Softening Instability: Part I —Localization Into a Planar Band

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Distributed damage such as cracking in heterogeneous brittle materials may be approximately described by a strain-softening continuum. To make analytical solutions feasible, the continuum is assumed to be local but localization of softening strain into a region of vanishing volume is precluded by requiring that the softening region, assumed to be in a state of homogeneous strain, must have a certain minimum thickness which is a material property. Exact conditions of stability of an initially uniform strain field against strain localization are obtained for the case of an infinite layer in which the strain localizes into an infinite planar band. First, the problem is solved for small strain. Then a linearized incremental solution is obtained taking into account geometrical nonlinearity of strain. The stability condition is shown to depend on the ratio of the layer thickness to the softening band thickness. It is found that if this ratio is not too large compared to 1, the state of homogeneous strain may be stable well into the softening range. Part II of this study applies Eshelby's theorem to determine the conditions of localization into ellipsoidal regions in infinite space, and also solves localization into circular or spherical regions in finite bodies.

Introduction

Distributed damage such as cracking or void nucleation and growth may be macroscopically described by a constitutive law for a continuum that exhibits strain-softening (Bažant, 1986). In the softening range, the matrix of incremental moduli of the material is not positive-definite. After Hadamard (1903) pointed out that such a condition implies an imaginary wave speed, strain-softening has been considered an unacceptable property for a continuum and some scholars have argued that strain-softening simply does not exist (Read and Hegemier, 1984; Sandler, 1984).

The various mathematical arguments for the nonexistence of strain-softening, however, overlooked one crucial experimental fact pointed out in 1974 by Bažant (1986): A material in the strain-softening state also has available to it another matrix of incremental moduli which applies for unloading and is positive-definite. This fact makes an essential difference. It causes that the material in a strain softening-state can propagate unloading waves, and that solutions to various dynamic and static problems with strain-softening exist, even for the classical, local continuum. Some solutions are unique, representing limits of finite-element discretizations, which converge quite rapidly (Bažant and Belytschko, 1985, 1987; Belytschko et al., 1986). In some cases, strain-softening apparently leads to chaos (Belytschko et al., 1986). Generally, the dynamic strain-softening problems (for a local continuum) belong to the class of "ill-posed" initial boundary-value problems, which are well known and accepted as realistic in other branches of physics (e.g., Joseph et al., 1985; Joseph, 1986, Yoo, 1985). Therefore, there is nothing fundamentally wrong with the concept of strain-softening from the mathematical viewpoint.

From the physical viewpoint, however, strain-softening in a classical local continuum is an unacceptable concept because structures are predicted to fail with zero energy dissipation. The reason is that the zone of softening damage often localizes into a line or surface while the energy dissipation per unit volume is finite. This difficulty, however, may be overcome by introducing some form of localization limiters (Bažant and Belytschko, 1987)—mathematical formulations that prevent the strain-softening damage to localize into a region of zero volume. Among numerous possibilities, the limitation to localization may be best introduced by treating the softening damage as nonlocal while the elastic behavior, including unloading, is treated as local (Pijaudier-Cabot and Bažant, 1986; Bažant, Lin, and Pijaudier-Cabot, 1987; Bažant and Pijaudier-Cabot, 1987). Physical justification of this nonlocal formulation has been given on the basis of homogenization of a quasi-periodic microcrack array (Bažant, 1987). Other possibilities are, e.g., the use of strain gradient or higher-order derivatives (or gradients) of the yield function in the constitutive equation (e.g., Floegl and Mang, 1981; Mang and Eberhardsteiner, 1985; Schreyer and Chen, 1986; Triantafyllidis and Aifantis, 1986).

In this paper we will seek analytical solutions to

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multidimensional localization. They appear to be possible with the well-known approach which uses the simplest and earliest type of localization limiter proposed in 1974 by Bažant (1976) and later implemented in the finite-element crack band model (Bažant and Cedolin, 1979, 1980; Bažant, 1982; Bažant and Oh, 1983, 1984). Despite the presence of strain-softening, the constitutive equation is in this approach treated as local, but the softening damage is not allowed to localize into a region whose size is less than a certain characteristic length l of the material. This length is determined empirically by fitting test data and roughly corresponds to the size of the representative volume used in statistical theory of heterogeneous materials (Kröner, 1967). For concrete, l is of the order of the maximum aggregate size. Although this approach is, admittedly, crude and does not permit resolving the distribution of damage density throughout the softening region, it has given some surprisingly good results for the overall structural response. This was confirmed by comparisons with extensive fracture test data (Bažant and Oh, 1983; Bažant 1982, 1984) and later also by comparisons with accurate nonlocal finite-element solutions (Bažant and Pijaudier-Cabot, 1987; Bažant, Pijaudier-Cabot, and Pan, 1986; Bažant and Zubelewicz, 1986; see the Appendix).

