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Solar oblateness and Mercury's perihelion precession

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ABSTRACT

The Keplerian laws of planetary motion are solutions of the two-body gravitational problem. Solar oblateness resulting from the rotation of the Sun distorts the gravitational force acting on a planet and disturbs its Keplerian motion. An analytic solution of a planetary orbit disturbed by the solar gravitational oblateness is derived. In addition to short- and long-periodic disturbances there are secular disturbances, which lead to a perihelion precession and a nodal regression as well as to a mean-motion advance. The magnitude of the short-periodic perihelion precession could disturb observations of the secular effect if the survey is shorter then one Julian year. Transformation of formulae from the solar equatorial plane to the ecliptic plane is discussed. Numerical estimates of the secular perihelion precessions of Mercury, Venus and Mars are in good agreement with published results, confirming our theory. Inversely, the solar oblateness could be determined through observation of the perihelion precession of a planet. The solution is also valid for satellite orbits in the solar gravitational field.

Key words: gravitation – methods: analytical – methods: numerical – celestial mechanics – Earth – planets and satellites: general.

1 INTRODUCTION

The observed perihelion precession of Mercury's orbit is affected by the Earth's coordinate system, general relativity and planetary gravitational attractions, as well as by the solar gravitational oblateness (Shapiro 1999; Hill et al. 1974). The agreement between observation and theory was remarkably good around the 1910s (and this agreement has been accepted by the scientific community as confirmation of the validity of the theory of general relativity). However, the problem of how much of the perihelion advance results from solar oblateness remains and has been studied by numerous scientists during the past century (e.g. Gilvarry & Sturrock 1967; Sturrock & Gilvarry 1967; Boehme 1970; Dicke 1970; Campbell & Moffat 1983; Campbell et al. 1983; Dicke, Kuhn & Libbrecht 1987; Kuhn et al. 1998; Godier & Rozelot 1999; Shapiro 1999; Godier & Rozelot 2000; Milani et al. 2001; Rozelot, Godier & Lefebvre 2001; Pireaux & Rozelot 2003; Rozelot et al. 2004; Pireaux, Barriot & Rosenblatt 2006; Fivian et al. 2008, 2009; Kuhn, Emilio & Bush 2009; Wayte 2010). The theoretical value of Mercury's perihelion precession was estimated from the disturbed equation so far and restricted to secular effects, without the disturbing problem exactly solved. Therefore, derivations of analytic solutions of the equations of planetary motion disturbed by the solar gravitational oblateness are also important for investigating non-secular effects.

The rotation of the Sun introduces solar oblateness, which distorts the gravitational force acting on a planet and disturbs the planetary Keplerian orbit. This is in principle similar to the orbit of a satellite round the Earth being perturbed by the geopotential. Studies show that the solar oblateness is very small (0.00001, Fivian et al. 2008; 0.00005, Sturrock & Gilvarry 1967). Compared with the oblateness of the Earth (1/298.245642), the Sun is a very spherical mass-body. This indicates that a good enough accuracy might be achieved by taking only the J_2 term of the heliopotential into account. Methods to solve the equations of satellite motion disturbed by solar and lunar gravitation, atmospheric drag, solar radiation pressure and geopotential have been developed, and the solutions are given in Xu (2008) and Xu et al. (2010a,b). Hence, an analytic solution of a planetary orbit disturbed by the zonal heliopotential terms of J_k, $k = 2, 3, 4, \ldots$ can be similarly derived.

The disturbing function of the solar gravitational oblateness and the Lagrangian equations of planetary motion are discussed in Section 2. The solutions for the oblateness disturbance are derived, and coordinate transformation as well as numerical discussions are addressed in Section 3, which is followed by a concluding summary in Section 4.

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2 DISTURBING FUNCTION OF SOLAR OBLATENESS AND DISTURBED EQUATIONS OF MOTION

The heliopotential disturbing function, caused by the solar oblateness, can be written as (Vallado David 2007; equation 4.35 of Xu 2008; Lynden-Bell 2009; Pal 2009)

$$V = \frac{-\mu_s a_s^2}{2r^3} J_2(3\sin^2\phi - 1) = b\frac{1}{r^3}(3\sin^2\phi - 1),$$
$$b = \frac{-1}{2}\mu_s a_s^2 J_2. \tag{1}$$

