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Sojourn Times in Feedback Queues

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This paper considers an M/M/1 queue with a very general feedback mechanism. When a customer completes his i -th service, he departs from the system with probability $1-p(i)$ and he cycles back with probability $p(i)$. The main result of the paper is a formula for the joint distribution of the successive sojourn times of a customer in the system. As a by-result, it is shown that the sojourn times in all individual cycles are identically, negative exponentially, distributed. Also, the correlation between the sojourn times of the j -th and k -th cycle of a customer is calculated; furthermore, the distribution of the total sojourn time is derived.

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1. INTRODUCTION

This paper considers an M/M/1 queue with a very general feedback mechanism. When a newly arriving customer, to be called a type-1 customer, has received his service, he departs from the system with probability $1-p(1)$ and is fed back to the end of the queue with probability $p(1)$; in the latter case he becomes a type-2 customer. When he has received his i -th service, he leaves with probability $1-p(i)$ and he cycles back with probability $p(i)$, in the latter case becoming a type- $i+1$ customer. The service times of each customer at all visits are identically, negative exponentially, distributed stochastic variables. The resulting queueing model has the property that the joint queue-length distribution of type- i customers, $i=1,2,\dots$, is of product-form type. This property is exploited to analyse the sojourn-time process. In particular, we are interested in the joint distribution of the sojourn times of a customer on his successive cycles.

Feedback queues are useful for modelling many phenomena in computer-communication systems. The following example in computer timesharing was communicated to us by R.D. Nelson (IBM Research Center, Hawthorne). A newly arriving job is considered to be a class 1 job. The scheduler first allocates some CPU time to the job. During this initial CPU allocation the job, if not completed, establishes its initial working set and typically changes class to class 2. The next time the job obtains the processor it is allocated a CPU slice for class 2 customers and, if not finished, becomes a class 3 job with its associated CPU slice. This procedure continues until the job leaves the system. This example gives rise to a single server queue with feedback. R.D. Nelson [7] considers both the case in which jobs of all classes form one queue which is processed in FCFS order, and the case of a head-of-the-line priority scheme. For both cases, allowing general service time distributions, he has obtained a set of equations for the mean sojourn times during each cycle. This set of equations can be solved numerically.

Other examples of feedback systems are found in telecommunications. E.g., a telephone call may generate several tasks for processing. Such tasks can sometimes be considered as feedbacks. From a customer's viewpoint, a very important performance measure is the response time or sojourn time, defined as the time spent by a particular sequence of tasks from its arrival to its service completion [6]; and not just its mean is of interest, but also (the tail of) its distribution.

In the queueing literature, research on feedback queues has been mainly restricted to single-server queues with so-called Bernoulli feedback [3,4,9]: when a customer (task) completes his service, he departs from the system with probability $1-p$ and is fed back with probability p . Fontana and Diaz

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Berzosa [5,6] extend some results obtained for the M/G/1 model with Bernoulli feedback to a more general feedback model with priorities. In [2] we have considered sojourn time distributions in an M/M/1 queue with deterministic feedback, viz., each customer makes exactly N passes through the system. Simultaneously, Doshi and Kaufman [4] have studied sojourn time distributions in an M/G/1 queue with Bernoulli feedback, employing an iteration procedure which is very similar to the one used in [2]. The feedback mechanisms in those studies are special cases of the one in the present study: for deterministic feedback, take $p(i)=1, i \leq N-1, p(N)=0$; for Bernoulli feedback, take $p(i) \equiv p$.

The organization of the rest of the paper is as follows. In Section 2 the model is described in detail. Section 3 contains our main result. We derive a formula for (the transform of) the joint distribution of the successive sojourn times of a customer in the system and the number of customers of the various types present at his successive departure epochs. As a by-result, it is shown that the sojourn times in all individual cycles are identically, negative exponentially, distributed. Also, the correlation between the sojourn times of the j -th and k -th cycle of a customer is calculated; furthermore, the distribution of the total sojourn time is derived. In Section 4 two special feedback mechanisms are studied: Bernoulli feedback (Subsection 4.1) and deterministic feedback (Subsection 4.2). In these two cases, a limiting procedure leads to the sojourn time distribution in the M/M/1 queue with processor sharing and the M/D/1 queue with processor sharing, respectively.

2. MODEL DESCRIPTION

We consider a single server queueing system with infinite waiting room, see Fig. 1. Customers arrive at the system according to a Poisson process with intensity $\lambda > 0$. After having received a service, a customer may either leave the system or be fed back. When a customer has completed his i -th service, he departs from the system with probability $1 - p(i)$ and is fed back with probability $p(i)$. Fed back customers return instantaneously, joining the end of the queue. A customer who is visiting the queue for the i -th time will be called a type- i customer. To avoid the problems that occur in dealing with an infinite number of customer types, it is assumed that after a certain number of services the feedback probabilities of a customer remain constant. Thus $p(i) = p(N) := p, i = N, N+1, \dots$ for some $N \geq 1$. The service discipline is First Come First Served (FCFS).

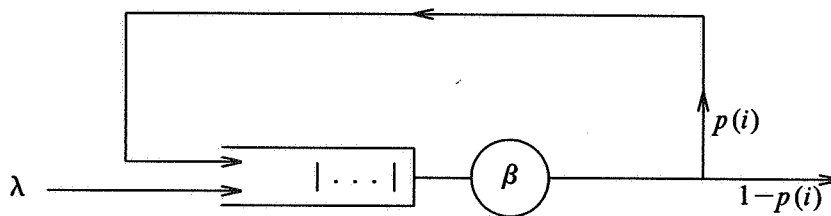


Fig. 1 M/M/1 queue with feedback.

The successive service times of a customer are independent, negative exponentially distributed, random variables, with mean β . These service times are also independent of the service times of other customers.

Introduce

$$\begin{aligned} q(0) &:= 1, \\ q(i) &:= \prod_{j=0}^{i-1} p(j), \quad i = 1, \dots, N-1, \end{aligned} \tag{2.1}$$

$$q(N) := \sum_{m=N}^{\infty} \prod_{j=0}^{m-1} p(j) = q(N-1)p(N-1)/(1-p),$$

with

$$p(0) := 1.$$

Note that $q(i)$ is the relative arrival rate of type- i customers, $i = 1, \dots, N$. The offered load to the queue per unit of time is $\rho := \lambda\beta \sum_{i=1}^N q(i)$. For stability it is required that $\rho < 1$.

We are interested in the following quantities:

- X_i : number of type- i customers in the system at an arbitrary epoch, $i = 1, \dots, N$;
- $X_i^{(j)}$: number of type- i customers in the system at the j -th service completion of a customer, $i = 1, \dots, N$, $j = 1, 2, \dots$;
- $X_i^{(0)}$: number of type- i customers in the system at the arrival of a new customer, $i = 1, \dots, N$;
- S_j : time required for the j -th pass through the system (j -th sojourn time), $j = 1, 2, \dots$.