The method of analysis of strain-localization instability due to softening and its thermodynamic basis were formulated in 1974 (Bažant, 1976) and the stability conditions for one-dimensional localization in a bar and flexural localization in a beam were derived. The equivalence to failure analysis on the basis of fracture energy was also demonstrated (Bažant, 1982). Several other studies reanalyzed the one-dimensional localization from other viewpoints yielding equivalent results (e.g., Ottosen, 1986).

Rudnicki and Rice (1975) and Rice (1976) made important pioneering studies of localization into a planar band in an infinite space. They focused their studies primarily on localization caused by the geometrically nonlinear effects of finite strain before the peak of the stress-strain diagram (i.e., in the plastic hardening range), but they also obtained some critical states for negative values of the plastic hardening modulus. Rudnicki (1977) further analyzed, on the basis of Eshelby's theorem, an infinite space that contains a weakened zone of ellipsoidal shape, determined for localization into such a zone the critical neutral equilibrium states of homogeneous stress and strain, and demonstrated various cases of localization instabilities in the hardening as well as softening regime. Although the studies of Rudnicki and Rice represented an important advance, they did not actually address the stability conditions but were confined to neutral equilibrium conditions for the critical state. They did not consider a general incremental stress-strain relation but were limited to von Mises plasticity (Rudnicki and Rice 1975) or Drucker-Prager plasticity (Rudnicki, 1977), in some cases enhanced with a vertex hardening term. They also did not consider bodies of finite dimensions, for which the size of the localization region usually has a major influence on the critical state and, in the case of planar localization bands, did not consider unloading to occur outside the localization band, which is important for finite bodies. The present study attempts to take all these conditions and effects into account.

The purpose of the present study is to obtain exact analytical solutions for some multidimensional localization problems with softening, using the method introduced in 1974 by Bažant (1976, 1979), which is based on a local continuum with an imposed lower bound on the size h of the softening region. In this paper, we will analyze localization of strain into a planar band, both without and with geometrical nonlinearity of strain and, in a subsequent companion paper, we will analyze in a similar manner the localization of strain into ellipsoidal regions, including the special case of spherical and circular regions. The solution for ellipsoidal regions will be based on the celebrated Eshelby's theorem for eigenstrain in an ellipsoidal inclusion in an infinite elastic solid.

The present study (including Part II) will deal only with stability of equilibrium states, and not with bifurcations of the equilibrium path. As is known from Shanley's column theory, such bifurcations can occur at increasing load and do not necessarily coincide with the limit of stable equilibrium.

Before embarking on our analysis, comments on several related aspects are in order. It has been widely believed that softening is properly treated by fracture mechanics, in particular, by line-crack fracture models with a cohesive zone characterized by a softening stress-displacement relation. One problem with this approach may be that the stress-displacement relation might not be unique and might depend on the fracture specimen geometry. This is suggested by nonlocal finite-element results as well as the difficulties in fitting test data for various specimen geometries on the basis of the same material properties. Another problem is that this approach is limited to single fractures or noninteracting multiple fractures, and becomes unobjective (with regard to the choice of mesh) when multiple interacting cracks are present (Bažant, 1986). The formulation could be made objective by imposing a certain minimum admissible spacing of cracks as a material property, but this runs into difficulty when the interacting cracks are not parallel. (Line cracks with a fixed minimum spacing h are, of course, macroscopically equivalent to distributed cracking if the cumulative cracking strain over distance h is made to be equal to the crack opening displacement.)