Here, solar variables are denoted by the index 's', μ_s is the solar gravitational constant, a_s is the mean equatorial radius of the Sun, and r and ϕ are the heliocentric radius and latitude of a planet. J_2 is an un-normalized coefficient, which has different values in different papers owing to the different measuring technologies and theoretical models used (e.g. Campbell et al. 1983; Kuhn et al. 1998; Rozelot et al. 2004; Kuhn et al. 2009; Rozelot et al. 2011). J_2 values are, for example, $0 \sim 1.08 \times 10^{-5}$ (Kislik 1983), 1.46×10^{-7} (Fivian et al. 2008), 2×10^{-7} (Pireaux & Rozelot 2003; Pitjeva 2005) and 2.3×10^{-7} (Shapiro 1999). A list of J_2 values determined by different authors can be found in Pireaux & Rozelot (2003) and Pitjeva (2005). A review of the solar oblateness can be found in Rozelot et al. (2011).

Function (1) can be represented in Keplerian variables using the following relationships (Kaula 2001; equation 5.2 of Xu 2008):

$$\sin\phi = \sin i \sin u, \quad u = \omega + f, \tag{2}$$

$$r = \frac{a(1-e^2)}{1+e\cos f}. \tag{3}$$

The Keplerian elements, namely a, e, ω, i, Ω, M, f, are the semi-major axis, the eccentricity of the ellipse, the argument of perihelion, the inclination angle, the right ascension of the ascending node, the mean anomaly and the true anomaly, respectively.

It then follows that

$$\frac{\partial V}{\partial a} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial a} = \frac{-3}{a}\frac{b}{r^3}(3\sin^2 i \sin^2 u - 1),$$

$$\frac{\partial V}{\partial \Omega} = 0,$$

$$\frac{\partial V}{\partial i} = \frac{b}{r^3}6\sin i \sin^2 u \cos i,$$

$$\frac{\partial V}{\partial \omega} = \frac{b}{r^3}6\sin u \cos u \sin^2 i,$$

$$\frac{\partial V}{\partial e} = \frac{-3b}{r^4}(3\sin^2 i \sin^2 u - 1)\frac{\partial r}{\partial e} + \frac{6b}{r^3}\sin u \cos u \sin^2 i \frac{\partial u}{\partial e}$$

and

$$\frac{\partial V}{\partial M} = \frac{-3b}{r^4}(3\sin^2 i \sin^2 u - 1)\frac{\partial r}{\partial M} + \frac{6b}{r^3}\sin u \cos u \sin^2 i \frac{\partial u}{\partial M}. \tag{4}$$

Here, the partial derivatives are (Kaula 2001; equation 4.24 of Xu 2008)

$$\frac{\partial f}{\partial(e, M)} = \left(\frac{2+e\cos f}{1-e^2}\sin f, \left(\frac{a}{r}\right)^2\sqrt{1-e^2}\right),$$
$$\frac{\partial r}{\partial(a, e, \omega, i, \Omega, M)} = \left(\frac{r}{a}, -a\cos f, 0, 0, 0, \frac{ae\sin f}{\sqrt{1-e^2}}\right). \tag{5}$$

Using mathematical expansion formulae (Wang et al. 1979; Bronstein & Semendjajew 1987) yields ($L = 2, 3$, truncation to e^3)

$$\frac{1}{r^L} = \frac{(1+e\cos f)^L}{a^L(1-e^2)^L}$$
$$\approx \frac{1}{a^L}\left(1 + Le\cos f + \frac{L(L-1)}{2}e^2\cos^2 f + \frac{L(L-1)(L-2)}{6}e^3\cos^3 f\right)(1+Le^2)$$
$$\approx \frac{1}{a^L}\left((1+Le\cos f)(1+Le^2) + \frac{L(L-1)}{2}e^2\cos^2 f + \frac{L(L-1)(L-2)}{6}e^3\cos^3 f\right). \tag{6}$$

These formulae can be substituted into the following Lagrangian equations of planetary motion (Battin 1999; Kaula 2001; equation 4.11 of Xu 2008):

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial V}{\partial M},$$
$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1-e^2}{na^2e}\frac{\partial V}{\partial M} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial V}{\partial \omega},$$
$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial V}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial V}{\partial i},$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{\partial V}{\partial \omega} - \frac{\partial V}{\partial \Omega}\right),$$
$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial V}{\partial i},$$
$$\frac{\mathrm{d}M}{\mathrm{d}t} = n - \frac{2}{na}\frac{\partial V}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial V}{\partial e}, \tag{7}$$

where n is the mean angular velocity. The first term 'n' on the right-hand side of the last equation of (7) represents the Keplerian mean motion and will be omitted later on. The true anomaly f in the above formulae can be further transformed to the mean anomaly M using (see, for example, equation 5.22 of Xu 2008, truncated to e^3)

$$\sin f = \left(1 - \frac{7}{8}e^2\right)\sin M + \left(e - \frac{7}{6}e^3\right)\sin 2M + \frac{9}{8}e^2\sin 3M + \frac{8}{6}e^3\sin 4M,$$
$$\cos f + e = \left(1 - \frac{9}{8}e^2\right)\cos M + \left(e - \frac{4}{3}e^3\right)\cos 2M + \frac{9}{8}e^2\cos 3M + \frac{4}{3}e^3\cos 4M. \tag{8}$$

