \beta) = \sigma(\alpha) \cup \rho(\beta)$ for all $(\alpha, \beta) \in C$.
(6) $(\sigma, A) \cup_{\mathcal{E}} (\rho, B) = (\delta, C)$, where $C = A \cup B$ and $\forall \alpha \in C$, $\delta(\alpha) = \begin{cases} \sigma(\alpha), & \text{if } \alpha \in A \backslash B, \\ \rho(\alpha), & \text{if } \alpha \in B \backslash A, \\ \sigma(\alpha) \cup \rho(\alpha), & \text{if } \alpha \in A \cap B. \end{cases}$
(7) $(\sigma, A) \cap_{\mathcal{E}} (\rho, B) = (\delta, C)$, where $C = A \cup B$ and $\forall \alpha \in C$, $\delta(\alpha) = \begin{cases} \sigma(\alpha), & \text{if } \alpha \in A \backslash B, \\ \rho(\alpha), & \text{if } \alpha \in B \backslash A, \\ \sigma(\alpha) \cap \rho(\alpha), & \text{if } \alpha \in A \cap B. \end{cases}$

The following lemma shows that every soft equivalence relation over a set is idempotent with respect to the restricted composition of soft binary relations.

Lemma 3.11. *If (σ, A) is a soft equivalence relation over X, then $(\sigma, A) \widehat{\circ} (\sigma, A) = (\sigma, A)$.*

Proof. Let $(\sigma, A) \widehat{\circ} (\sigma, A) = (\delta, A)$, where $\delta(\alpha) = \sigma(\alpha) \diamond \sigma(\alpha)$ for all $\alpha \in A$. Assume that (σ, A) is a soft equivalence relation over X. It is easy to see that $(\delta, A) = (\sigma, A) \widehat{\circ} (\sigma, A) \subseteq (\sigma, A)$ since (σ, A) is a soft transitive relation over X.

To show the reverse inclusion, note first that if $\sigma(\alpha) = \emptyset$, then $\delta(\alpha) = \sigma(\alpha) \diamond \sigma(\alpha) = \emptyset$. Next, we assume that $\sigma(\alpha) \neq \emptyset$. Then it follows that $\sigma(\alpha)$ is an equivalence relation on X. For any $(b, c) \in \sigma(\alpha)$, we have $(c, c) \in \sigma(\alpha)$ since $\sigma(\alpha) \neq \emptyset$ is reflexive. This implies $(b, c) \in \sigma(\alpha) \diamond \sigma(\alpha) = \delta(\alpha)$. Hence we deduce that $\sigma(\alpha) \subseteq \delta(\alpha)$ for all $\alpha \in A$. Therefore $(\sigma, A) \subseteq (\sigma, A) \widehat{\circ} (\sigma, A) = (\delta, A)$. □

Proposition 3.12. *If $\{(\sigma_i, A) : i \in I\}$ is a non-empty family of soft equivalence relations over a set X, then the following soft binary relations are soft equivalence relations over X:*

(a) $\bigcap_{\mathcal{R}} \{(\sigma_i, A) : i \in I\}$;
(b) $\bigcap_{\mathcal{E}} \{(\sigma_i, A) : i \in I\}$;
(c) $\bigwedge_{i \in I} (\sigma_i, A)$.

Proof. (a) Let $(\eta, A) = \bigcap_{\mathcal{R}} \{(\sigma_i, A) : i \in I\}$. By definition, we have $\eta(\alpha) = \bigcap_{i \in I} \sigma_i(\alpha)$ for all $\alpha \in A$. If $\eta(\alpha) \neq \emptyset$, then it is obvious that $\sigma_i(\alpha) \neq \emptyset$ for all $i \in I$. Then by Corollary 3.9, $\sigma_i(\alpha)$ is an equivalence relation on X for all $i \in I$. Hence it follows that $\eta(\alpha) = \bigcap_{i \in I} \sigma_i(\alpha)$ is also an equivalence relation on X. Using Corollary 3.9 again, we conclude that $(\eta, A) = \bigcap_{\mathcal{R}} \{(\sigma_i, A) : i \in I\}$ is a soft equivalence relation over X as required.

(b) In this case, it is obvious that $\bigcap_{\mathcal{E}} \{(\sigma_i, A) : i \in I\} = \bigcap_{\mathcal{R}} \{(\sigma_i, A) : i \in I\}$.

(c) The proof of (c) is similar to that of (a) and so omitted here. □

Definition 3.13. Let (σ, A) be a soft binary relation over a set X and let $\{(\rho, B) \in \mathfrak{S}_{Eq}(X) : (\sigma, A) \subseteq (\rho, B)\}$ be the family of all soft equivalence relations over X containing (σ, A). Then $(\sigma, A)^e = \bigcap_{\mathcal{R}}\{(\rho, B) \in \mathfrak{S}_{Eq}(X) : (\sigma, A) \subseteq (\rho, B)\}$ is called the *soft equivalence relation generated by* (σ, A).

It is easy to verify that $(\sigma, A)^e$ is the smallest soft equivalence relation over X containing (σ, A). We know that the concept of transitive closure of binary relations plays an important role in the study of equivalence relations. So we introduce the concept of soft transitive closure of a soft binary relation over a set X.

Definition 3.14. Let (σ, A) be a soft binary relation over X. Then the *soft transitive closure* of (σ, A) is defined as $(\sigma^\infty, A) = \bigcup_{n=1}^{\infty} (\sigma, A)^n$ where $(\sigma, A)^n = (\sigma, A)\widehat{\circ}(\sigma, A)\widehat{\circ}\cdots\widehat{\circ}(\sigma, A) = (\sigma^n, A)$ and $\sigma^n(\alpha) = \sigma(\alpha)\diamond\sigma(\alpha)\diamond\cdots\diamond\sigma(\alpha)$ (both contain n factors) for all $\alpha \in A$.

It is worth noting that the notation $\cup$ in the above definition could be interpreted as $\cup_{\mathcal{R}}$ or $\cup_{\mathcal{E}}$ since in this particular case the two soft union operations will always lead to the same results.

Proposition 3.15. *Let (σ, A) be a soft binary relation over X. Then (σ^∞, A) is the smallest soft transitive relation over X containing (σ, A).*

Proof. It is clear that (σ, A) is contained in (σ^∞, A) since $(\sigma, A) = (\sigma, A)^1 \subseteq (\sigma^\infty, A)$. Now let $\alpha \in A$ such that $\sigma^\infty(\alpha) \neq \emptyset$ and let $(x, y), (y, z) \in \sigma^\infty(\alpha) = \bigcup_{n=1}^{\infty} \sigma^n(\alpha)$. Then there exist some $m, n \in N$, such that $(x, y) \in \sigma^n(\alpha), (y, z) \in \sigma^m(\alpha)$. It follows that $(x, z) \in \sigma^{m+n}(\alpha) \subseteq \sigma^\infty(\alpha)$; hence $\sigma^\infty(\alpha) \neq \emptyset$ is transitive. By Lemma 3.8, we deduce that (σ^∞, A) is a soft transitive relation over X containing (σ, A).