3. MAIN RESULTS

In this section we present, in the form of Laplace-Stieltjes transforms and generating functions, an expression for the joint distribution of the successive sojourn times S_j , $j = 1, \dots, k$, and the number of type- i customers, $X_i^{(j)}$, $i = 1, \dots, N$, present at the j -th service completion of a customer who is fed back at least $k-1$ times, $k = 1, 2, \dots$. First note that the system described above can be considered as a queueing network consisting of one queue with N types of customers. Type- i customers are fed back with probability $p(i)$ after service, and then change into type- $(i+1)$ customers, $i = 1, \dots, N-1$. Type- N customers are fed back with probability p after service, and do not change their type. Because the service times are assumed to be exponentially distributed, the results obtained by Baskett et al. [1] can be applied to find the joint distribution of the number of type- i customers in the system at an arbitrary epoch. It is found that, for $x_1, \dots, x_N = 0, 1, \dots$,

$$P(x_1, \dots, x_N) := Pr\{X_1 = x_1, \dots, X_N = x_N\} = \quad (3.1)$$

$$(1 - \lambda\beta \sum_{i=1}^N q(i)) \prod_{i=1}^N (\lambda\beta q(i))^{x_i} \frac{(x_1 + \dots + x_N)!}{x_1! \dots x_N!}.$$

The PASTA property ([10]) implies the equality of the joint queue length distribution at the epoch of a new arrival and at an arbitrary epoch:

$$Pr\{X_1^{(0)} = x_1, \dots, X_N^{(0)} = x_N\} = P(x_1, \dots, x_N), \quad x_1, \dots, x_N = 0, 1, \dots \quad (3.2)$$

We follow a customer from the moment he arrives as a type-1 customer until he completes his k -th service. It holds that

$$E\{e^{-(\omega_1 S_1 + \dots + \omega_k S_k)} (z_{1,0}^{X_1^{(0)}} \dots z_{N,0}^{X_N^{(0)}}) \dots (z_{1,k}^{X_1^{(k)}} \dots z_{N,k}^{X_N^{(k)}})\} = \quad (3.3)$$

$$\sum_{x_1=0}^{\infty} \dots \sum_{x_N=0}^{\infty} P(x_1, \dots, x_N) E\{e^{-(\omega_1 S_1 + \dots + \omega_k S_k)} (z_{1,0}^{X_1^{(0)}} \dots z_{N,0}^{X_N^{(0)}}) \dots (z_{1,k}^{X_1^{(k)}} \dots z_{N,k}^{X_N^{(k)}}) | X_1^{(0)} = x_1, \dots, X_N^{(0)} = x_N\}.$$

Using the property that S_j is fully determined by the number of fed back customers and the number of external arrivals during S_{j-1} , it is shown in the Appendix that

$$E\{e^{-(\omega_1 S_1 + \dots + \omega_k S_k)} (z_{1,0}^{X_1^{(0)}} \dots z_{N,0}^{X_N^{(0)}}) \dots (z_{1,k}^{X_1^{(k)}} \dots z_{N,k}^{X_N^{(k)}}) \mid \mathbf{X}^{(0)} = x_1, \dots, \mathbf{X}_N^{(0)} = x_N\} = \quad (3.4)$$

$$\prod_{j=1}^k A_k^N(j, \omega, z) \prod_{i=1}^N (z_{i,0} f_k^N(i, \omega, z))^{x_i},$$

with $\omega := (\omega_1, \dots, \omega_k)$, $z := ((z_{1,0}, \dots, z_{N,0}), \dots, (z_{1,k}, \dots, z_{N,k}))$, and

$$A_k^N(1, \omega, z) := [1 + \beta\{\omega_k + \lambda(1 - z_{1,k})\}]^{-1}, \quad (3.5)$$

$$A_k^N(2, \omega, z) := [1 + \beta\{\omega_{k-1} + \lambda - \lambda z_{1,k-1} A_k^N(1, \omega, z)[p(1)z_{2,k} + 1 - p(1)]\}]^{-1},$$

$$A_k^N(i, \omega, z) := [1 + \beta\{\omega_{k-i+1} + \lambda - \lambda z_{1,k-i+1} A_k^N(i-1, \omega, z)[A_k^N(i-2, \omega, z)[\dots [A_k^N(2, \omega, z)[A_k^N(1, \omega, z) \\ [p(i-1)z_{i,k} + 1 - p(i-1)]p(i-2)z_{i-1,k-1} + 1 - p(i-2)]p(i-3)z_{i-2,k-2} + 1 - p(i-3)] \dots] \\ p(1)z_{2,k-i+2} + 1 - p(1)]\}]^{-1}, \quad i=3, \dots, k,$$

$$f_k^N(i, \omega, z) := A_k^N(k, \omega, z)[A_k^N(k-1, \omega, z)[\dots [A_k^N(2, \omega, z)[A_k^N(1, \omega, z) \\ [p(k+i-1)z_{k+i,k} + 1 - p(k+i-1)]p(k+i-2)z_{k+i-1,k-1} + 1 - p(k+i-2)] \\ p(k+i-3)z_{k+i-2,k-2} + 1 - p(k+i-3)] \dots [p(i)z_{i+1,1} + 1 - p(i)], \quad i=1, \dots, N. \quad (3.6)$$

Here we have defined $z_{i,j} := z_{N,j}$, $i = N+1, \dots, N+k$.

Substituting (3.4) and (3.1) into (3.3) we obtain our main result:

THEOREM

$$E\{e^{-(\omega_1 S_1 + \dots + \omega_k S_k)} (z_{1,0}^{X_1^{(0)}} \dots z_{N,0}^{X_N^{(0)}}) \dots (z_{1,k}^{X_1^{(k)}} \dots z_{N,k}^{X_N^{(k)}})\} = \frac{(1 - \lambda\beta \sum_{i=1}^N q(i)) \prod_{j=1}^k A_k^N(j, \omega, z)}{1 - \lambda\beta \sum_{i=1}^N q(i) z_{i,0} f_k^N(i, \omega, z)}, \quad (3.7)$$

$$\text{Re } \omega_j \geq 0, \quad |z_{i,j}| \leq 1, \quad i=1, \dots, N, \quad j=1, \dots, k.$$