Using software for mathematical symbolic operations, equation (7) can be transformed and reduced to a Fourier series of the form

$$\frac{\mathrm{d}\sigma_j}{\mathrm{d}t} = d_j + \sum_{k=1}^{6}(g_{jk}\cos kM + c_{jk}\sin kM), \tag{9}$$

where σ_j is the jth Keplerian element, and d, g and c are functions of a, e, ω, i and Ω. All terms in M are short-periodic perturbations, and d-terms are long-periodic and linear ones. We define the individual factor h_j for the j th equation of (9) [i.e. (7)] as

$$h_1 = \frac{2b}{na^4} = 2h_6, \quad h_2 = h_3 = \frac{\sqrt{1-e^2}}{na^5 e}b, \quad h_4 = h_5 = \frac{b}{na^5\sqrt{1-e^2}}. \tag{10}$$

They are omitted later on and are related by

$$h_3 \approx h_6/ae \approx h_5/e. \tag{11}$$

3 SOLUTIONS FOR OBLATENESS DISTURBANCE

The short-periodic terms in equation (9) can be easily integrated with respect to M by using the relationship $M = nt$ and the mean value theorem for integration (Wang et al. 1979; Bronstein & Semendjajew 1987):

$$\int_0^T p(y(t))q(M(t))\mathrm{d}t = p(y(\xi))\int_0^T q(M(t))\,\mathrm{d}t, \quad 0 \le \xi \le T. \tag{12}$$

Here, functions p [i.e. g and c in (9)] and q [i.e. $\sin kM$ and $\cos kM$ in (9)] can be integrated. The time interval $[0, T]$ can be transformed into the mean-motion angular intervals of $\{[2(j-1)\pi/k, 2j\pi/k], j = 1, ...J\}$ and $[2J\pi/k, nT]$ (for any k, the integer J can be obtained from the relationship $nT - 2J\pi/k < 2\pi/k$). The integrals of the trigonometric functions q (short-periodic terms) are zero, except over the remaining (non-full cycle) intervals $[2J\pi/k, nT]$. Therefore, it holds that

$$\int_0^T p(y(t))q(M(t))\,\mathrm{d}t = p(y(\xi))\int_{2J\pi/k}^{nT} q(M)\frac{1}{n}\mathrm{d}M. \tag{13}$$

Here, $y(t)$ are slow-changing Keplerian elements (not including M). Selecting T so that p can be considered constant over $[2J\pi/k, nT]$ [i.e. (13) is valid for $\xi = T$)], the integral (13) is generally valid. For integration over time intervals longer than T, the integration can be made in a step-wise fashion and then accumulated.

The indefinite integrals (i.e. the solutions) of equation (9) disturbed by the solar gravitational oblateness are then

$$\Delta\sigma_j = \Delta d_j + \sum_{k=1}^{6}\frac{1}{kn}(g_{jk}\sin kM - c_{jk}\cos kM), \tag{14}$$

where Δd_j denotes symbolically the integrals of the secular and long-periodic terms. Or, explicitly (truncated to the order of e for terms of M and to e^2 for the others, with the factor h_j omitted),

$$\begin{aligned}\Delta a = {} & \frac{-3e}{8n}\sin^2 i\sin 2\omega\sin M + \frac{3}{4n}\sin^2 i\sin 2\omega\sin 2M \\ & + \frac{21e}{8n}\sin^2 i\sin 2\omega\sin 3M - \frac{3e}{2n}\left(1 - \sin^2 i\left(\frac{5}{4} + \frac{1}{2}\cos^2\omega\right)\right)\cos M \\ & - \frac{3}{4n}\sin^2 i\cos 2\omega\cos 2M - \frac{21e}{8n}\sin^2 i\cos 2\omega\cos 3M,\end{aligned} \tag{15}$$