Softening Band Within a Finite Layer or Infinite Solid

Let us analyze the stability of softening that is localized in an infinite layer which is called the softening band (or localization band) and forms inside an infinite layer of thickness L ($L \geq h$); see Fig. 1. The minimum possible thickness h of the band is assumed to be a material property, proportional to the characteristic length l. The layer is initially in equilibrium under a uniform (homogeneous) state of strain ϵ_{ij}^0 and stress σ_{ij}^0 assumed to be in the strain-softening range (Fig. 2). Latin lower case subscripts refer to Cartesian coordinates x_i (i = 1, 2, 3) of material points in the initial state. The initial

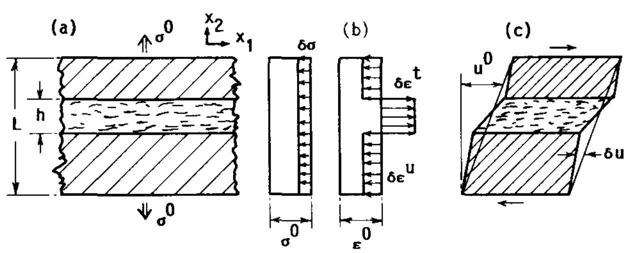

Fig. 1 Planar localization band in a layer

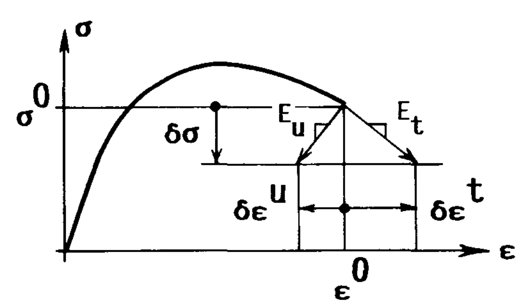

Fig. 2 Stress-strain diagram with softening and unloading

equilibrium is disturbed by small incremental displacements δu_i whose gradients $\delta u_{i,j}$ are uniform both inside and outside the band (subscripts preceded by a comma denote derivatives). The values of $\delta u_{i,j}$ inside and outside the band are denoted as $\delta u^t_{i,j}$ and $\delta u^u_{i,j}$ and are assumed to represent further loading and unloading, respectively, and so the increments $\delta u_{i,j}$ represent strain localization. We choose axis x_2 to be normal to the layer (Fig. 1). As the boundary conditions, we assume that the surface points of the layer are fixed during the incremental deformation, i.e., $\delta u_i = 0$ at the surfaces $x_2 = 0$ and $x_2 = L$ of the layer. Assuming homogeneous strains inside and outside of the band, compatibility of displacements at the band surface requires that

$$\delta u^u_{i,1} = \delta u^t_{i,1} = 0, \quad \delta u^u_{i,3} = \delta u^t_{i,3} = 0 \tag{1}$$

$$h\delta u^t_{i,2} + (L-h)\delta u^u_{i,2} = 0 \quad (i = 1,2,3). \tag{2}$$

We assume that the incremental material properties are characterized by incremental moduli tensors D^t_{ijpq} and D^u_{ijpq} for loading and unloading (Fig. 2). These two tensors may be either prescribed by the given constitutive law directly as functions of ϵ^0_{ij} and possibly other state variables, or they may be implied indirectly. The latter case occurs, e.g., for continuum damage mechanics. The crucial fact is that D^u_{ijpq} differs from D^t_{ijpq} and is always positive-definite, even if there is strain-softening. Noting that $\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$ and $D_{ijkm} = D_{jimk}$, we may write the incremental stress-strain relations as follows (repetition of subscripts implies Einstein's summation rule)

$$\delta\sigma^t_{ji} = D^t_{jikm}\delta\epsilon^t_{km} = D^t_{jikm}\delta u^u_{k,m} \quad \text{for loading} \tag{3}$$

$$\delta\sigma^u_{ji} = D^u_{jikm}\delta\epsilon^u_{km} = D^u_{jikm}\delta u^u_{k,m} \quad \text{for unloading.} \tag{4}$$

Stability of the initial equilibrium state may be decided on the basis of the work ΔW that must be done on the layer per unit area in the (x_1, x_3)-plane in order to produce the increments δu_i. This work, which represents the Helmholtz free energy under isothermal conditions and the total energy under adiabatic conditions, may be expanded as $\Delta W = \delta W + \delta^2 W + \ldots$ where δW = first variation (first-order work done by σ^0_{ij} on $\delta u_{i,j}$) and $\delta^2 W$ = second variation (the second-order work). If the initial state is an equilibrium state, δW must vanish. We, therefore, need to calculate only $\delta^2 W$. Using equations (3), (4), and (1), as well as the relation $\delta u^t_{i,2} = -\delta u^u_{i,2}(L-h)/h$ which follows from equation (2), we get

$$\delta^2 W = \frac{h}{2}\,\delta\sigma^t_{2i}\delta u^t_{i,2} + \frac{L-h}{2}\,\delta\sigma^u_{2i}\delta u^u_{i,2}$$

$$= \frac{h}{2}(\delta\sigma^t_{2i} - \delta\sigma^u_{2i})\delta u^t_{i,2} = \frac{h}{2}(D^t_{2ij2}\delta u^t_{j,2} - D^u_{2ij2}\delta u^u_{j,2})\delta u^t_{i,2} \tag{5}$$

$$\delta^2 W = \frac{h}{2}\left(D^t_{2ij2} + \frac{h}{L-h}\,D^u_{2ij2}\right)\delta u^t_{j,2}\delta u^t_{i,2}. \tag{6}$$