COROLLARY 1

The Laplace-Stieltjes transform of the joint distribution of the first k successive sojourn times of a customer, who is fed back at least $k-1$ times, is given by

$$E\{e^{-(\omega_1 S_1 + \dots + \omega_k S_k)}\} = \frac{(1 - \lambda\beta \sum_{i=1}^N q(i)) \prod_{j=1}^k A_k^N(j, \omega)}{1 - \lambda\beta \sum_{i=1}^N q(i) f_k^N(i, \omega)}, \quad (3.8)$$

with,

$$A_k^N(1, \omega) := [1 + \beta\omega_k]^{-1}, \quad (3.9)$$

$$A_k^N(2, \omega) := [1 + \beta\{\omega_{k-1} + \lambda - \lambda A_k^N(1, \omega)\}]^{-1},$$

$$A_k^N(i, \omega) := [1 + \beta\{\omega_{k-i+1} + \lambda - \lambda A_k^N(i-1, \omega)[A_k^N(i-2, \omega)] \cdots [A_k^N(2, \omega)[A_k^N(1, \omega)] \\ p(i-2) + 1 - p(i-2)]p(i-3) + 1 - p(i-3)] \cdots [p(1) + 1 - p(1)]]^{-1}, \quad i = 3, \dots, k,$$

$$f_k^N(i, \omega) := A_k^N(k, \omega)[A_k^N(k-1, \omega)] \cdots [A_k^N(2, \omega)[A_k^N(1, \omega)p(k+i-2) + 1 - p(k+i-2)] \\ p(k+i-3) + 1 - p(k+i-3)] \cdots [p(i) + 1 - p(i)], \quad i = 1, \dots, N. \quad (3.10)$$

PROOF: Straightforward substitution.

COROLLARY 2

$$E\{z_1^{X_1^0} \cdots z_N^{X_N^0}\} = E\{z_1^{X_1} \cdots z_N^{X_N}\} = \frac{1 - \lambda\beta \sum_{i=1}^N q(i)}{1 - \lambda\beta \sum_{i=1}^N q(i)z_i}, \quad |z_i| \leq 1, \quad i = 1, \dots, N, \quad j = 0, 1, \dots, k. \quad (3.11)$$

PROOF: It follows from (3.1) and (3.2) that (3.11) holds for $j=0$. If (3.11) also holds for $j=1$, then it clearly holds for all $j=0, 1, \dots$. The validity of (3.11) for $j=1$ follows by a simple calculation.

REMARKS

(i) The Laplace-Stieltjes transform of the joint sojourn time distribution can be presented in a form which is more suitable for obtaining sojourn time moments. For this purpose we first rewrite (3.9) and (3.10):

$$A_k^N(i, \omega) := [1 + \beta\omega_{k-i+1} + \lambda\beta\{1 - \tilde{q}(i-1) \prod_{j=1}^{i-1} A_k^N(j, \omega) - \sum_{l=2}^{i-1} \tilde{q}(i-l)(1-p(i-l)) \prod_{j=l}^{i-1} A_k^N(j, \omega)\}]^{-1}, \quad i = 1, \dots, k,$$

$$f_k^N(i, \omega) := \frac{1}{\tilde{q}(i)} [\tilde{q}(k+i-1) \prod_{j=1}^k A_k^N(j, \omega) + \sum_{l=2}^k \tilde{q}(k+i-l)(1-p(k+i-l)) \prod_{j=l}^k A_k^N(j, \omega)], \quad i = 1, \dots, N,$$

with

$$\tilde{q}(0) := 1,$$

$$\tilde{q}(i) := \prod_{j=0}^{i-1} p(j), \quad i = 1, 2, \dots$$

Introducing

$$M(i, \omega) := \prod_{j=1}^i \frac{1}{A_k^N(j, \omega)}, \quad i = 1, \dots, k, \quad (3.12)$$

$$M(0, \omega) := 1,$$

and

$$H(k, i, \omega) := M(k, \omega) f_k^N(i, \omega), \quad i = 1, \dots, N,$$

it is easily seen that (3.8) can be written as

$$E\{e^{-(\omega_1 S_1 + \dots + \omega_k S_k)}\} = \frac{1 - \lambda \beta \sum_{i=1}^N q(i)}{M(k, \omega) - \lambda \beta \sum_{i=1}^N q(i) H(k, i, \omega)}, \quad (3.13)$$

with

$$M(i, \omega) = (1 + \beta \omega_{k-i+1}) M(i-1, \omega) + \lambda \beta \left[M(i-1, \omega) - \tilde{q}(i-1) - \sum_{j=2}^{i-1} \tilde{q}(i-j)(1-p(i-j)) M(j-1, \omega) \right], \quad i = 1, \dots, k, \quad (3.14)$$

$$H(k, i, \omega) = \frac{1}{\tilde{q}(i)} \left[\tilde{q}(k+i-1) + \sum_{j=2}^k \tilde{q}(k+i-j)(1-p(k+i-j)) M(j-1, \omega) \right], \quad i = 1, \dots, N. \quad (3.15)$$

(ii) The fact that the joint queue length distribution at the arrival of a customer and after each of his passes is the same (cf. Corollary 2), implies that the sojourn times S_j , $j = 1, \dots, k$ have the same marginal distribution. S_1 can easily be obtained from (3.13), (3.14) and (3.15) by taking $k = 1$ (or taking $\omega_j = 0$, $j = 2, \dots, k$). It is found, with offered load $\rho = \lambda \beta \sum_{i=1}^N q(i)$, that the sojourn times are negative exponentially distributed with mean $\beta / (1 - \rho)$:

$$E\{e^{-\omega_j S_j}\} = \frac{1 - \rho}{1 - \rho + \beta \omega_j}, \quad j = 1, \dots, k. \quad (3.16)$$

This coincides with the sojourn time transform in an ordinary M/M/1 queue with mean service time β and arrival rate $\lambda \sum_{i=1}^N q(i)$.

(iii) In order to investigate the dependence between the i -th and j -th sojourn times we have computed the Laplace-Stieltjes transform of the joint distribution of S_i and S_j , $1 \leq i < j \leq k$. It is found from (3.13), (3.14) and (3.15) that

$$E\{e^{-(\omega_i S_i + \omega_j S_j)}\} = \frac{1 - \rho}{1 - \rho + \beta \omega_i + \beta \omega_j + \beta^2 \omega_i \omega_j C_{j-i}}, \quad 1 \leq i < j \leq k, \quad (3.17)$$

where C_{j-i} is determined by

$$C_1 = 1, \quad (3.18)$$

$$C_n = (1 + \lambda \beta) C_{n-1} - \lambda \beta \sum_{l=2}^{n-1} \tilde{q}(n-l)(1-p(n-l)) C_{l-1}, \quad n = 2, \dots, k-1.$$

Note that $E\{e^{-(\omega_i S_i + \omega_j S_j)}\}$ only depends on i and j through the difference $j - i$. This property might also have been derived from Corollary 2.

From (3.17) the correlation coefficient, $\text{corr}(S_i, S_j)$, can easily be obtained:

$$\text{corr}(S_i, S_j) = 1 - C_{j-i}(1 - \rho), \quad 1 \leq i < j \leq k. \quad (3.19)$$

It follows from (3.18) and (3.19) that $\text{corr}(S_i, S_j)$ as a function of i and j only depends on $j-i$, and that it decreases if $j-i$ grows. For $j-i=1$, $\text{corr}(S_i, S_j)=\rho$.