Now assume that (ρ, A) is a soft transitive relation over X containing (σ, A). Then one can observe that

$$(\sigma, A)^2 = (\sigma, A)\widehat{\circ}(\sigma, A) \subseteq (\rho, A)\widehat{\circ}(\rho, A) \subseteq (\rho, A),$$

and more generally we obtain $(\sigma, A)^n \subseteq (\rho, A)$ for all $n \in \mathbb{N}$. Therefore $(\sigma^\infty, A) \subseteq (\rho, A)$. □

Proposition 3.16. *If (σ, A) is a soft symmetric relation over X, then so is its soft transitive closure (σ^∞, A).*

Proof. Suppose that (σ, A) is a soft symmetric relation over X. Let $\alpha \in A$ be such that $\sigma^\infty(\alpha) \neq \emptyset$. Then there exist $x, y \in X$ and some $n \in N$ with $(x, y) \in \sigma^n(\alpha) \neq \emptyset$. Thus we can find a sequence

$$x = z_1 \to z_2 \to \cdots \to z_{n-1} \to z_n = y,$$

where $(z_i, z_{i+1}) \in \sigma(\alpha)$ for $i = 1, 2, \cdots, n-1$. Since $\sigma(\alpha) \neq \emptyset$, by Lemma 3.6 we have $\sigma(\alpha)$ is a symmetric relation on X. It follows that $(z_{i+1}, z_i) \in \sigma(\alpha)$ for $i = 1, 2, \cdots, n-1$. Hence we can deduce that $(y, x) \in \sigma^n(\alpha)$. This shows that $\sigma^\infty(\alpha) \neq \emptyset$ is a symmetric relation on X. Therefore, (σ^∞, A) is a soft symmetric relation over X by Lemma 3.6. □

Let (σ, A) be a soft binary relation over X and let 1_X denote the *identity relation* on X. We can define a soft binary relation (σ^{1_X}, A) over X such that for any $\alpha \in A$,

$$\sigma^{1_X}(\alpha) = \begin{cases} 1_X, & \text{if } \sigma(\alpha) \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We refer to (σ^{1_X}, A) as the *pseudo-equality soft binary relation* over X corresponding to (σ, A).

From the above definition, we can verify that (σ, A) is a soft reflexive relation over X if and only if $(\sigma^{1_X}, A) \subseteq (\sigma, A)$.

Theorem 3.17. *Let (σ, A) be a soft binary relation over X. Then we have $(\sigma, A)^e = \left[(\sigma, A) \cup_{\mathcal{E}} (\sigma, A)^{-1} \cup_{\mathcal{E}} \left(\sigma^{1_X}, A\right)\right]^\infty$.*

Proof. Let $(\sigma, A) \cup_{\mathcal{E}} (\sigma, A)^{-1} \cup_{\mathcal{E}} \left(\sigma^{1_X}, A\right) = (\delta, A)$. Note first that by Proposition 3.15, $(\delta, A)^\infty$ is a soft transitive relation over X containing (δ, A). If $\delta(\alpha) \neq \emptyset$ then we have $\sigma(\alpha) \neq \emptyset$. Otherwise, $\sigma(\alpha) = \emptyset$ implies that $\sigma^{-1}(\alpha) = \sigma^{1_X}(\alpha) = \emptyset$, and so $\delta(\alpha) = \sigma(\alpha) \cup \sigma^{-1}(\alpha) \cup \sigma^{1_X}(\alpha) = \emptyset$, which is a contradiction. Since $\sigma(\alpha) \neq \emptyset$, we deuce that $\sigma^{1_X}(\alpha) = 1_X \subseteq \delta(\alpha)$, which shows that $\delta(\alpha) \neq \emptyset$ is a reflexive relation on X. Thus (δ, A) is a soft reflexive relation over X. Next, let us assume that $\delta^{\infty}(\alpha) = \bigcup_{n=1}^{\infty} \sigma^n(\alpha) \neq \emptyset$. It follows that $\delta^n(\alpha) \neq \emptyset$ for some $n \in N$. If $\delta(\alpha) = \emptyset$, then $\delta^n(\alpha) = \emptyset$ for all $n \in N$; this leads to a contradiction. Hence $\delta(\alpha) \neq \emptyset$ and thus as shown above, we have that $\delta(\alpha)$ is a reflexive relation on X. Therefore, $1_X \subseteq \delta(\alpha) \subseteq \delta^{\infty}(\alpha)$. This shows that $\delta^{\infty}(\alpha) \neq \emptyset$ is a reflexive relation on X. Hence $(\delta, A)^{\infty}$ is a soft reflexive relation over X. In addition, one can observe that $(\delta, A) = (\sigma, A) \cup_{\mathcal{E}} (\sigma, A)^{-1} \cup_{\mathcal{E}} \left(\sigma^{1_X}, A\right)$ is a soft symmetric relation over X. By Proposition 3.16 it follows that the soft transitive closure $(\delta, A)^{\infty}$ is also a soft symmetric relation over X. Hence we conclude that $(\delta, A)^{\infty}$ is a soft equivalence relation over X.

Now, let (ρ, A) be a soft equivalence relation over X such that $(\sigma, A) \subseteq (\rho, A)$. Then it is easy to see that $(\sigma, A)^{-1} \subseteq (\rho, A)^{-1} = (\rho, A)$. For any $\alpha \in A$, if $\rho(\alpha) = \emptyset$ then $\sigma(\alpha) = \emptyset$ since $\sigma(\alpha) \subseteq \rho(\alpha)$. In this case, we have $\sigma^{1_X}(\alpha) = \emptyset = \sigma(\alpha) \subseteq \rho(\alpha)$. On the other hand, if $\rho(\alpha) \neq \emptyset$ then $\rho(\alpha)$ is an equivalence relation over X by Corollary 3.9. Thus we also have $\sigma^{1_X}(\alpha) \subseteq 1_X \subseteq \rho(\alpha)$. Therefore, $(\sigma^{1_X}, A) \subseteq (\rho, A)$, and so we deduce that $(\delta, A) = (\sigma, A) \cup_{\mathcal{E}} (\sigma, A)^{-1} \cup_{\mathcal{E}} \left(\sigma^{1_X}, A\right) \subseteq (\rho, A)$. Moreover, by lemma 3.11 we can obtain

$$(\delta, A)^2 = (\delta, A)\widehat{\circ}(\delta, A) \subseteq (\rho, A)^2 = (\rho, A)\widehat{\circ}(\rho, A) = (\rho, A).$$