$$\begin{aligned}\Delta e = {} & \frac{3e}{8n}\sin^2 i\sin 2\omega\sin M - \frac{3e}{2n}\left(1 - \sin^2 i\left(\frac{5}{4} + \frac{1}{2}\sin^2\omega\right)\right)\cos M \\ & + \frac{7e}{8n}\sin^2 i\sin 2\omega\sin 3M - \frac{7e}{8n}\sin^2 i\cos 2\omega\cos 3M,\end{aligned} \tag{16}$$

$$\begin{aligned}\Delta\omega = &\left(\frac{-9e}{8} - \frac{15}{8} e\cos 2i\right) t + \frac{1}{n}\left(\frac{-3}{2} + \sin^2 i\left(\frac{3}{4}\cos^2\omega + \frac{15}{8}\right)\right)\sin M \\ &+ \frac{3e}{4n}\left(-3 + \cos^2 i\cos 2\omega + \sin^2 i\left(5\cos^2\omega + \frac{1}{2}\right)\right)\sin 2M \\ &- \frac{7}{8n}\sin^2 i\cos 2\omega\sin 3M - \frac{51e}{16n}\sin^2 i\cos 2\omega\sin 4M \\ &+ \frac{3}{8n}\sin^2 i\sin 2\omega\cos M + \frac{3e}{4n}\left(1 + \frac{3}{2}\sin^2 i\right)\sin 2\omega\cos 2M \\ &- \frac{7}{8n}\sin^2 i\sin 2\omega\cos 3M - \frac{51e}{16n}\sin^2 i\sin 2\omega\cos 4M,\end{aligned} \tag{17}$$

$$\begin{aligned}\Delta i = &\frac{-3e}{8n}\sin 2i\sin 2\omega\sin M + \frac{3}{8n}\sin 2i\sin 2\omega\sin 2M \\ &+ \frac{7e}{8n}\sin 2i\sin 2\omega\sin 3M + \frac{3}{8n}\sin 2i\cos 2\omega\cos M \\ &- \frac{3}{8n}\sin 2i\cos 2\omega\cos 2M - \frac{7e}{8n}\sin 2i\cos 2\omega\cos 3M,\end{aligned} \tag{18}$$

$$\begin{aligned}\Delta\Omega = &\left(\frac{3}{2} + \frac{9}{4}e^2\right) t\cos i + \frac{3e}{4n}\cos i(2\cos^2\omega + 5)\sin M \\ &- \frac{3}{4n}\cos i\cos 2\omega\sin 2M - \frac{7e}{4n}\cos i\cos 2\omega\sin 3M \\ &+ \frac{3e}{4n}\cos i\sin 2\omega\cos M - \frac{3}{4n}\cos i\sin 2\omega\cos 2M \\ &- \frac{7e}{4n}\cos i\sin 2\omega\cos 3M,\end{aligned} \tag{19}$$

$$\begin{aligned}\Delta M = &\frac{-3}{8a}\left(\left(1 + \frac{21}{4}e^2\right) + \left(3 + \frac{3}{4}e^2\right)\cos 2i\right) t + \int \frac{3}{16a}e^2\sin^2 i\cos 2\omega\,\mathrm{d}t \\ &+ \frac{1}{an}\left(\frac{3}{2e} - \frac{87e}{16} + \sin^2 i\left(\left(\frac{-21}{8e} + \frac{351e}{32}\right)\cos^2\omega + \left(\frac{-15}{8e} + \frac{171e}{32}\right)\sin^2\omega\right)\right)\sin M \\ &+ \frac{1}{2an}\left(\frac{9}{2} + \sin^2 i\left(-15\cos^2\omega + \frac{3}{2}\sin^2\omega\right)\right)\sin 2M \\ &+ \frac{1}{an}\left(\frac{53e}{16} + \sin^2 i\left(\left(\frac{7}{8e} - \frac{1079e}{64}\right)\cos^2\omega + \left(\frac{-7}{8e} + \frac{443e}{64}\right)\sin^2\omega\right)\right)\sin 3M \\ &+ \frac{51}{16an}\sin^2 i\cos 2\omega\sin 4M + \frac{507e}{64an}\sin^2 i\cos 2\omega\sin 5M \\ &- \frac{1}{n}\sin^2 i\left(\frac{3}{8ae} - \frac{87e}{32a}\right)\sin 2\omega\cos M - \frac{33}{8na}\sin^2 i\sin 2\omega\cos 2M \\ &- \frac{1}{n}\sin^2 i\left(\frac{-7}{8ae} + \frac{761e}{64a}\right)\sin 2\omega\cos 3M + \frac{51}{16na}\sin^2 i\sin 2\omega\cos 4M \\ &+ \frac{507e}{64an}\sin^2 i\sin 2\omega\cos 5M.\end{aligned} \tag{20}$$