We may denote

$$Z_{ij} = D^t_{2ij2} + \frac{h}{L-h}\,D^u_{2ij2} \tag{7}$$

or

$$\mathbf{Z} = [Z_{ij}] = \begin{bmatrix} D^t_{2112}, & D^t_{2122}, & D^t_{2132} \\ D^t_{2212}, & D^t_{2222}, & D^t_{2232} \\ D^t_{2312}, & D^t_{2322}, & D^t_{2332} \end{bmatrix} + \frac{h}{L-h}\begin{bmatrix} D^u_{2112}, & D^u_{2122}, & D^u_{2132} \\ D^u_{2212}, & D^u_{2222}, & D^u_{2232} \\ D^u_{2312}, & D^u_{2322}, & D^u_{2332} \end{bmatrix} \tag{8}$$

Matrix $\mathbf{Z}$ is symmetric ($Z_{ij} = Z_{ji}$) if and only if $D^t_{2ij2} = D^t_{2ji2}$ and $D^u_{2ij2} = D^u_{2ji2}$. This is assured if D^t_{kijm} as well as D^u_{kijm} has a symmetric matrix. Our derivation, however, is valid in general for nonsymmetric Z_{ij} or D^t_{kijm}. The necessary stability condition may be stated according to equation (6) as follows

$$\delta^2 W = \frac{h}{2}\,\delta u^t_{i,2} Z_{ij} \delta u^t_{j,2} > 0 \text{ for any } \delta u^t_{i,2}. \tag{9}$$

The expression in equation (9) is a quadratic form. If $\delta^2 W$ is positive for all possible variations $\delta u^t_{i,2}$, no change of the initial state can occur if no work is done on the body, and so the initial state of uniform strain is stable. On the other hand, if $\delta^2 W$ is negative for some variation $\delta u^t_{i,2}$, the change of the initial state releases energy, which is ultimately dissipated as heat, thus increasing the entropy S of the system by $\Delta S = -\delta^2 W/T$ where T = absolute temperature. Hence, due to the second law of thermodynamics, such a change of initial state must happen spontaneously. Therefore, we must conclude that the intial state of homogeneous strain is unstable if $\delta^2 W$ (or Z_{ij}) ceases to be positive-definite. Positive-definiteness of the 3×3 matrix Z_{ij} is a necessary condition of stability. (We cannot claim equation (9) to be sufficient for stability since we have not analyzed all possible localization modes. However, changing $>$ to $<$ yields a sufficient condition for instability.)

Positive-definiteness of the 6×6 matrix $\mathbf{Z}^* = \mathbf{D}_t + \mathbf{D}_u h/(L-h)$ (where $\mathbf{D}_t$ and $\mathbf{D}_u$ are the 6×6 matrices of incremental moduli for loading and unloading) implies stability. However, the body can be stable even if $\mathbf{Z}^*$ is not positive-definite.

If the softening band is infinitely thin $(h/L \to 0)$, or the layer is infinitely thick $(L/h \to \infty)$, we have $Z_{ij} = D^t_{2ij2}$, and so matrix $\mathbf{Z}$ loses positive-definiteness when the (3×3) matrix of D^t_{2i2} ceases to be positive-definite. This condition, whose special case for von Mises plasticity was obtained by Rudnicki and Rice (1975) and Rice (1976), indicates that instability may occur right at the peak of the stress-strain diagram. However, for a continuum approximation of a heterogeneous material, for which h must be finite, the loss of stability can occur only after the strain undergoes a finite increment beyond the peak of the stress-strain diagram. For $L \to h$, the softening band is always stable.