(iv) The Laplace-Stieltjes transform of the distribution of a customer's total time spent in the system until the end of his k -th pass $S^{(k)}:=S_1+\dots+S_k$, can be obtained from (3.13) by substituting $\omega_j=\omega_0$, $j=1, \dots, k$. To derive an expression for the variance of this sojourn time, $\text{var}(S^{(k)})$, it is convenient to use the formula

$$\text{var}(S^{(k)}) = k \text{var}(S_1) + 2 \sum_{i=1}^k \sum_{j=i+1}^k \text{cov}(S_i, S_j).$$

The covariance of S_i and S_j , $\text{cov}(S_i, S_j)$, and $\text{var}(S_1)$ can easily be obtained from the results given in (ii) and (iii). It is found that

$$\text{var}(S^{(k)}) = \left[\frac{\beta}{1-\rho} \right]^2 \left[k^2 - 2(1-\rho) \sum_{j=1}^{k-1} j C_{k-j} \right], \quad (3.20)$$

with C_1, \dots, C_{k-1} given by (3.18).

The Laplace-Stieltjes transform of the distribution of the total sojourn time S of an (arbitrary) customer is now given by

$$E\{e^{-\omega_0 S}\} = \sum_{k=1}^{\infty} \tilde{q}(k)(1-p(k))E\{e^{-\omega_0 S^{(k)}}\}. \quad (3.21)$$

(v) Let the mean service time $\beta \downarrow 0$, and choose the feedback probabilities such that the total required mean service time, $\tilde{\beta}$, remains a positive constant. It is intuitively clear that this limiting procedure reduces the feedback queue to an M/G/1 queue with processor sharing. Different choices of the feedback probabilities lead to different total service time distributions with mean $\tilde{\beta}$. This opens the possibility for a new derivation of sojourn time results for the M/G/1 queue with processor sharing (cf. Ott [8]). We shall not pursue this here; in Section 4 the M/D/1 processor sharing queue will thus be analyzed.

EXAMPLE

In this example we examine the case $k=2$ for which explicit closed form results are easily obtained. From (3.17) and (3.18) it follows that for this case

$$E\{e^{-(\omega_1 S_1 + \omega_2 S_2)}\} = \frac{1-\rho}{1-\rho + \beta\omega_1 + \beta\omega_2 + \beta^2\omega_1\omega_2}. \quad (3.22)$$

Note that the feedback probabilities $p(i)$, $i=1, \dots, N$, enter into the joint distribution of S_1 and S_2 only via the offered load ρ . Thus, as long as ρ remains constant, the joint distribution of S_1 and S_2 is independent of the individual values of $p(i)$, $i=1, \dots, N$. Doshi and Kaufman [4] derived (3.22) for the (special) case of Bernoulli feedback ($p(i) \equiv p$).

From (3.19) it follows that

$$\text{corr}(S_1, S_2) = \rho. \quad (3.23)$$

Let $F_2(\cdot)$ denote the distribution function of the sojourn time until the end of the second pass:

$$F_2(t) := Pr\{S_1 + S_2 < t\}, \quad t \geq 0.$$

From (3.22) we find

$$E\{e^{-\omega_0(S_1+S_2)}\} = \frac{1+\sqrt{\rho}}{2\sqrt{\rho}} \frac{1-\sqrt{\rho}}{1-\sqrt{\rho}+\beta\omega_0} - \frac{1-\sqrt{\rho}}{2\sqrt{\rho}} \frac{1+\sqrt{\rho}}{1+\sqrt{\rho}+\beta\omega_0}.$$

Hence

$$F_2(t) = \frac{1+\sqrt{\rho}}{2\sqrt{\rho}}(1-e^{-t(1-\sqrt{\rho})/\beta}) - \frac{1-\sqrt{\rho}}{2\sqrt{\rho}}(1-e^{-t(1+\sqrt{\rho})/\beta}), \quad t \geq 0. \quad (3.24)$$

In [4] Doshi and Kaufman compare $F_2(\cdot)$ with the distribution of $S_1 + S_2$ that results when one assumes that S_1 and S_2 are independent. Due to the positive correlation between S_1 and S_2 (cf. (3.23)), it is found that $F_2(\cdot)$ has a longer "tail" than this approximate distribution.

Finally, the variance of $S_1 + S_2$ is obtained from (3.20):

$$\text{var}(S_1 + S_2) = 2(1+\rho) \left(\frac{\beta}{1-\rho} \right)^2.$$

4. SPECIAL CASES

In this section we study two feedback systems which are special cases of the general model described in Section 2, viz., Bernoulli feedback (Subsection 4.1) and deterministic feedback (Subsection 4.2). It will appear that these models yield simple, explicit expressions for most of the quantities analyzed in Section 3. At the end of Subsection 4.2 we show that the sojourn time distribution in the M/D/1 queue with processor sharing can easily be obtained from the sojourn time distribution in the deterministic feedback model.

The Laplace-Stieltjes transform of the joint sojourn time distribution in the Bernoulli feedback model has also been derived by Doshi and Kaufman [4]. The results for the deterministic feedback model were already obtained in a previous paper [2]. In fact, those results have led to the study of the general model.

4.1 Bernoulli feedback

The Bernoulli feedback model is obtained from the general model by taking $N=1$: when a customer completes his service he departs from the system with probability $1-p$ and is fed back with probability p .

Obviously

$$\rho = \frac{\lambda\beta}{1-p}.$$

The Laplace-Stieltjes transform of the joint distribution of the successive sojourn times S_1, \dots, S_k can be obtained from (3.8)-(3.10) by substituting $N=1$, $p(i) \equiv p$. The resulting expression has also been derived by Doshi and Kaufman [4].

To obtain explicit expressions for $E\{e^{-(\omega_i S_i + \omega_j S_j)}\}$, $\text{corr}(S_i, S_j)$, and $\text{var}(S^{(k)})$, (see (3.17), (3.19) and (3.20)) we have to derive C_n , $n=1, \dots, k-1$, from the set of difference equations (3.18). After the substitution $\tilde{q}(j) = p^{j-1}$, $j=1, 2, \dots$, (3.18) reduces to

$$C_1 = 1, \quad (4.1)$$

$$C_n = (1+\lambda\beta)C_{n-1} - \lambda\beta \sum_{l=2}^{n-1} (p^{n-l-1} - p^{n-l})C_{l-1}, \quad n=2, \dots, k-1.$$

From (4.1) it follows that

$$\begin{aligned} C_1 &= 1, \\ C_2 &= 1 + \lambda\beta, \\ C_n - pC_{n-1} &= (1 + \lambda\beta)C_{n-1} - p(1 + \lambda\beta)C_{n-2} - \lambda\beta(1-p)C_{n-2}, \quad n = 3, \dots, k-1. \end{aligned}$$

Hence

$$\begin{aligned} C_1 &= 1, \\ C_2 &= 1 + \lambda\beta, \\ C_n &= (1 + \lambda\beta + p)C_{n-1} - (\lambda\beta + p)C_{n-2}, \quad n = 3, \dots, k-1. \end{aligned} \tag{4.2}$$