In general, we have $(\delta, A)^n \subseteq (\rho, A)$ for all $n \in N$. This implies $(\delta, A)^{\infty} \subseteq (\rho, A)$ and so $(\delta, A)^{\infty}$ is the smallest soft equivalence relation over X containing (σ, A). Thus $(\sigma, A)^e = (\delta, A)^{\infty}$, completing the proof. □

Note that the two soft union operations $\cup_{\mathcal{R}}$ and $\cup_{\mathcal{E}}$ will always lead to the same results when the soft sets involved have the same set of parameters. Thus by Theorem 3.17, one can deduce the following:

Corollary 3.18. *Let (σ, A) be a soft binary relation over X. Then we have* $(\sigma, A)^e = \left[(\sigma, A) \cup_{\mathcal{R}} (\sigma, A)^{-1} \cup_{\mathcal{R}} \left(\sigma^{1_X}, A\right)\right]^{\infty}$.

4. Soft congruence relations over semigroups

In this section we shall study soft equivalence relations over semigroups, which are compatible to the algebraic structure of their underlying semigroups. We begin with the following basic notion.

Definition 4.1. Let (σ, A) be a soft binary relation over a semigroup S. Then (σ, A) is said to be (*right, left*) *compatible* if $\sigma(\alpha)$ is a (right, left) compatible relation on S for all $\alpha \in A$.

Definition 4.2. A soft equivalence relation (σ, A) over a semigroup S is said to be a (*right, left*) *soft congruence* over S if (σ, A) is a (right, left) compatible soft binary relation over S.

Proposition 4.3. *A soft binary relation (σ, A) over a semigroup S is a soft congruence relation over S if and only if $\sigma(\alpha) \neq \emptyset$ is a congruence relation on X for all $\alpha \in A$.*

Proof. Straightforward. □

Since the intersection of two congruence relations on a semigroup S is a congruence relation on S, one can verify the following.

Proposition 4.4. *Let (σ, A) and (ρ, B) be two soft congruences over a semigroup S. Then we have*

(a) *$(\sigma, A) \cap_{\mathcal{R}} (\rho, B)$ is a soft congruence over S contained in both (σ, A) and (ρ, B) whenever $A \cap B \neq \emptyset$.*
(b) *$(\sigma, A) \cup_{\mathcal{E}} (\rho, B)$ is a soft congruence over S whenever $A \cap B = \emptyset$.*
(c) *$(\sigma, A) \wedge (\rho, B)$ is a soft congruence over S.*
(d) *$(\sigma, A) \cap_{\mathcal{E}} (\rho, B)$ is a soft congruence over S.*

Proposition 4.5. *If $\{(\sigma_i, A) : i \in I\}$ is a non-empty family of soft congruence relations over a semigroup S, then the following soft binary relations are soft congruence relations over S:*

(a) $\bigcap_{\mathcal{R}}\{(\sigma_i, A) : i \in I\}$;
(b) $\bigcap_{\mathcal{E}}\{(\sigma_i, A) : i \in I\}$;

(c) $\bigwedge_{i\in I}(\sigma_i, A)$.

Proof. The proof is similar to that of Proposition 3.12 and thus is omitted here. □

Theorem 4.6. *Let* (σ, A) *and* (ρ, B) *be two soft congruence relations over a semigroup* S *with* $A \cap B \neq \emptyset$. *If* $(\sigma, A)\widehat{\circ}(\rho, B) = (\rho, B)\widehat{\circ}(\sigma, A)$, *then* $(\sigma, A)\widehat{\circ}(\rho, B)$ *is also a soft congruence over* S.

Proof. Suppose that (σ, A) and (ρ, B) are two soft congruences over a semigroup S such that $A \cap B \neq \emptyset$ and $(\sigma, A)\widehat{\circ}(\rho, B) = (\rho, B)\widehat{\circ}(\sigma, A)$. Let $(\sigma, A)\widehat{\circ}(\rho, B) = (\delta, C)$, where $C = A \cap B$ and $\delta(c) = \sigma(c) \diamond \rho(c)$ for all $c \in C$. For any $c \in C$, if $\delta(c) = \sigma(c) \diamond \rho(c) \neq \emptyset$ then $\sigma(c) \neq \emptyset$ and $\rho(c) \neq \emptyset$. Hence we have $\sigma(c)$ and $\rho(c)$ are reflexive relations on S since (σ, A) and (ρ, B) are soft reflexive relations over S. Thus $1_S \subseteq \sigma(c)$ and $1_S \subseteq \rho(c)$. It follows that $1_S = 1_S \diamond 1_S \subseteq \sigma(c) \diamond \rho(c) = \delta(c)$, which shows that $\delta(c) \neq \emptyset$ is a reflexive relation on S. Therefore, we obtain that (δ, C) is a soft reflexive relation over S.

Next, we show that (δ, C) is a soft symmetric relation over S. To see this, note first that for all $c \in C$ $\delta(c) = \sigma(c) \diamond \rho(c) = \rho(c) \diamond \sigma(c)$ since $(\sigma, A)\widehat{\circ}(\rho, B) = (\rho, B)\widehat{\circ}(\sigma, A)$. For any $c \in C$, if $\delta(c) = \sigma(c) \diamond \rho(c) \neq \emptyset$ then $\sigma(c) \neq \emptyset$ and $\rho(c) \neq \emptyset$, which are both symmetric relations on S. Thus we have

$$\delta^{-1}(c) = [\sigma(c) \diamond \rho(c)]^{-1} = [\rho(c) \diamond \sigma(c)]^{-1} = \sigma^{-1}(c) \diamond \rho^{-1}(c) = \sigma(c) \diamond \rho(c) = \delta(c).$$

That is, $\delta(c) \neq \emptyset$ is a symmetric relation on S. By Lemma 3.6, we thus deduce that (δ, C) is a soft symmetric relation over S.