All Keplerian elements are subject to short-periodic disturbances. The mean anomaly is also disturbed long-periodically by (see equation 20; h_6 taken into account)

$$\Delta M = h_6 \int \frac{3}{16a}e^2\sin^2 i\cos 2\omega\,\mathrm{d}t = \frac{3h_6 e^2}{32an_\omega}\sin^2 i\sin 2\omega. \tag{21}$$

The integration is performed using the relationship $\omega = n_\omega t$ (n_ω is the secular motion of the perihelion). The magnitude of the secular disturbance (21) depends on the value of n_ω and will be discussed below. The secular effects are (see equations 17, 19, 20; h_j taken into account)

$$\Delta\omega = h_3\frac{3e}{8}(-3 - 5\cos 2i)t = n_\omega t, \tag{22}$$

$$\Delta\Omega = h_5\frac{3}{2}\left(1 + \frac{3}{2}e^2\right) t\cos i = n_\Omega t, \tag{23}$$

$$\Delta M = h_6\frac{-3}{8a}\left(\left(1 + \frac{21}{4}e^2\right) + \left(3 + \frac{3}{4}e^2\right)\cos 2i\right) t = n_M t. \tag{24}$$

Because of the small inclinations of solar planets, $\cos i$ and $\cos 2i$ are both positive. Recalling that the b in factor h_j is negative, (22)–(24) show that the solar oblateness (rotational ellipsoid form) will lead to a perihelion precession and an advancing of the mean motion of all planets. The right ascension of the ascending node will experience a retrograde motion. Taking the values of the factors h_j in (10) and their relationships (11) into account, the perihelion precession and the nodal regression as well as the mean motion advance are of the same order of magnitude.

Comparing equations (15)–(20) with (14), the coefficients of g and c can be easily obtained; therefore, the differential equation (9) is also explicitly given.

The above-derived solution is also valid for satellite motion in the solar gravitational field disturbed by solar oblateness.

3.1 Orbital variables in ecliptic and equatorial coordinate systems

The solutions (15)–(20) are given in the solar equatorial coordinate system, whereas the planetary orbital elements and the results (for example the observed perihelion precession from the Earth) are usually represented in the ecliptic coordinate system [denoted as $(a_1, e_1, \omega_1, i_1, \Omega_1, M_1)$]. The relationships of the Keplerian elements in the two above-mentioned coordinate systems can be obtained from the orbital sphere geometry (cf. Fig. 1, Wang et al. 1979; Bronstein & Semendjajew 1987):

$$\begin{aligned}
&(a, e, \omega, M) = (a_1, e_1, \omega_1 + \omega_2, M_1),\\
&\tan\left(\frac{\Omega+\omega_2}{2}\right) = \left(\cos\left(\frac{i_1-\varepsilon}{2}\right) \Big/ \cos\left(\frac{i_1+\varepsilon}{2}\right)\right) \tan\left(\frac{\Omega_1}{2}\right),\\
&\tan\left(\frac{\Omega-\omega_2}{2}\right) = \left(\sin\left(\frac{i_1-\varepsilon}{2}\right) \Big/ \sin\left(\frac{i_1+\varepsilon}{2}\right)\right) \tan\left(\frac{\Omega_1}{2}\right),\\
&\tan\left(\frac{\pi-i}{2}\right) = \left(\cos\left(\frac{\Omega-\omega_2}{2}\right) \Big/ \cos\left(\frac{\Omega+\omega_2}{2}\right)\right) \operatorname{ctan}\left(\frac{i_1+\varepsilon}{2}\right).
\end{aligned} \tag{25}$$

Here, ε is the inclination of the Sun's equator to the ecliptic and can be found in, for example, Bate et al. (1971). Therefore the orbital elements in the equatorial system can be obtained using (25), and the secular effects of (22)–(24) can be computed. Taking only the secular effects into account and making a full derivative operation on (25) yields

$$\begin{aligned}
&(\Delta\omega_1, \Delta M_1) = (\Delta\omega - \Delta\omega_2, \Delta M),\\
&\frac{\Delta\Omega+\Delta\omega_2}{2}\cos^{-2}\left(\frac{\Omega+\omega_2}{2}\right) = \frac{\Delta\Omega_1}{2}\left(\cos\left(\frac{i_1-\varepsilon}{2}\right) \Big/ \cos\left(\frac{i_1+\varepsilon}{2}\right)\right)\cos^{-2}\left(\frac{\Omega_1}{2}\right),\\
&\frac{\Delta\Omega-\Delta\omega_2}{2}\cos^{-2}\left(\frac{\Omega-\omega_2}{2}\right) = \frac{\Delta\Omega_1}{2}\left(\sin\left(\frac{i_1-\varepsilon}{2}\right) \Big/ \sin\left(\frac{i_1+\varepsilon}{2}\right)\right)\cos^{-2}\left(\frac{\Omega_1}{2}\right).
\end{aligned} \tag{26}$$