The general solution of (4.2) is given by

$$C_n = U_1 y_1^n + U_2 y_2^n,$$

where $y_1 = 1$ and $y_2 = \lambda\beta + p$ are the roots of

$$y^2 - (1 + \lambda\beta + p)y + (\lambda\beta + p) = 0,$$

and U_1 and U_2 are determined by

$$\begin{aligned} U_1 y_1 + U_2 y_2 &= 1, \\ U_1 y_1^2 + U_2 y_2^2 &= 1 + \lambda\beta. \end{aligned}$$

After some calculations it is found that

$$C_n = \frac{1-p - \lambda\beta(\lambda\beta+p)^{n-1}}{1-p - \lambda\beta}, \quad n = 1, \dots, k-1. \tag{4.3}$$

Substitution of (4.3) in (3.17), (3.19) and (3.20) yields

$$E\{e^{-(\omega_i S_i + \omega_j S_j)}\} = \frac{1 - \frac{\lambda\beta}{1-p}}{1 - \frac{\lambda\beta}{1-p} + \beta\omega_i + \beta\omega_j + \beta^2\omega_i\omega_j \frac{1-p - \lambda\beta(\lambda\beta+p)^{j-i-1}}{1-p - \lambda\beta}}, \quad 1 \leq i < j \leq k, \tag{4.4}$$

$$\text{corr}(S_i, S_j) = \frac{\lambda\beta}{1-p} (\lambda\beta + p)^{j-i-1}, \quad 1 \leq i < j \leq k, \tag{4.5}$$

$$\text{var}(S^{(k)}) = \left[\frac{\beta}{1 - \frac{\lambda\beta}{1-p}} \right]^2 \left[k + 2 \frac{\lambda\beta}{(1-p)^2} \left\{ \frac{k}{1 - \frac{\lambda\beta}{1-p}} - \frac{1}{1-p} \frac{1}{(1 - \frac{\lambda\beta}{1-p})^2} (1 - (p + \lambda\beta)^k) \right\} \right], \tag{4.6}$$

$k = 1, 2, \dots$

It follows from (4.5) that $\lim_{j-i \uparrow \infty} \text{corr}(S_i, S_j) = 0$ (cf. Section 3, Remarks (iii)). It is also seen that $\text{corr}(S_i, S_j)$ is an increasing function of $\lambda\beta$ for fixed i and j . These intuitively appealing properties are illustrated in Fig. 2.

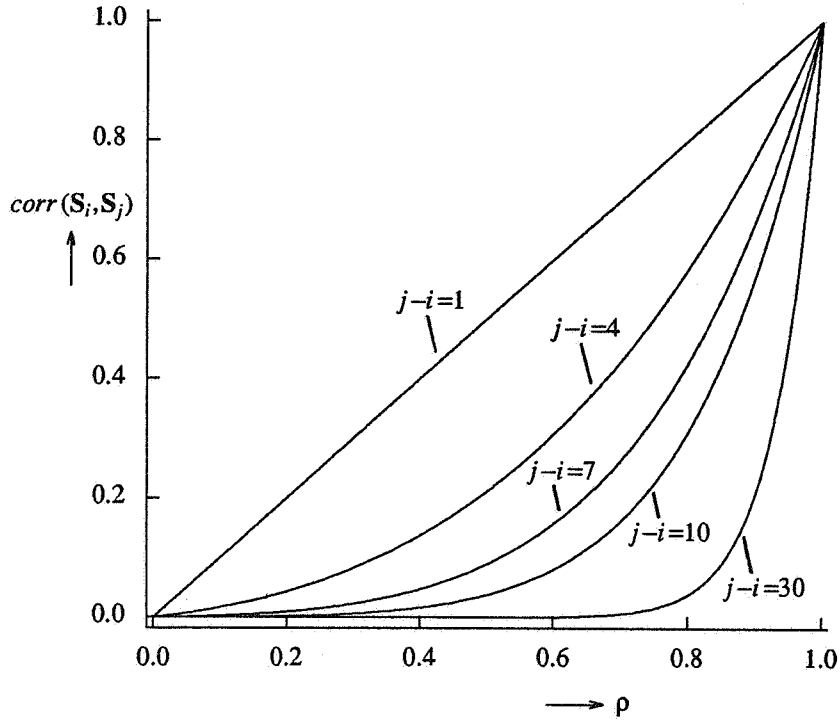


Fig. 2 $\text{corr}(S_i, S_j)$ as a function of offered load $\rho = \frac{\lambda\beta}{1-p}$, with $p=0.5$.

The Laplace-Stieltjes transform of $S^{(k)}$ can be obtained from (3.13)-(3.15) by substituting $\omega_j = \omega_0$, $j=1, \dots, k$. The resulting set of difference equations (3.14) can be solved in the same way as (4.2). After extensive but straightforward calculations it is found that

$$E\{e^{-\omega_0 S^{(k)}}\} = \frac{1-p-\lambda\beta}{Q_1(1-x_2)x_1^k + Q_2(1-x_1)x_2^k}, \quad \text{Re } \omega_0 \geq 0, \quad k=1, 2, \dots, \quad (4.7)$$

where,

$$x_1 = \frac{1 + \beta\omega_0 + \lambda\beta + p + \sqrt{(1 + \beta\omega_0 + \lambda\beta + p)^2 - 4(p + p\beta\omega_0 + \lambda\beta)}}{2},$$

$$x_2 = \frac{1 + \beta\omega_0 + \lambda\beta + p - \sqrt{(1 + \beta\omega_0 + \lambda\beta + p)^2 - 4(p + p\beta\omega_0 + \lambda\beta)}}{2},$$

$$Q_1 = \frac{x_2 - (1 + \beta\omega_0)}{x_2 - x_1},$$

$$Q_2 = \frac{x_1 - (1 + \beta\omega_0)}{x_1 - x_2}.$$

Letting $p \uparrow 1$ and $\lambda\beta \downarrow 0$ such that $\lambda\beta/(1-p)$ remains constant, reduces the feedback model to the M/M/1 queue with processor sharing. Application of this limiting procedure to the total sojourn time distribution, which readily follows from (4.7), yields the sojourn time distribution in the M/M/1 queue with processor sharing (Ott [8]). The calculations, which are quite involved, are omitted here.