For soft transitivity, we consider

$$\begin{aligned}(\delta, C)^2 = [(\sigma, A)\widehat{\circ}(\rho, B)]\widehat{\circ}[(\sigma, A)\widehat{\circ}(\rho, B)] &= (\sigma, A)\widehat{\circ}[(\rho, B)\widehat{\circ}(\sigma, A)]\widehat{\circ}(\rho, B)\\ &= (\sigma, A)\widehat{\circ}[(\sigma, A)\widehat{\circ}(\rho, B)]\widehat{\circ}(\rho, B)\\ &= [(\sigma, A)\widehat{\circ}(\sigma, A)]\widehat{\circ}[(\rho, B)\widehat{\circ}(\rho, B)]\\ &= (\sigma, A)\widehat{\circ}(\rho, B) = (\delta, C).\end{aligned}$$

This shows that (δ, C) is also a soft transitive relation over S. Therefore, (δ, C) is a soft equivalence relation over S.

To complete our proof, by Proposition 4.3 we only need to show that for any $c \in C$ such that $\delta(c) \neq \emptyset$, the equivalence relation $\delta(c)$ is indeed a congruence relation on S. In fact, if $\delta(c) = \sigma(c) \diamond \rho(c) \neq \emptyset$ then $\sigma(c) \neq \emptyset$ and $\rho(c) \neq \emptyset$, which are both congruence relations on S by Proposition 4.3. Now assume that $(x, y) \in \delta(c) = \sigma(c) \diamond \rho(c)$. Then there exists $z \in S$ such that $(x, z) \in \sigma(c)$ and $(z, y) \in \rho(c)$. Then for any $t \in S$, we have $(tx, tz) \in \sigma(c)$ and $(tz, ty) \in \rho(c)$. This implies that $(tx, ty) \in \delta(c)$. Similarly we get $(xt, yt) \in \delta(c)$. Hence we have shown that $\delta(c) \neq \emptyset$ is indeed a congruence relation on S. Therefore, we conclude that (δ, C) is a soft congruence relation over S. □

Proposition 4.7. *Let* (σ, A) *and* (ρ, B) *be two soft congruence relations over a semigroup* S *with* $A \cap B \neq \emptyset$. *Then the following are equivalent:*

(a) $(\sigma, A)\widehat{\circ}(\rho, B)$ *is a soft congruence relation over* S.
(b) $(\sigma, A)\widehat{\circ}(\rho, B)$ *is a soft equivalence relation over* S.
(c) $(\sigma, A)\widehat{\circ}(\rho, B)$ *is a soft symmetric relation over* S.
(d) $(\sigma, A)\widehat{\circ}(\rho, B) = (\rho, B)\widehat{\circ}(\sigma, A)$

Proof. It is clear that $(a) \Rightarrow (b) \Rightarrow (c)$.

We show that $(c) \Rightarrow (d)$. So let us suppose that $(\delta, C) = (\sigma, A)\widehat{\circ}(\rho, B)$ is a soft symmetric relation over S, where $C = A \cap B$ and $\delta(c) = \sigma(c) \diamond \rho(c)$ for all $c \in C$. Note first that if either $\sigma(c) = \emptyset$ or $\rho(c) = \emptyset$, then clearly we have $\sigma(c) \diamond \rho(c) = \rho(c) \diamond \sigma(c) = \emptyset$. Now, assume that $\sigma(c) \neq \emptyset$ and $\rho(c) \neq \emptyset$. Then both of them are congruence relations on S by Proposition 4.3. In particular, $\sigma^{-1}(c) = \sigma(c)$ and $\rho^{-1}(c) = \rho(c)$. But by hypothesis, we also have $\delta^{-1}(c) = \delta(c)$. Thus it follows that

$$\sigma(c) \diamond \rho(c) = \delta(c) = \delta^{-1}(c) = [\sigma(c) \diamond \rho(c)]^{-1} = \rho^{-1}(c) \diamond \sigma^{-1}(c) = \rho(c) \diamond \sigma(c).$$

Therefore, we have shown that $(\sigma, A)\widehat{\circ}(\rho, B) = (\rho, B)\widehat{\circ}(\sigma, A)$ as required.

$(d) \Rightarrow (a)$ is easily obtained by Theorem 4.6. □

Definition 4.8. Let (σ, A) and (ρ, B) be two soft congruence relations over a semigroup S. Then we define $(\sigma, A)\widehat{\vee}(\rho, B) = (\delta, C)$, where $C = A \cap B \neq \emptyset$ and $\delta(\gamma) = \sigma(\gamma) \vee \rho(\gamma)$ for all $\gamma \in C$. Note that $\sigma(\gamma) \vee \rho(\gamma)$ denotes the smallest congruence containing both $\sigma(\gamma)$ and $\rho(\gamma)$.

By definition, one can verify that $(\sigma, A)\widehat{\vee}(\rho, B)$ is the smallest soft congruence relation over X, which contains the soft congruence relations (σ, A) and (ρ, B).

Proposition 4.9. *Let (σ, A) and (ρ, A) be two soft congruence relations over a semigroup S. If $(\sigma, A)\widehat{\circ}(\rho, A)$ is a soft congruence relation over S, then we have $(\sigma, A)\widehat{\vee}(\rho, A) = [(\sigma, A)\widehat{\circ}(\rho, A)] \cup_{\mathcal{E}} (\sigma, A) \cup_{\mathcal{E}} (\rho, A)$.*

Proof. Suppose that (σ, A) and (ρ, A) are two soft congruence relations over a semigroup S such that $(\sigma, A)\widehat{\circ}(\rho, A)$ is again a congruence relation over S. Let $[(\sigma, A)\widehat{\circ}(\rho, A)] \cup_{\mathcal{E}} (\sigma, A) \cup_{\mathcal{E}} (\rho, A) = (\delta, A)$. Then one can verify that for any $\alpha \in A$, the following equality is valid:

$$\delta(\alpha) = \begin{cases} \sigma(\alpha) \diamond \rho(\alpha), & \text{if } \sigma(\alpha) \neq \emptyset \text{ and } \rho(\alpha) \neq \emptyset, \\ \sigma(\alpha), & \text{if } \rho(\alpha) = \emptyset, \\ \rho(\alpha), & \text{if } \sigma(\alpha) = \emptyset. \end{cases}$$

By hypothesis, we know that (σ, A), (ρ, A) and $(\sigma, A)\widehat{\circ}(\rho, A)$ are soft congruence relations over S. It follows that $\delta(\alpha)$ is a congruence relation on S for all $\alpha \in A$ such that $\delta(\alpha) \neq \emptyset$. Then by Proposition 4.3, we deduce that $(\delta, A) = [(\sigma, A)\widehat{\circ}(\rho, A)] \cup_{\mathcal{E}} (\sigma, A) \cup_{\mathcal{E}} (\rho, A)$ is a soft congruence relation over S containing (σ, A) and (ρ, A).