From the last two equations of (26), it follows that

$$\begin{aligned}
&\Delta\omega_2 = -\Delta\Omega\frac{\cos^{-2}\left(\frac{\Omega+\omega_2}{2}\right) - \beta\cos^{-2}\left(\frac{\Omega-\omega_2}{2}\right)}{\cos^{-2}\left(\frac{\Omega+\omega_2}{2}\right) + \beta\cos^{-2}\left(\frac{\Omega-\omega_2}{2}\right)},\\
&\beta = \operatorname{ctan}\left(\frac{i_1-\varepsilon}{2}\right) \Big/ \operatorname{ctan}\left(\frac{i_1+\varepsilon}{2}\right).
\end{aligned} \tag{27}$$

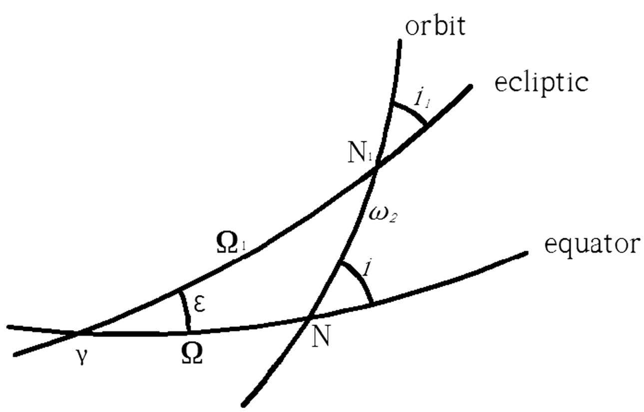

Figure 1. Orbital sphere geometry of the ecliptic and the equator. Here γ is the equinox of date; N and N_1 are ascending nodes of the planetary orbit on the equator and the ecliptic, respectively; and ε is the inclination of the solar equator to the ecliptic.

Using $\Delta\omega_2$ from (27), $\Delta\Omega_1$ can be computed using one of the last two equations of (26). The traditional perihelion precession is related to the vernal equinox of date and has the form (Campbell et al. 1983; Iorio 2005)

$$\Delta\bar{\omega}_1 = \Delta\omega_1 + \Delta\Omega_1 \cos i_1. \tag{28}$$

Then we have all the formulae of the final secular effects of ($\Delta\bar{\omega}_1$, $\Delta\Omega_1$, ΔM_1).

3.2 Secular perihelion precession of Mercury's orbit

The perihelion precession of Mercury's orbit can be computed as follows. The masses of the Sun and Mercury are 1.99×10^{30} and 3.30×10^{23} kg, and the gravitational constant is 6.67×10^{-11} m^3 kg^{-1} s^{-2}, so that $\mu_s \approx 1.33 \times 10^{20}$ and $\mu_{\mathrm{Mercury}} \approx 2.20 \times 10^{13}$ m^3 s^{-2}. In J2000.0, Mercury's Keplerian elements in ecliptic coordinates (Xu 2008) are: $a = 0.38709831$ au, $e = 0.20563175$, $i_1 = 7^\circ\!.0049860$, $\omega_1 = 29^\circ\!.1252260$, $\Omega_1 = 48^\circ\!.3308930$, the mean longitude $L_1 = 252^\circ\!.250906$. 1 au $= 1.49597870700 \times 10^{11}$ m, $a_s = 6.96 \times 10^8$ m, $t = 1$ (Julian century (Jc)) $= 36525 \times 86400$ s, $\varepsilon = 7^\circ\!.25$ (Bate et al. 1971). This yields $\omega_2 = 24^\circ\!.7748$, $\Omega = 23^\circ\!.8888$, $i = 13^\circ\!.0005$. The mean angular velocity n of Mercury is represented by Kepler's third law as

$$n = a^{-3/2}\mu^{1/2}. \tag{29}$$

Here, μ is the total gravitational constant of the Sun and Mercury. The secular effects of Mercury's orbit are thus (units: arcsec Jc^{-1})