4.2 Deterministic feedback

Taking $p(i)=1$, $i=1, \dots, N-1$, $p(N)=p=0$, we obtain the deterministic feedback model in which each customer is fed back exactly $N-1$ times. After the N -th service the customer leaves the system. Obviously

$$\rho = N\lambda\beta.$$

Noting that

$$\begin{aligned} q(j) &= \tilde{q}(j) = 1, & 0 \leq j \leq N, \\ &= 0, & j > N, \end{aligned}$$

it is easily seen from (3.13)-(3.15) that

$$E\{e^{-(\omega_1 S_1 + \dots + \omega_N S_N)}\} = \frac{1 - N\lambda\beta}{M(N, \omega) - \lambda\beta \sum_{i=0}^{N-1} M(i, \omega)}, \quad (4.8)$$

with

$$\begin{aligned} M(0, \omega) &= 1, \\ M(i, \omega) &= (1 + \lambda\beta + \beta\omega_{N-i+1})M(i-1, \omega) - \lambda\beta, \quad i=1, \dots, N. \end{aligned} \quad (4.9)$$

At the end of this section we shall use (4.8) and (4.9) to obtain the Laplace-Stieltjes transform of the total sojourn time distribution.

As in Section 4.1 we solve the set of difference equations (3.18) to obtain explicit expressions for $E\{e^{-(\omega_1 S_1 + \omega_j S_j)}\}$, $\text{corr}(S_i, S_j)$, and $E\{e^{-\omega_0 S^{(N)}}\}$ from the general formulas (3.17), (3.19) and (3.20). Substituting in (3.18) $p(i)=\tilde{q}(i)=1$, $i=1, \dots, N-1$, we get

$$\begin{aligned} C_1 &= 1, \\ C_n &= (1 + \lambda\beta)C_{n-1}, \quad n=2, \dots, N-1. \end{aligned}$$

Hence

$$C_n = (1 + \lambda\beta)^{n-1}, \quad n=1, \dots, N-1. \quad (4.10)$$

Now it follows from (3.17), (3.19) and (3.20) that

$$E\{e^{-(\omega_i S_i + \omega_j S_j)}\} = \frac{1 - N\lambda\beta}{1 - N\lambda\beta + \beta\omega_i + \beta\omega_j + \beta^2 \omega_i \omega_j (1 + \lambda\beta)^{j-i-1}}, \quad 1 \leq i < j \leq N, \quad (4.11)$$

$$\text{corr}(S_i, S_j) = 1 - (1 - N\lambda\beta)(1 + \lambda\beta)^{j-i-1}, \quad 1 \leq i < j \leq N, \quad (4.12)$$

$$\text{var}(\mathbf{S}^{(N)}) = \text{var}(\mathbf{S}) = \left[\frac{\beta}{1-N\lambda\beta} \right]^2 \left[\frac{2}{(\lambda\beta)^2} - N^2 - 2(1-N\lambda\beta) \frac{(1+\lambda\beta)^N}{(\lambda\beta)^2} \right]. \quad (4.13)$$

The Laplace-Stieltjes transform of the distribution of the total sojourn time \mathbf{S} is obtained from (4.8) and (4.9) by substituting $\omega_j = \omega_0$, $j = 1, \dots, N$. This substitution yields (cf.(4.9))

$$\begin{aligned} M(0, \omega) &= 1, \\ M(i, \omega) &= (1 + \lambda\beta + \beta\omega_0)M(i-1, \omega) - \lambda\beta, \quad i = 1, \dots, N. \end{aligned}$$

Hence

$$M(i, \omega) = (1 + \lambda\beta + \beta\omega_0)^i - \lambda\beta \sum_{j=0}^{i-1} (1 + \lambda\beta + \beta\omega_0)^j = \frac{\beta\omega_0}{\lambda\beta + \beta\omega_0} (1 + \lambda\beta + \beta\omega_0)^i + \frac{\lambda\beta}{\lambda\beta + \beta\omega_0}, \quad i = 1, \dots, N.$$

Using this result it follows from (4.8) that

$$E\{e^{-\omega_0 \mathbf{S}}\} = \frac{(1-N\lambda\beta)(\lambda + \omega_0)^2}{\omega_0^2(1 + \lambda\beta + \beta\omega_0)^N + \lambda(\lambda + \omega_0)(1-N\lambda\beta) + \lambda\omega_0}, \quad \text{Re } \omega_0 \geq 0. \quad (4.14)$$

Finally, we shall show that (4.14) can be used to obtain the sojourn time distribution in the M/D/1 queue with processor sharing (cf. Section 3, Remarks (v)).

Let $N \uparrow \infty$ and $\beta \downarrow 0$ in such a way that $\tilde{\beta} := N\beta$ remains constant. Then the distribution of the total service time received by each customer approaches the deterministic distribution fixed at $\tilde{\beta}$:

$$\lim_{\substack{N \uparrow \infty \\ \beta \downarrow 0}} \left[\frac{1}{1 + \beta\omega_0} \right]^N = \lim_{N \uparrow \infty} \left[\frac{1}{1 + \frac{\tilde{\beta}}{N}\omega_0} \right]^N = e^{-\tilde{\beta}\omega_0}, \quad \text{Re } \omega_0 \geq 0.$$

This limiting procedure apparently reduces the deterministic feedback model to the M/D/1 queueing model with processor sharing. Indeed, in the limit (4.14) equals the Laplace-Stieltjes transform of the sojourn time distribution in the M/D/1 system with processor sharing:

$$\lim_{\substack{N \uparrow \infty \\ \beta \downarrow 0}} E\{e^{-\omega_0 \mathbf{S}}\} = \frac{(\lambda + \omega_0)^2 (1 - \lambda\tilde{\beta}) e^{-\tilde{\beta}(\lambda + \omega_0)}}{\omega_0^2 + \lambda(\lambda + 2\omega_0 - \lambda\tilde{\beta}(\lambda + \omega_0)) e^{-\tilde{\beta}(\lambda + \omega_0)}}, \quad \text{Re } \omega_0 \geq 0, \quad (4.15)$$

a result previously obtained by Ott [8].

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APPENDIX

In this appendix we prove Formula (3.4). For ease of notation we introduce the service time distribution function $B(t) := 1 - e^{-t/\beta}$.

The derivation of (3.4) is based on the fact that $(\mathbf{X}_i^{(i+1)}, \dots, \mathbf{X}_N^{(i+1)})$, which determines the distribution of \mathbf{S}_{i+2} , is conditionally independent of $\{(\mathbf{X}_1^{(0)}, \dots, \mathbf{X}_N^{(0)}), \dots, (\mathbf{X}_1^{(i-1)}, \dots, \mathbf{X}_N^{(i-1)}); \mathbf{S}_1, \dots, \mathbf{S}_i\}$ given $\{(\mathbf{X}_1^{(i)}, \dots, \mathbf{X}_N^{(i)}); \mathbf{S}_{i+1}\}$, $i = 1, \dots, k-1$. Using this property it is easily seen that:

$$E\{e^{-(\omega_1 \mathbf{S}_1 + \dots + \omega_k \mathbf{S}_k)} (z_{1,0}^{\mathbf{X}_1^{(0)}} \dots z_{N,0}^{\mathbf{X}_N^{(0)}}) \dots (z_{1,k}^{\mathbf{X}_1^{(k)}} \dots z_{N,k}^{\mathbf{X}_N^{(k)}}) \mid \mathbf{X}_1^{(0)} = x_1, \dots, \mathbf{X}_N^{(0)} = x_N\} =$$