Now, let (μ, A) be any soft congruence over S containing (σ, A) and (ρ, A). Then for every $\alpha \in A$, we have $\sigma(\alpha) \subseteq \mu(\alpha)$ and $\rho(\alpha) \subseteq \mu(\alpha)$. It follows that $\sigma(\alpha) \diamond \rho(\alpha) \subseteq \mu(\alpha) \diamond \mu(\alpha) = \mu(\alpha)$. Hence we deduce that $\delta(\alpha) \subseteq \mu(\alpha)$ for all $\alpha \in A$. This shows that (δ, A) is the smallest soft congruence over S, containing both (σ, A) and (ρ, A). Therefore, $(\sigma, A)\widehat{\vee}(\rho, A) = [(\sigma, A)\widehat{\circ}(\rho, A)] \cup_{\mathcal{E}} (\sigma, A) \cup_{\mathcal{E}} (\rho, A)$. □

It is worth noting that under the condition of Proposition 4.9, $(\sigma, A)\widehat{\vee}(\rho, A) \neq (\sigma, A)\widehat{\circ}(\rho, A)$ in the general case. This can be illustrated by an example as follows:

Example 4.10. Suppose that S is a semigroup and $A = \{a, b, c\}$ is a set of parameters. Let (σ, A) be a soft binary relation on S such that $\sigma(a) = 1_S$, $\sigma(b) = \emptyset$ and $\sigma(c) = \nabla_S$ (the universal relation on S). Let (ρ, A) be a soft binary relation on S such that $\rho(a) = \emptyset$, $\rho(b) = 1_S$ and $\rho(c) = \emptyset$. Let $(\sigma, A)\widehat{\vee}(\rho, A) = (\delta, A)$. Then it is clear that $\delta(a) = 1_S$ and $\delta(b) = 1_S$ and $\delta(c) = \nabla_S$. One can observe that $(\sigma, A)\widehat{\circ}(\rho, A) = (\Phi, A)$, where $\Phi(\alpha) = \emptyset$ for all $\alpha \in A$. Thus we have $(\sigma, A)\widehat{\vee}(\rho, A) \neq (\sigma, A)\widehat{\circ}(\rho, A)$. On the other hand, it is easy to see that $(\sigma, A) \cup_{\mathcal{E}} (\rho, A) = (\delta, A)$. Hence $(\sigma, A)\widehat{\vee}(\rho, A) = [(\sigma, A)\widehat{\circ}(\rho, A)] \cup_{\mathcal{E}} (\sigma, A) \cup_{\mathcal{E}} (\rho, A)$.

Corollary 4.11. *Let (σ, A) and (ρ, A) be two soft congruence relations over a semigroup S. If $(\sigma, A)\widehat{\circ}(\rho, A)$ is a soft congruence relation over S, then we have $(\sigma, A)\widehat{\vee}(\rho, A) = [(\sigma, A)\widehat{\circ}(\rho, A)] \cup_{\mathcal{R}} (\sigma, A) \cup_{\mathcal{R}} (\rho, A)$.*

Proof. This follows directly from Proposition 4.9 since in this case the two soft union operations $\cup_{\mathcal{R}}$ and $\cup_{\mathcal{E}}$ coincide for a fixed parameter set A. □

Let S be a semigroup. We consider the collection of all soft congruence relations over S with a fixed set of parameters A, which is denoted by $\mathfrak{S}^A_{Cr}(S)$. One can verify that $\mathfrak{S}^A_{Cr}(S)$ is partially ordered by the inclusion of soft binary relations (see Definition 3.10). Moreover, we have the following result:

Theorem 4.12. *$(\mathfrak{S}^A_{Cr}(S), \subseteq, \cap_{\mathcal{R}}, \widehat{\vee})$ is a lattice.*

Proof. Let (σ, A) and (ρ, A) be two soft congruence relations over a semigroup S. Then it is obvious that $(\sigma, A) \cap_{\mathcal{R}} (\rho, A)$ is the greatest soft congruence relation over S contained in both (σ, A) and (ρ, A). That is, $(\sigma, A) \cap_{\mathcal{R}} (\rho, A)$ is the greatest lower bound of (σ, A) and (ρ, A) in $(\mathfrak{S}^A_{Cr}(S), \subseteq)$.

In addition, by Proposition 4.9 we know that $(\sigma, A)\widehat{\vee}(\rho, A) = [(\sigma, A)\widehat{\circ}(\rho, A)] \cup_{\mathcal{E}} (\sigma, A) \cup_{\mathcal{E}} (\rho, A)$, which is the smallest soft congruence relation over S containing both (σ, A) and (ρ, A). Hence $(\sigma, A)\widehat{\vee}(\rho, A)$ is the least upper bound of (σ, A) and (ρ, A) in $(\mathfrak{S}^A_{Cr}(S), \subseteq)$. Therefore, we conclude that $\mathfrak{S}^A_{Cr}(S)$ is a lattice with respect to the inclusion of soft binary relations. □

As an immediate consequence of the above theorem, we obtain an assertion as follows:

Corollary 4.13. $(\mathfrak{S}^{A}_{Cr}(S), \subseteq, \cap_{\mathcal{E}}, \widehat{\vee})$ *is a lattice.*

5. Relationship between fuzzy relations and soft binary relations

Fuzzy relations were introduced by Zadeh in his famous paper [33]. Fuzzy relations, especially fuzzy equivalence/similarity relations, have received much attention since the advent of fuzzy sets [15, 19, 22, 34]. In this section we shall show that every fuzzy relation may be considered as a soft binary relation.

A fuzzy relation R between two sets X and Y is regarded as a fuzzy subset μ_R of $X \times Y$, where μ_R is a mapping from $X \times Y$ to $[0, 1]$. The mapping μ_R associates the grade of membership $\mu_R(x, y)$ to each $(x, y) \in X \times Y$ in R.

Definition 5.1. ([34]) For any $\alpha \in [0, 1]$, an α-level set of a fuzzy relation R is denoted by R_α and is defined as $R_\alpha = \{(x, y) : \mu_R(x, y) \geq \alpha\}$.