$$\begin{aligned}\Delta\omega &= \mu_s^{1/2} a_s^2 J_2 \frac{3\sqrt{1-e^2}}{16} a^{-7/2}(3 + 5\cos 2i)t = n_\omega t \\ &\approx 3.38726 \times 10^{-5} J_2 t = 1.06894 \times 10^5 J_2,\end{aligned} \tag{30}$$

$$\begin{aligned}\Delta\Omega &= -\mu_s^{1/2} a_s^2 J_2 \frac{3}{4\sqrt{1-e^2}} a^{-7/2}\left(1 + \frac{3}{2}e^2\right) t\cos i = n_\Omega t \\ &\approx -1.95611 \times 10^{-5} J_2 t = -6.17301 \times 10^4 J_2,\end{aligned}$$

$$\Delta\Omega_1 = -1.25410 \times 10^5 J_2, \tag{31}$$

$$\Delta\omega_2 = -6.43262 \times 10^4 J_2,$$

$$\Delta\omega_1 = \Delta\omega - \Delta\omega_2 = 1.71220 \times 10^5 J_2,$$

$$\Delta M_1 = 5.87838 \times 10^4 J_2. \tag{32}$$

It follows that the perihelion precession of Mercury's orbit is

$$\Delta\bar{\omega}_1 = 2.95694 \times 10^5 J_2. \tag{33}$$

$J_2 = 1.47 \times 10^{-7}$ (Fivian et al. 2008) indicates a perihelion advance (related to the vernal equinox and the ecliptic plane) of 0.0435 arcsec Jc^{-1}; $J_2 = 2 \times 10^{-7}$ (Pireaux & Rozelot 2003; Pitjeva 2005) indicates a perihelion advance of 0.0591 arcsec Jc^{-1}; $J_2 = 2.3 \times 10^{-7}$(Shapiro 1999) indicates an advance of 0.0680 arcsec Jc^{-1}. The perihelion precessions related to references (30)–(33) are functions of J_2, which could be useful for determining the solar oblateness by surveying the perihelion precession of a planet's orbits.

3.3 Perihelion precessions of the orbits of Venus and Mars

For Venus, the gravitational constant and orbital Keplerian elements in J2000.0 are (Xu 2008): $\mu_{\mathrm{Venus}} \approx 3.248585 \times 10^{14}$ m^3 s^{-2}, $a = 0.72332982$ au, $e = 0.00677118$, $i_1 = 3^\circ\!.3946620$, $\omega_1 = 54^\circ\!.883787$, $\Omega_1 = 76^\circ\!.6799200$, $L_1 = 181^\circ\!.979801$, yielding $\omega_2 = 54^\circ\!.4478$, $\Omega = 22^\circ\!.4415$, $i = 8^\circ\!.68149$. The mean angular velocity n of Venus is represented by (29), where μ is the total gravitational constant of the Sun and Venus. The secular effects of Venus' orbit are then (units: arcsec Jc^{-1}):

$$\Delta\omega = 1.27015 \times 10^4 J_2, \tag{34}$$

$$\Delta\Omega = -6.46275 \times 10^3 J_2,$$

$$\Delta\Omega_1 = -2.83334 \times 10^4 J_2, \tag{35}$$

$$\Delta\omega_2 = -2.1895 \times 10^4 J_2,$$

$$\Delta\omega_1 = \Delta\omega - \Delta\omega_2 = 3.45964 \times 10^4 J_2,$$

$$\Delta M_1 = 6.31424 \times 10^3 J_2. \tag{36}$$

The perihelion precession of Venus' orbit follows:

$$\Delta\bar{\omega}_1 = 6.28801 \times 10^4 J_2. \tag{37}$$

$J_2 = 1.47 \times 10^{-7}$ (Fivian et al. 2008) indicates a perihelion advance (related to the vernal equinox and the ecliptic plane) of 0.00924 arcsec Jc^{-1}; $J_2 = 2 \times 10^{-7}$ (Pireaux & Rozelot 2003; Pitjeva 2005) indicates a perihelion advance of 0.01258 arcsec Jc^{-1}; and $J_2 = 2.3 \times 10^{-7}$(Shapiro 1999) indicates an advance of 0.01446 arcsec Jc^{-1}.