$$z_{1,0}^{x_1} \dots z_{N,0}^{x_N} \int_{t_1=0}^{\infty} e^{-\omega_1 t_1} \int_{t_2=0}^{\infty} e^{-\omega_2 t_2} \dots \int_{t_{k-1}=0}^{\infty} e^{-\omega_{k-1} t_{k-1}} \int_{t_k=0}^{\infty} e^{-\omega_k t_k}$$

$$\sum_{n_1^{(0)}=0}^{\infty} e^{-\lambda t_1} \frac{(\lambda t_1)^{n_1^{(0)}}}{n_1^{(0)}!} z_{1,1}^{n_1^{(0)}} \left[\prod_{j=1}^N \sum_{n_{j+1}^{(0)}=0}^{x_j} \binom{x_j}{n_{j+1}^{(0)}} p(j)^{n_{j+1}^{(0)}} (1-p(j))^{x_j - n_{j+1}^{(0)}} z_{j+1,1}^{n_{j+1}^{(0)}} \right]$$

$$\prod_{m=2}^k \left\{ \sum_{n_1^{(m)}=0}^{\infty} e^{-\lambda t_m} \frac{(\lambda t_m)^{n_1^{(m)}}}{n_1^{(m)}!} z_{1,m}^{n_1^{(m)}} \left[\prod_{j=1}^{N+m-1} \sum_{n_{j+1}^{(m)}=0}^{n_j^{(m-1)}} \binom{n_j^{(m-1)}}{n_{j+1}^{(m)}} p(j)^{n_{j+1}^{(m)}} (1-p(j))^{n_j^{(m-1)} - n_{j+1}^{(m)}} z_{j+1,m}^{n_{j+1}^{(m)}} \right] \right\}$$

$$dB(t_k)^{(1+n_1^{(k-1)} + \dots + n_{N+k-1}^{(k-1)})^*} dB(t_{k-1})^{(1+n_1^{(k-2)} + \dots + n_{N+k-2}^{(k-2)})^*} \dots dB(t_2)^{(1+n_1^{(0)} + \dots + n_{N+1}^{(0)})^*} dB(t_1)^{(1+x_1 + \dots + x_N)^*}.$$

Note that by definition $z_{i,j} := z_{N,j}$, $i = N+1, \dots, N+k$, $j = 1, \dots, k$.

We first evaluate the integral with respect to t_k , obtaining

$$E\{e^{-(\omega_1 \mathbf{S}_1 + \dots + \omega_k \mathbf{S}_k)} (z_{1,0}^{\mathbf{X}_1^{(0)}} \dots z_{N,0}^{\mathbf{X}_N^{(0)}}) \dots (z_{1,k}^{\mathbf{X}_1^{(k)}} \dots z_{N,k}^{\mathbf{X}_N^{(k)}}) \mid \mathbf{X}_1^{(0)} = x_1, \dots, \mathbf{X}_N^{(0)} = x_N\} =$$

$$z_{1,0}^{x_1} \dots z_{N,0}^{x_N} \int_{t_1=0}^{\infty} e^{-\omega_1 t_1} \int_{t_2=0}^{\infty} e^{-\omega_2 t_2} \dots \int_{t_{k-1}=0}^{\infty} e^{-\omega_{k-1} t_{k-1}}$$

$$\sum_{n_1^{(0)}=0}^{\infty} e^{-\lambda t_1} \frac{(\lambda t_1)^{n_1^{(0)}}}{n_1^{(0)}!} z_{1,1}^{n_1^{(0)}} \left[\prod_{j=1}^N \sum_{n_{j+1}^{(0)}=0}^{x_j} \binom{x_j}{n_{j+1}^{(0)}} p(j)^{n_{j+1}^{(0)}} (1-p(j))^{x_j - n_{j+1}^{(0)}} z_{j+1,1}^{n_{j+1}^{(0)}} \right]$$

$$\prod_{m=2}^{k-1} \left\{ \sum_{n_1^{(m)}=0}^{\infty} e^{-\lambda t_m} \frac{(\lambda t_m)^{n_1^{(m)}}}{n_1^{(m)}!} z_{1,m}^{n_1^{(m)}} \left[\prod_{j=1}^{N+m-1} \sum_{n_{j+1}^{(m)}=0}^{n_j^{(m-1)}} \binom{n_j^{(m-1)}}{n_{j+1}^{(m)}} p(j)^{n_{j+1}^{(m)}} (1-p(j))^{n_j^{(m-1)} - n_{j+1}^{(m)}} z_{j+1,m}^{n_{j+1}^{(m)}} \right] \right\}$$

$$[1 + \beta\{\omega_k + \lambda(1 - z_{1,k})\}]^{-(1+n_1^{(k-1)} + \dots + n_{N+k-1}^{(k-1)})} \prod_{j=1}^{N+k-1} (p(j)z_{j+1,k} + 1 - p(j))^{n_j^{(k-1)}}$$

$$dB(t_{k-1})^{(1+n_1^{(k-2)} + \dots + n_{N+k-2}^{(k-2)})^*} \dots dB(t_2)^{(1+n_1^{(0)} + \dots + n_{N+1}^{(0)})^*} dB(t_1)^{(1+x_1 + \dots + x_N)^*} =$$

$$\begin{aligned}
& z_{1,0}^{x_1} \cdots z_{N,0}^{x_N} A_k^N(1, \omega, z) \int_{t_1=0}^{\infty} e^{-\omega_1 t_1} \int_{t_2=0}^{\infty} e^{-\omega_2 t_2} \cdots \int_{t_{k-1}=0}^{\infty} e^{-\omega_{k-1} t_{k-1}} \\
& \sum_{n_1^{(0)}=0}^{\infty} e^{-\lambda_1} \frac{(\lambda_1)^{n_1^{(0)}}}{n_1^{(0)}!} z_{1,1}^{n_1^{(0)}} \left[\prod_{j=1}^N \sum_{n_{j+1}^{(0)}=0}^{x_j} \binom{x_j}{n_{j+1}^{(0)}} p(j)^{n_{j+1}^{(0)}} (1-p(j))^{x_j - n_{j+1}^{(0)}} z_{j+1,1}^{n_{j+1}^{(0)}} \right] \\
& \prod_{m=2}^{k-2} \left\{ \sum_{n_1^{(m)}=0}^{\infty} e^{-\lambda_m} \frac{(\lambda_m)^{n_1^{(m)}}}{n_1^{(m)}!} z_{1,m}^{n_1^{(m)}} \left[\prod_{j=1}^{N+m-1} \sum_{n_{j+1}^{(m)}=0}^{n_j^{(m-1)}} \binom{n_j^{(m-1)}}{n_{j+1}^{(m)}} p(j)^{n_{j+1}^{(m)}} (1-p(j))^{n_j^{(m-1)} - n_{j+1}^{(m)}} z_{j+1,m}^{n_{j+1}^{(m)}} \right] \right\} \\
& e^{-\lambda_{k-1}(1-z_{1,k-1})} A_k^N(1, \omega, z) [p(1)z_{2,k} + 1 - p(1)] \\
& \prod_{j=1}^{N+k-2} (A_k^N(1, \omega, z) [p(j+1)z_{j+2,k} + 1 - p(j+1)] p(j)z_{j+1,k-1} + 1 - p(j))^{n_j^{(k-2)}} \\
& dB(t_{k-1})^{(1+n_1^{(k-2)} + \dots + n_{N+k-2}^{(k-2)})^*} \cdots dB(t_2)^{(1+n_1^{(0)} + \dots + n_{N+1}^{(0)})^*} dB(t_1)^{(1+x_1 + \dots + x_N)^*}.
\end{aligned}$$