Note that $\{R_\alpha : \alpha \in [0, 1]\}$ form a nested sequence of crisp binary relations such that $\alpha_1 \geq \alpha_2 \Rightarrow R_{\alpha_1} \subseteq R_{\alpha_2}$. The notion of α-level sets provides a connection between a fuzzy relation and a family of crisp binary relations. Moreover, we have the following assertion:

Proposition 5.2. *Every fuzzy relation may be considered as a soft binary relation.*

Proof. Let R be a fuzzy relation and μ_R be the membership function of R. We consider the family $\{R_\alpha : \alpha \in [0, 1]\}$ consisting of α-level sets of the fuzzy relation R. Note also the fuzzy relation R can be reconstructed from the family of α-level sets by means of the following formula $\mu_R(x, y) = \bigvee\{\alpha : (x, y) \in R_\alpha\}$, for all $(x, y) \in X \times Y$. Now, let us define a soft binary relation $(\sigma_R, [0, 1])$ over $Z = X \cup Y$ such that $\sigma_R(\alpha) = R_\alpha$ for all $\alpha \in [0, 1]$. Then it is clear that the fuzzy relation R can be identified with the soft binary relation $(\sigma_R, [0, 1])$ over Z. In this sense, we assert that every fuzzy relation may be considered as a soft binary relation. □

6. Soft homomorphisms and soft quotient semigroups

In this section we introduce soft homomorphisms of soft semigroups and soft quotient semigroups. We then establish several soft homomorphism theorems for soft semigroups.

Definition 6.1. Let (F, A) and (G, B) be two soft semigroups over the semigroups S and T, respectively. Let $f : S \to T$ and $g : A \to B$ be two functions. Then we say that $(f, g) : (F, A) \to (G, B)$ is a *soft homomorphism* and (F, A) is *soft homomorphic to* (G, B) if the following conditions hold:

(1) f is an epimorphism from S onto T.
(2) g is a surjective mapping from A onto B.
(3) $f(F(\alpha)) = G(g(\alpha))$ for all $\alpha \in A$.

If f is a monomorphism from S to T and g is an injective mapping from A onto B then (f, g) is called a *soft isomorphism*. Given $(f, g) : (F, A) \to (G, B)$ and $(h, p) : (G, B) \to (H, C)$, then *composition* of (f, g) and (h, p), denoted $(h, p) \circ (f, g)$, is defined as $(h, p) \circ (f, g) = (k, q)$ where $q = p \circ g$ and $k = h \circ f$. Next, we introduce the concept of soft quotient structures of a semigroup S with respect to soft congruence relations over S.

Proposition 6.2. *Let* (σ, A) *be a soft congruence relation over a semigroup* S *and* $S/\sigma(\alpha) = \left\{[x]_{\sigma(\alpha)} : x \in S\right\}$ *where* $\alpha \in A$. *Then for any* $\alpha \in A$, $S/\sigma(\alpha)$ *is a semigroup under the binary operation induced by* S, *which is given by* $[x]_{\sigma(\alpha)}[y]_{\sigma(\alpha)} = [xy]_{\sigma(\alpha)}$ *for all* $x, y \in S$. *Moreover,* S *is homomorphic to* $S/\sigma(\alpha)$ *for each* $\alpha \in A$.

Proof. First of all we check the binary operation is well defined consider $[a]_{\sigma(\alpha)} = [a']_{\sigma(\alpha)}$ and $[b]_{\sigma(\alpha)} = [b']_{\sigma(\alpha)}$ for all $a, a', b, b' \in S$. This implies $(a, a') \in \sigma(\alpha)$ and $(b, b') \in \sigma(\alpha)$ for all $\alpha \in A$. This implies $(ab, a'b') \in \sigma(\alpha)$ for all $\alpha \in A$.

Since (σ, A) is a soft congruence relation on S, so for all $a, b, c \in S$ consider

$$\begin{aligned}\left([a]_{\sigma(\alpha)}[b]_{\sigma(\alpha)}\right)[c]_{\sigma(\alpha)} &= \left([ab]_{\sigma(\alpha)}\right)[c]_{\sigma(\alpha)} \\ &= [(ab)c]_{\sigma(\alpha)} \\ &= [a(bc)]_{\sigma(\alpha)} \\ &= [a]_{\sigma(\alpha)}\left([bc]_{\sigma(\alpha)}\right) \\ &= [a]_{\sigma(\alpha)}\left([b]_{\sigma(\alpha)}[c]_{\sigma(\alpha)}\right).\end{aligned}$$

Therefore $S/\sigma(\alpha)$ is a semigroup for all $\alpha \in A$. Above discussion shows that for all $\alpha \in A$, we have a semigroup of classes of S and each semigroup of classes is homomorphic image of S, with homomorphism $f_\alpha : S \to S/\sigma(\alpha)$ defined as $f_\alpha(x) = [x]_{\sigma(\alpha)}$. □

If each $\sigma(\alpha)$ is distinct then each of $S/\sigma(\alpha)$ is distinct. We can denote $S/(\sigma, A)$ as the collection of semigroups each of which is a homomorphic image of S. Furthermore, if S is regular then each $S/\sigma(\alpha)$ is regular; hence $S/(\sigma, A)$ is a collection of regular semigroups.

Let σ be a congruence on a semigroup S. Let (F, A) be a soft semigroup over S. Denote the soft set $(F, A)/\sigma$ by (K, A) where $K(\alpha) = \{[a]_\sigma : a \in F(\alpha)\}$ for all $\alpha \in A$. Since $F(\alpha)$ is a subsemigroup of S, it is clear that $K(\alpha)$ is a subsemigroup of S/σ for all $\alpha \in A$. Hence $(F, A)/\sigma$ is a soft semigroup over S/σ. Now let $i_A : A \to A$ be the *identity mapping* over A and $\sigma^\top$ be the *natural homomorphism* of semigroups given by $\sigma^\top(a) = [a]_\sigma$ for all $\alpha \in A$. Then one can verify that $(\sigma^\top, i_A)$ is a soft homomorphism from (F, A) to $(F, A)/\sigma$, which will be called the *natural soft homomorphism* in what follows.

If (F, A) is a soft semigroup over a semigroup S and (σ, A) is a soft binary relation over S such that $\sigma(\alpha) \neq \emptyset$ is a congruence relation on $F(\alpha) \neq \emptyset$ for all $\alpha \in A$, then we say that (σ, A) is a soft congruence relation on the soft semigroup (F, A).

Definition 6.3. Let (F, A) and (G, B) be two soft semigroups over S and T, respectively. Let (f, g) be a soft homomorphism from (F, A) to (G, B). We define $\ker(f, g)$ as the soft binary relation (δ, A) over S, where

$$\delta(\alpha) = \{(a, b) \in F(\alpha) \times F(\alpha) : f(a) = f(b)\} = (\ker f)|_{F(\alpha)},$$

for all $\alpha \in A$.