Strain-localization instability in a uniaxially stressed bar represents a one-dimensional problem. It also may be obtained from the present three-dimensional solution as the special case for which the softening material is incrementally orthotropic, with $D^t_{2222} = E_t$ (<0) and $D^u_{2222} = E_u$ (>0) as the only nonzero incremental moduli. Equations (7) and (9) then yield the stability condition presented in 1974 (Bažant, 1976)

$$-\frac{E_t}{E_u} < \frac{h}{L-h} \quad \text{or} \quad \frac{E_t}{h} + \frac{E_u}{L-h} > 0. \tag{10}$$

This simple condition clearly illustrates that for finite L/h the localization instability can occur only at a finite slope $|E_t|$, i.e., some finite distance beyond the peak of the stress-strain diagram.

If the end of the bar at $x = L$ is not fixed but has an elastic support with spring constant C_s, one may obtain the solution by imagining the bar length to be augmented to length L', the additional length L' having the same stiffness as the spring; i.e., $L' - L = C_s/E_u$. Therefore, the stability condition is $-E_t/E_u < h/(L' - h)$, which yields the condition given before (Bažant, 1976).

The stability condition for a layer whose surface points are supported by an elastic foundation (Fig. 3) may be treated similarly, i.e., by adding to the layer of thickness L another layer of thickness $L' - L$ such that its stiffness is equivalent to the given foundation modulus.

The case when the outer surfaces of the layer are kept at

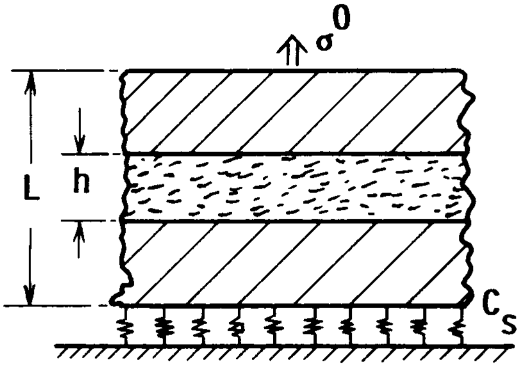

Fig. 3 Layer with localization band and an elastic foundation

constant load $p_i^0 = \sigma_{ij}^0 n_j$ during localization is equivalent to adding a layer of inifinite thickness $L' - L$. Therefore, the necessary stability condition is that the 3×3 matrix of D_{2ij2}^t be positive-definite, i.e., no softening can occur.

Another simple type of localization may be caused by softening in pure shear, Fig. 2(*c*). In this case, $\delta u_{1,2} \neq 0$, $\delta u_{2,2} = \delta u_{3,2} = 0$, and $D_{2112}^t = G_t$, $D_{2112}^u = G_u$ are the shear moduli. According to equations (7) and (9), the necessary stability condition is

$$-\frac{G_t}{G_u} < \frac{h}{L-h}. \tag{11}$$

Consider now the limiting case of an infinite space. While in a layer an infinitely long softening band must be parallel to the layer surface, for an infinite space, the softening band can have any orientation. Since $L \to \infty$, we have $Z_{ij} = D_{2ij2}^t$, and if the band is normal to axis x_2, stability requires that $\delta^2 W = \delta u_{i,2}^t D_{2ij2}^t \delta u_{j,2}^t h/2$ = positive-definite. To generalize this condition to a band of arbitrary orientation, we may carry out an arbitrary rotation transformation of coordinates from x_i to x_i'. The transformation relations are $x_i = c_{ij} x_j'$ where c_{ij} are the direction cosines of the old coordinate base vectors in the new coordinates. According to the rules of transformation of tensors, we now have $2\delta^2 W = h(c_{ik} c_{2m} \delta u_{k,m}') \, (c_{2p} c_{iq} c_{jr} c_{2s} D_{pqrs}^{t'}) \, (c_{ju} c_{2v} \delta u_{u,v}')$ where the primes refer to the new coordinates x_i'. Noting that $c_{ik} c_{iq} = \delta_{kq}$, $c_{jr} c_{ju} = \delta_{ru}$, we obtain

$$2\delta^2 W = h \delta a_{pq} D_{pqsr}^{t'} \delta a_{sr} \quad \text{with} \quad \delta a_{pq} = c_{2p} \delta u_{q,m} c_{2m}. \tag{12}$$

For arbitrary rotations, δa_{pq} can have any values. Thus the 6×6 matrix of moduli $D_{pqsr}^{t'}$, and also D_{ijkm}^t, must be positive-definite in order to insure that the strain cannot localize into an infinite planar band of any orientation. The same requirement was stated by Hadamard (1903), who derived it from the condition that the wave speed would not become imaginary. Hadamard's analysis, however, implied that $\mathbf{D}^u = \mathbf{D}^t$.