Next the integral with respect to t_{k-1} is evaluated, yielding

$$E\{e^{-(\omega_1 S_1 + \dots + \omega_k S_k)} (z_{1,0}^{X_1^{(0)}} \cdots z_{N,0}^{X_N^{(0)}}) \cdots (z_{1,k}^{X_1^{(k)}} \cdots z_{N,k}^{X_N^{(k)}}) \mid \mathbf{X}_1^{(0)} = x_1, \dots, \mathbf{X}_N^{(0)} = x_N\} =$$

$$\begin{aligned}
& z_{1,0}^{x_1} \cdots z_{N,0}^{x_N} A_k^N(1, \omega, z) \int_{t_1=0}^{\infty} e^{-\omega_1 t_1} \int_{t_2=0}^{\infty} e^{-\omega_2 t_2} \cdots \int_{t_{k-2}=0}^{\infty} e^{-\omega_{k-2} t_{k-2}} \\
& \sum_{n_1^{(0)}=0}^{\infty} e^{-\lambda_1} \frac{(\lambda_1)^{n_1^{(0)}}}{n_1^{(0)}!} z_{1,1}^{n_1^{(0)}} \left[\prod_{j=1}^N \sum_{n_{j+1}^{(0)}=0}^{x_j} \binom{x_j}{n_{j+1}^{(0)}} p(j)^{n_{j+1}^{(0)}} (1-p(j))^{x_j - n_{j+1}^{(0)}} z_{j+1,1}^{n_{j+1}^{(0)}} \right] \\
& \prod_{m=2}^{k-2} \left\{ \sum_{n_1^{(m)}=0}^{\infty} e^{-\lambda_m} \frac{(\lambda_m)^{n_1^{(m)}}}{n_1^{(m)}!} z_{1,m}^{n_1^{(m)}} \left[\prod_{j=1}^{N+m-1} \sum_{n_{j+1}^{(m)}=0}^{n_j^{(m-1)}} \binom{n_j^{(m-1)}}{n_{j+1}^{(m)}} p(j)^{n_{j+1}^{(m)}} (1-p(j))^{n_j^{(m-1)} - n_{j+1}^{(m)}} z_{j+1,m}^{n_{j+1}^{(m)}} \right] \right\} \\
& [1 + \beta\{\omega_{k-1} + \lambda - \lambda z_{1,k-1}\} A_k^N(1, \omega, z) [p(1)z_{2,k} + 1 - p(1)]]^{-(1+n_1^{(k-2)} + \dots + n_{N+k-2}^{(k-2)})} \\
& \prod_{j=1}^{N+k-2} (A_k^N(1, \omega, z) [p(j+1)z_{j+2,k} + 1 - p(j+1)] p(j)z_{j+1,k-1} + 1 - p(j))^{n_j^{(k-2)}} \\
& dB(t_{k-2})^{(1+n_1^{(k-3)} + \dots + n_{N+k-3}^{(k-3)})^*} \cdots dB(t_2)^{(1+n_1^{(0)} + \dots + n_{N+1}^{(0)})^*} dB(t_1)^{(1+x_1 + \dots + x_N)^*} =
\end{aligned}$$

$$\begin{aligned}
& z_{1,0}^{x_1} \cdots z_{N,0}^{x_N} A_k^N(1, \omega, z) A_k^N(2, \omega, z) \int_{t_1=0}^{\infty} e^{-\omega_1 t_1} \int_{t_2=0}^{\infty} e^{-\omega_2 t_2} \cdots \int_{t_{k-2}=0}^{\infty} e^{-\omega_{k-2} t_{k-2}} \\
& \sum_{n_1^{(0)}=0}^{\infty} e^{-\lambda_1} \frac{(\lambda_1)^{n_1^{(0)}}}{n_1^{(0)}!} z_{1,1}^{n_1^{(0)}} \left[\prod_{j=1}^N \sum_{n_{j+1}^{(0)}=0}^{x_j} \binom{x_j}{n_{j+1}^{(0)}} p(j)^{n_{j+1}^{(0)}} (1-p(j))^{x_j - n_{j+1}^{(0)}} z_{j+1,1}^{n_{j+1}^{(0)}} \right] \\
& \prod_{m=2}^{k-2} \left\{ \sum_{n_1^{(m)}=0}^{\infty} e^{-\lambda_m} \frac{(\lambda_m)^{n_1^{(m)}}}{n_1^{(m)}!} z_{1,m}^{n_1^{(m)}} \left[\prod_{j=1}^{N+m-1} \sum_{n_{j+1}^{(m)}=0}^{n_j^{(m-1)}} \binom{n_j^{(m-1)}}{n_{j+1}^{(m)}} p(j)^{n_{j+1}^{(m)}} (1-p(j))^{n_j^{(m-1)} - n_{j+1}^{(m)}} z_{j+1,m}^{n_{j+1}^{(m)}} \right] \right\}
\end{aligned}$$

$$\prod_{j=1}^{N+k-2} (A_k^N(2, \omega, z) [A_k^N(1, \omega, z) [p(j+1)z_{j+2,k} + 1 - p(j+1)] p(j)z_{j+1,k-1} + 1 - p(j)])^{n_j^{(k-2)}}$$

$$dB(t_{k-2})^{(1+n_1^{(k-3)}+\dots+n_{N+k-3}^{(k-3)})^*} \dots dB(t_2)^{(1+n_1^{(1)}+\dots+n_{N+1}^{(1)})^*} dB(t_1)^{(1+x_1+\dots+x_N)^*}.$$

We now sum over $n_1^{(k-2)}, \dots, n_{N+k-2}^{(k-2)}$ and subsequently integrate with respect to t_{k-2} , thus obtaining $A_k^N(3, \omega, z)$ terms; etc.; finally the summations over $n_1^{(1)}, \dots, n_N^{(1)}$ and the integration over t_1 are performed, which give rise to the $(f_k^N(i, \omega, z))^{x_i}$ contribution in Formula (3.4).