Proposition 6.4. *Let (F, A) and (G, B) be two soft semigroups over S and T, respectively. If (f, g) is a soft homomorphism from (F, A) to (G, B), then $\ker(f, g)$ is a soft congruence relation on (F, A).*

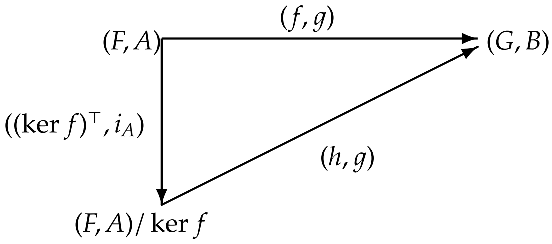
Proof. Since (f, g) is a soft homomorphism from (F, A) to (G, B), by definition $f : S \to T$ is an epimorphism of semigroups and $g : A \to B$ is a surjective map such that $f(F(\alpha)) = G(g(\alpha))$ for all $\alpha \in A$. We shall write the soft binary relation $\ker(f, g)$ as (δ, A) where $\delta(\alpha) = (\ker f)|_{F(\alpha)}$ for all $\alpha \in A$. Since $\ker f$ is clearly an equivalence relation over S, it is easy to see that $\delta(\alpha)$ is an equivalence relation on the subsemigroup $F(\alpha)$ for all $\alpha \in A$.

Furthermore, assume that $(a, a') \in \delta(\alpha)$ and $(b, b') \in \delta(\alpha)$. Then $f(a) = f(a')$ and $f(b) = f(b')$, whence we have

$$f(ab) = f(a) f(b) = f(a') f(b') = f(a'b').$$

This shows that $(ab, a'b') \in \delta(\alpha)$, and $\delta(\alpha)$ is a congruence on $F(\alpha)$ for all $\alpha \in A$. Consequently, $\ker(f, g)$ is a soft congruence relation on (F, A). □

Theorem 6.5. *Let (F, A) and (G, B) be two soft semigroups over S and T, respectively. If $(f, g) : (F, A) \to (G, B)$ is a soft homomorphism and $g : A \to B$ is an injective map, then there exists a unique soft isomorphism $(h, g) : (F, A)/\ker f \to (G, B)$ such that $h(F(\alpha)/\ker f) = G(g(\alpha))$ for all $\alpha \in A$, and the diagram:*



commutes, where $((\ker f)^\top, i_A) : (F, A) \to (F, A)/\ker f$ is a natural soft homomorphism.

Proof. Since $(f, g) : (F, A) \to (G, B)$ is a soft homomorphism, so $f : S \to T$ is an epimorphism of semigroups, and $g : A \to B$ is a surjective map such that $f(F(\alpha)) = G(g(\alpha))$ for all $\alpha \in A$. Let $h : S/\ker f \to T$ be a map defined by $h\left([a]_{\ker f}\right) = f(a)$, where $a \in S$. Since

$$\begin{aligned}[a]_{\ker f} = [b]_{\ker f} &\Leftrightarrow (a, b) \in \ker f \\ &\Leftrightarrow f(a) = f(b) \\ &\Leftrightarrow h\left([a]_{\ker f}\right) = h\left([b]_{\ker f}\right),\end{aligned}$$

we immediately deduce that h is both well-defined and injective. Moreover, one can verify that $h : S/\ker f \to T$ is an epimorphism of semigroups.

Denote the soft set $(F, A)/\ker f$ by (K, A) where $K(\alpha) = F(\alpha)/\ker f$ for all $\alpha \in A$. Then we have

$$\begin{aligned}h(K(\alpha)) &= h(F(\alpha)/\ker f) \\ &= \left\{h\left([a]_{\ker f}\right) : a \in F(\alpha)\right\} \\ &= \{f(a) : a \in F(\alpha)\} \\ &= f(F(\alpha)) \\ &= G(g(\alpha)),\end{aligned}$$

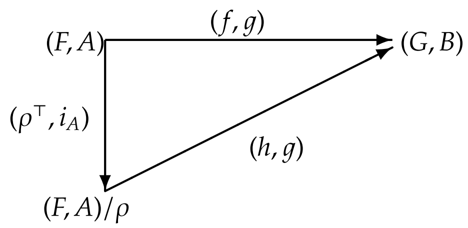
for all $\alpha \in A$. Also, we deduce from the hypothesis that g is indeed a bijection and so (h, g) is a soft isomorphism from $(F, A)/\ker f$ to (G, B).

To show that the diagram is commutative, note first that the soft composition of the natural soft homomorphism $((\ker f)^\top, i_A)$ and the soft homomorphism (h, g) defined as above is evidently a soft homomorphism from (F, A) to (G, B). Now, let $(h, g) \cdot ((\ker f)^\top, i_A) = (p, q)$. It is clear that $q = g$ since $i_A : A \to A$ is the identity map on A. Thus it suffices to prove that $p = h \cdot (\ker f)^\top = f$. In fact, let $a \in S$. Then

$$p(a) = h\left((\ker f)^\top(a)\right) = h\left([a]_{\ker f}\right) = f(a),$$

whence $p = h \cdot (\ker f)^\top = f$ and $(p, q) = (f, g)$ as required. Finally, one easily sees that the soft homomorphism (h, g) is unique since $h : S/\ker f \to T$ is completely determined by f. □

Theorem 6.6. *Let (F, A) and (G, B) be two soft semigroups over S and T, respectively. Let $(f, g) : (F, A) \to (G, B)$ be a soft homomorphism and ρ be a congruence on S such that $\rho \subseteq \ker f$. Then there exists a unique soft homomorphism $(h, g) : (F, A)/\rho \to (G, B)$ such that $h(F(\alpha)/\rho) = G(g(\alpha))$ for all $\alpha \in A$, and the diagram:*



commutes, where $(\rho^\top, i_A) : (F, A) \to (F, A)/\rho$ is a natural soft homomorphism.

Proof. Since (f, g) is a soft homomorphism from (F, A) to (G, B), it follows that $f : S \to T$ is an epimorphism of semigroups, and $g : A \to B$ is a surjective map such that $f(F(\alpha)) = G(g(\alpha))$ for all $\alpha \in A$. Let $h : S/\rho \to T$ be a map defined by $h\left([a]_\rho\right) = f(a)$, where $a \in S$. Note that

$$[a]_\rho = [b]_\rho \Rightarrow (a, b) \in \rho \subseteq \ker f \Rightarrow f(a) = f(b),$$

we immediately deduce that h is a well-defined map. Moreover, we can show that $h : S/\rho \to T$ is an epimorphism of semigroups. In fact, let $a, b \in S$. Clearly,

$$h\left([a]_\rho [b]_\rho\right) = h\left([ab]_\rho\right) = f(ab) = f(a) f(b) = h\left([a]_\rho\right) h\left([b]_\rho\right),$$

whence h is a homomorphism from S/ρ to T. Also, it is easy to see that h is surjective since by hypothesis $f : S \to T$ is an epimorphism.