Finally, consider another case of boundary conditions: The case when the plane surfaces of an infinite layer slide freely over rigid bodies during the localization. The boundary conditions at $x_2 = 0$ and $x_2 = L$ now are $\delta\sigma_{21} = \delta\sigma_{23} = 0$. Similar to equation (6), we now have

$$\delta^2 W = \frac{h}{2}\left(D_{2222}^t + \frac{h}{L-h} D_{2222}^u\right)\delta u_{2,2}^t \delta u_{2,2}^t + \tfrac{1}{2}(\delta N_{11} \delta u_{1,1} + \delta N_{33} \delta u_{3,3} + 2\delta N_{13} \delta u_{1,3}) \tag{13}$$

in which δN_{11}, δN_{33}, δN_{13} are homogeneously distributed inplane incremental normal and shear force resultants over the whole thickness of the layer. Overall equilibrium now requires that $\delta N_{11} = \delta N_{33} = \delta N_{13} = 0$. Hence, the necessary condition of stability against localization is

$$D_{2222}^t + \frac{h}{L-h} D_{2222}^u > 0. \tag{14}$$

Extension to Geometrically Nonlinear Effects in Finite Strain

Strain localization in an infinite planar band can be also easily solved even when the geometrically nonlinear effects are taken into account. The geometrical nonlinearity we consider is due to finite strains or finite rotations, not to changes in the geometrical configuration of the structure. The deviations δu_i and $\delta\epsilon_{ij}$ from the initial state, characterized by homogeneous stresses σ_{ij}^0, are again considered to be infinitely small, and the energy expression ΔW that governs stability is second-order small. However, the contribution to $\delta^2 W$, which arises from the geometrically nonlinear finite-strain expression, is also second-order small, and so it must be included. This may be done if the material stress increment $\delta\sigma_{ij}$ in the work expression is replaced by the mixed (first) Piola-Kirchhoff stress increment $\delta\tau_{ij}$, which is referred to the initial state and is nonsymmetric (Malvern, 1969). As is well known, $\delta\tau_{ij} = \delta T_{ij} - \sigma_{ik}^0 \delta u_{j,k} + \sigma_{ij}^0 \delta u_{k,k}$ where δT_{ij} = material increment of the true (Cauchy) stress. Since neither $\delta\sigma_{ij}$ nor $\delta\tau_{ij}$ is invariant at coordinate rotations, one must use in the incremental stress-strain relation the objective stress increment $\delta\sigma_{ij}$ (representing the objective stress rate times the increment of time); $\delta\sigma_{ij}$ is symmetric.

The relationship between $\delta\sigma_{ij}$ and $\delta\tau_{ij}$ may be written (Bažant, 1971) in the general form

$$\delta\tau_{ij} = \delta\sigma_{ij} + R_{ij\,km\,rs} \sigma_{rs}^0 \delta u_{k,m} = H_{ijkm} \delta u_{k,m} \tag{15}$$

$$H_{ijkm} = D_{ijkm} + R_{ij\,km\,rs} \sigma_{rs}^0 \tag{16}$$

where we substituted $\delta\sigma_{ij} = D_{ijkm} \delta u_{k,m}$. Coefficients $R_{ij\,pq\,rs}$ are certain constants which take into account the geometrical nonlinearity of finite strain. The values of $R_{ij\,km\,pq}$ are different for various possible choices of the objective stress rate and the associated type of the finite-strain tensor. The expressions which are admissible according to the requirements of tensorial invariance and objectivity are (Bažant 1971)

$$R_{ij\,pq\,rs} \sigma_{rs}^0 = \delta_{ip} \sigma_{qj}^0 - \alpha(\delta_{ip} \sigma_{qj}^0 + \delta_{iq} \sigma_{pj}^0 + \delta_{jp} \sigma_{qi}^0 + \delta_{jq} \sigma_{pi}^0) \tag{17}$$

where α can be an arbitrary constant. The case $\alpha = 0$ corresponds to Truesdell's objective stress rate and Green's (Lagrangian) finite-strain tensor; the case $\alpha = 1/4$ to the finite-strain theory of Biot; the case $\alpha = 1/2$ to the Jaumann's objective stress rate and the finite strain theory of Southwell, Biezeno-Hencky, and Neuber (and to the logarithmic strain); and the case $\alpha = 3/4$ to Cotter-Rivlin's convected stress rate (see, equations (14*a*), (15), (17*a*), and (22) in Bažant, 1971). In general, α can be any real number. For each different α-value, however, different values of incremental moduli D_{ijpq} must be used so as to obtain physically equivalent results; generally [cf. Bažant, 1971, equation (19)]

$$D_{ijpq} = [D_{ijpq}]_{\alpha=0} + \alpha(\delta_{ip} \sigma_{qj}^0 + \delta_{iq} \sigma_{pj}^0 + \delta_{jp} \sigma_{qi}^0 + \delta_{jq} \sigma_{pi}^0). \tag{18}$$