We shall write the soft set $(F, A)/\rho$ as (K, A) where $K(\alpha) = F(\alpha)/\rho$ for all $\alpha \in A$. Then we have

$$\begin{aligned} h(K(\alpha)) &= h(F(\alpha)/\rho) \\ &= \left\{h\left([a]_\rho\right) : a \in F(\alpha)\right\} \\ &= \{f(a) : a \in F(\alpha)\} \\ &= f(F(\alpha)) \\ &= G(g(\alpha)), \end{aligned}$$

for all $\alpha \in A$. Therefore, we conclude that (h, g) is a soft isomorphism from $(F, A)/\rho$ to (G, B).

Now we show that the diagram is commutative. Let $(h, g) \cdot (\rho^\top, i_A) = (p, q)$. Note that (p, q) is a soft homomorphism from (F, A) to (G, B) since it is the soft composition of the natural soft homomorphism $(\rho^\top, i_A)$ and the soft homomorphism (h, g) defined as above. Also, it is clear that $q = g$ since i_A is the identity map on A. Thus it remains to show that $p = h \cdot \rho^\top = f$. In fact, let $a \in S$. Then

$$p(a) = h(\rho^\top(a)) = h\left([a]_\rho\right) = f(a),$$

whence $p = h \circ \rho^{\top} = f$ and $(p, q) = (f, g)$ as required. Finally, one easily sees that the soft homomorphism (h, g) is unique since $h : S/\rho \to T$ is completely determined by f. □

7. Conclusions

This study was devoted to the discussion of the relations among binary relations, soft sets, and semigroups. We generalized the concepts of both crisp binary relations and fuzzy binary relations to soft binary relations. Some basic concepts, operations, characterizations and related properties with regard to soft binary relations were proposed. We also considered soft congruence relations over semigroups and obtained certain lattice structures related to them. Finally, we introduced soft homomorphisms and established several isomorphism theorems for soft semigroups using soft congruence relations.

8. Acknowledgements

The authors are highly grateful to the anonymous referees and Professor Dragan S. Djordjević for their valuable comments and suggestions. The authors also wish to thank Professor Ljubiša Kočinac for his kind help.

References

[1] M.I. Ali, F. Feng, X.Y. Liu, W.K. Min, M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.
[2] M.I. Ali, A note on soft sets, rough soft sets and fuzzy soft sets, Appl. Soft Computing 11 (2011) 3329–3332.
[3] M.I. Ali, M. Shabir, M. Naz, Algebraic structures of soft sets associated with new operations, Comput. Math. Appl. 61 (2011) 2647–2654.
[4] N. Ajmal, K.V. Thomas, A complete study of the lattices of fuzzy congruences and fuzzy normal subgroups, Inform. Sci. 82 (1995) 197–218.
[5] H. Aktaş, N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.
[6] R. Biswas, S. Nanda, Rough groups and rough subgroups, Bull. Polish Acad. Sci. Math 42 (1994) 251–254.
[7] D. Chen, E.C.C. Tsang, D.S. Yeung, X. Wang, The parameterization reduction of soft sets and its applications, Comput. Math. Appl. 49 (2005) 757–763.
[8] P. Das, Lattice of fuzzy congruences in inverse semigroups, Fuzzy Sets Syst. 91 (1997) 399–409.
[9] M. Demirci, J. Recasens, Fuzzy groups, fuzzy functions and fuzzy equivalence relations, Fuzzy Sets Syst. 144 (2004) 441–458.
[10] F. Feng, Y.B. Jun, X.Z. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621–2628.
[11] F. Feng, Y.B. Jun, X.Y. Liu, L.F. Li, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. 234 (2010) 10–20.
[12] F. Feng, Y.M. Li, V. Leoreanu-Fotea, Application of level soft sets in decision making based on interval-valued fuzzy soft sets, Comput. Math. Appl. 60 (2010) 1756–1767.
[13] F. Feng, C.X. Li, B. Davvaz, M. Irfan Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Computing 14 (2010) 899–911.
[14] F. Feng, X.Y. Liu, V. Leoreanu-Fotea, Y.B. Jun, Soft sets and soft rough sets, Inform. Sci. 181 (2011) 1125–1137.
[15] K.C. Gupta, R.K. Gupta, Fuzzy equivalence relation redefined, Fuzzy Sets Syst. 79 (1996) 227–233.
[16] J.M. Howie, An Introduction to Semigroup Theory, Academic Press, 1976.
[17] Y.B. Jun, Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (2008) 1408–1413.
[18] Y.B. Jun, C.H. Park, Application of soft sets in ideal theory of BCK/BCI-algeras, Inform. Sci. 178 (2008) 2466–2475.
[19] J.P. Kim, D.R. Bae, Fuzzy congruences in groups, Fuzzy Sets Syst. 85 (1997) 115–120.
[20] N. Kuroki, On fuzzy semigroups, Inform. Sci. 53 (1991) 203–236.
[21] N. Kuroki, Rough ideals in semigroups, Inform. Sci. 100 (1997) 139–163.
[22] N. Kuroki, Fuzzy congruences on T^*-pure semigroups, Inform. Sci. 84 (1995) 239–246.
[23] P.K. Maji, A.R. Roy, R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.
[24] P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[25] P.K. Maji, A.R. Roy, A fuzzy set theoretic approach to decsion making problems, J. Comput. Appl. Math. 203 (2007) 412–418.
[26] D. Molodstov, Soft set theory–first results, Comput. Math. Appl. 37 (1999) 19–31.
[27] V. Murali, Fuzzy equivalence relations, Fuzzy Sets Syst. 30 (1989) 155–163.
[28] Z. Pawlak, Rough sets, Int. J. Comput. Inform. Sci. 11 (1982) 341–356.
[29] K.Y. Qin, Z.Y. Hong, On soft equality, J. Comput. Appl. Math. 234 (2010) 1347–1355.
[30] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[31] Y.J. Tan, Fuzzy congruences on a regular semigroup, Fuzzy Sets Syst. 117 (2001) 447–453.
[32] Q.M. Xiao, Z.L. Zhang, Rough prime ideals and rough fuzzy prime ideals in semigroups, Inform. Sci. 176 (2006) 725–733.
[33] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.
[34] L.A. Zadeh, Similarity relation and fuzzy ordering, Inform. Sci. 3 (1971) 177–200.