From equation (5), in which $\delta\sigma_{ij}$ must now be replaced by $\delta\tau_{ij}$, the necessary stability condition is

$$\begin{aligned} 2\delta^2 W &= h \delta\tau_{2i}^t \delta u_{i,2}^t + (L-h)\delta\tau_{2i}^u \delta u_{i,2}^u \\ &= h(\delta\tau_{2i}^t - \delta\tau_{2i}^u) u_{i,2}^t = h(H_{2ij2}^t \delta u_{j,2}^t - H_{2ij2}^u \delta u_{j,2}^u) \delta u_{i,2}^t \\ &= h \delta u_{i,2}^t Z_{ij} \delta u_{j,2}^t > 0 \quad \text{for any } \delta u_{i,2}^t \end{aligned} \tag{19}$$

in which

$$Z_{ij} = H_{2ij2}^t + \frac{h}{L-h} H_{2ij2}^u = D_{2ij2}^t + \frac{h}{L-h} D_{2ij2}^u + \frac{L}{L-h} R_{2ij2rs} \sigma_{rs}^0. \tag{20}$$

In particular, if we use the Lagrangian (Green's) finite strain which is associated with Truesdell's objective stress rate ($\alpha = 0$), we have

$$Z_{ij} = \left[D^{l}_{2ij2} + \frac{h}{L-h} D^{u}_{2ij2} \right]_{\alpha=0} + \frac{L}{L-h} \sigma^0_{i2}\delta_{2j}. \tag{21}$$

If we use Jaumann's objective stress rate ($\alpha = 1/2$), we have

$$Z_{ij} = \left[D^{l}_{2ij2} + \frac{h}{L-h} D^{u}_{2ij2} \right]_{\alpha=1/2}$$

$$+ \frac{L}{2(L-h)} (\sigma^0_{i2}\delta_{2j} - \sigma^0_{j2}\delta_{2i} - \sigma^0_{22}\delta_{ij} - \sigma^0_{ij}). \tag{22}$$

It must be emphasized that the conditions of positive-definiteness of matrix $[Z_{ij}]$ for various possible values of α are all physically equivalent [even though for each different α-value the incremental moduli D^{l}_{2ij2} and D^{u}_{2ij2} are different, as dictated by equation (18)]. This fact often has been forgotten. The use of various types of objective stress rates often has been discussed in the literature on the basis of some imagined numerical convenience, and the fact that the incremental moduli cannot be the same for different objective stress rates has been overlooked (despite the analysis in Bažant, 1971). These practices, widespread in the finite-element literature, are of course incorrect.

As for the identification of the incremental moduli from test data, it can be done only in reference to a certain chosen objective stress rate. Different values of the moduli must result for different choices although the results are equivalent physically (see, Bažant, 1971).

The condition det $[Z_{ij}] = 0$ obviously represents the critical state. The special case of this condition for which the constitutive law consists of von Mises plasticity enhanced by vertex hardening, $L/h \rightarrow \infty$ (infinite space), $\alpha = 1/2$ (Jaumann's rate), and the initial stress σ^0_{ij} is pure shear σ^0_{12}, was derived by Rudnicki and Rice (1975). Their analysis was concerned only with the critical state of neutral equilibrium, rather than with stability. The values of the unloading moduli D^{u}_{ijkm} and the fact that they are different from D^{l}_{ijkm} and positive-definite were irrelevant for their analysis.

Rudnicki and Rice (1975) showed that, due to geometrical nonlinearity, the critical state of strain localization can develop in plastic materials while the matrix of D^{l}_{ijkm} is still positive-definite, i.e., before the final yield plateau of the stress-strain diagram is reached. This may explain the formation of shear bands in plastic (nonsoftening) materials. The destabilizing effect is then due exclusively to geometrical nonlinearity. According to a geometrically linear analysis (small strain theory), strain localization could develop in plastic (nonsoftening) materials only upon reaching the yield plateau but not earlier.

From equation (20), it appears that the geometrical nonlinearity can have a significant effect on strain localization only if the incremental moduli for loading are of the same order of magnitude as the initial stresses σ^0_{ij}; precisely, if max $|D^{l}_{ijkm}|$ and max $|\sigma^0_{ij}|$ are of the same order of magnitude. (The unloading moduli D^{u}_{ijkm} of structural materials are always several orders of magnitude larger.) Thus the importance of geometrical nonlinearity depends on D^{l}_{ijkm}. For strain-softening type of localization, the geometrical nonlinearity can be important only if the instability develops at a very small negative slope of the stress-strain diagram, which occurs very close to the peak stress point. This can occur only if the layer thickness L is much larger than l. If $L - h \ll l$, instability occurs when the downward slope of the stress-strain diagram (Fig. 2) is of the same order of magnitude as the initial elastic modulus E, and this is inevitably orders of magnitude larger than the initial stresses σ^0_{ij}.

Therefore, we must conclude that the role of geometrical nonlinearity in localization due to strain-softening cannot be significant except if the localization occurs immediately after the peak point of the stress-strain diagram (Fig. 2). Then the values of the strains at localization instability obtained by geometrically nonlinear and linear analyses cannot differ significantly since they must both be close to the peak point of the stress-strain diagram. For the postcritical large deformations, however, geometrical nonlinearity is no doubt always important.

Conclusions

The condition of stability of a layer against localization of strain into a band reduces to the condition of positive-definiteness of a certain matrix which represents a weighted average of the matrices of loading and unloading moduli of the material. The weight of the loading (softening) moduli increases with the ratio of the band thickness to the layer thickness. If this ratio tends to zero (infinite space), the instability is determined solely by the loading moduli and occurs as soon as this matrix ceases to be positive-definite. When the band thickness is finite, instability does not occur until the initial strain exceeds the strain at peak stress by a finite amount. Geometrical nonlinearity of strain has a significant effect only for a very small band-to-layer thickness ratio, and unstable localization then occurs near the peak stress state.

Extension to localization that is not unidirectional and numerical examples are relegated to a subsequent companion paper.

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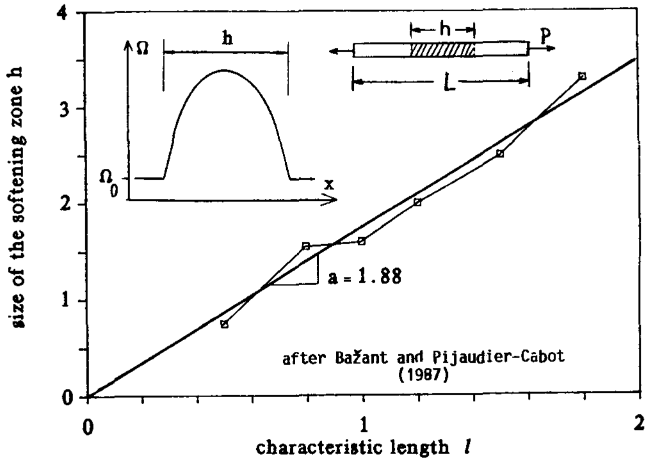

Fig. 4 Comparison of approximate local solution based on prescribed softening segment length *h* (dashed lines) with the exact nonlocal solution (solid lines), obtained for a bar of length *L* by Bažant and Pijaudier-Cabot (1987)

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APPENDIX

In the crack band model, the minimum possible size h of the softening zone in a local continuum is considered to be a material constant. The degree to which this approach can capture nonlocal behavior is illustrated by the comparison in Fig. 4, taken from Bažant and Pijaudier-Cabot (1987). The data points show the length h of the strain-softening segment in a bar of length $L > h$, made of a nonlocal strain-softening material with characteristic length l. Nonlocal is only the fracturing (or damage) strain while the elastic strain, including unloading, is local. The results represent accurate solutions of an integral equation for the static uniaxial localization instability in a bar that is initially strained uniformly, with a strain in the strain-softening range. In contrast to the present analysis, the length h of the localization segment is unknown and is solved as a function of the assumed characteristic length l. The results show that approximately $h = a$ where $a \approx$ constant ($a \approx 1.88$). Since l is a material constant, h may also be approximately considered to be a constant, as assumed in the crack band model.