For Mars, the gravitational constant and orbital Keplerian elements in J2000.0 are (Xu 2008): $\mu_{\mathrm{Mars}} \approx 4.28283 \times 10^{13}$ m ^{3} s^{-2}, $a =$ 1.523679342 au, $e = 0.09340062$, $i_1 = 1\overset{\circ}{.}8497260$, $\omega_1 = 286\overset{\circ}{.}502141$, $\Omega_1 = 49\overset{\circ}{.}558093$, $L_1 = 355\overset{\circ}{.}4332750$, yielding $\omega_2 = 40\overset{\circ}{.}154$, $\Omega =$ $9\overset{\circ}{.}49336$, $i = 8\overset{\circ}{.}56572$. The mean angular velocity n of Mars is given by (29), where μ is the total gravitational constant of the Sun and Mars. The secular effects of the orbit of Mars are then (units: arcsec Jc^{-1}):

$$\Delta\omega = 9.32923 \times 10^2 J_2, \tag{38}$$

$$\Delta\Omega = -4.84852 \times 10^2 J_2,$$

$$\Delta\Omega_1 = -2.92718 \times 10^3 J_2, \tag{39}$$

$$\Delta\omega_2 = -2.44622 \times 10^3 J_2,$$

$$\Delta\omega_1 = \Delta\omega - \Delta\omega_2 = 3.37914 \times 10^3 J_2,$$

$$\Delta M_1 = 4.74181 \times 10^2 J_2. \tag{40}$$

The perihelion precession of Mars' orbit follows:

$$\Delta\bar{\omega}_1 = 6.3048 \times 10^3 J_2. \tag{41}$$

$J_2 = 1.47 \times 10^{-7}$ (Fivian et al. 2008) indicates a perihelion advance (related to the vernal equinox and the ecliptic plane) of 0.0009 arcsec Jc^{-1}; $J_2 = 2 \times 10^{-7}$ (Pireaux & Rozelot 2003; Pitjeva 2005) indicates a perihelion advance of 0.0013 arcsec Jc^{-1}; and $J_2 = 2.3 \times 10^{-7}$ (Shapiro 1999) indicates an advance of 0.0015 arcsec Jc^{-1}.

3.4 Comparisons with results of Iorio (2005)

The solutions of (15)–(20) are newly derived and include the effects of short- and long-periodic terms. The secular terms of (22)–(24), including (28), are nearly identical to equations (12)–(14) of Iorio (2005), except for differences of a coefficient of (1/2) and the factor functions of e. Our formulae are truncated to e^3, which indicates that the formulae of Iorio have a precision of e. It seems that Iorio omitted the difference between the solar equatorial plane and the ecliptic one. The coefficient -125410 of J_2 in (31) has very good agreement with the value -126878.626 given in table 2 of Iorio. The coefficient 58783.8 of J_2 in (32) is about 1/2 of the value of 123703.132 in Iorio because of the difference in the original formulae. The coefficient 295694 of J_2 in (33) is different from the value of 126404.437 in Iorio because of the difference between the equator and the ecliptic. In general, the agreements are very good. The disagreements show that the difference between the solar equator and the ecliptic plane has to be dealt with precisely, as outlined in this paper. Comparisons between the results computed for Venus and Mars and those given in Iorio show a systematic consistency with the case of Mercury stated above.

3.5 Short-periodic perihelion precession

For simplicity, the following discussions consider the solar equator as the ecliptic. Because $\sin^2 i_1$ (≈ 0.015 and $e^2 \approx 0.042$) can be omitted in (17), the short-periodic perihelion precession is dominated by

$$\begin{aligned}\Delta\omega &= \frac{3h_3}{2n}\left(-\sin M - \frac{e}{2}((3-\cos^2 i\cos 2\omega)\sin 2M - \sin 2\omega\cos 2M)\right)\\ &= \frac{3a_s^2 J_2}{4a^2 e}\left(\sin M + \frac{e}{2}((3-\cos^2 i\cos 2\omega)\sin 2M - \sin 2\omega\cos 2M)\right)\\ &\approx 108.673 J_2(\sin M + 0.25515\sin 2M - 0.08743\cos 2M).\end{aligned} \tag{42}$$

Comparing the magnitude of (42) with the mean velocity of the perihelion (22), it can be seen that the magnitude of the short-periodic effect is much larger than the secular one (by a factor of 3×10^6). Within one Julian year, the perihelion precession is 0.000254 arcsec. For $J_2 = 2.26804 \times 10^{-7}$ the amplitude of (42) is 0.0000246 arcsec and this is about 1/10 of the yearly advance. Therefore, observations for the perihelion precession must be performed for a long enough time to be able to separate the short-periodic effects from the secular effects.

3.6 Secular mean-motion advancing

From (24) it follows that

$$\begin{aligned}n_M &= \frac{3na_s^2 J_2}{16a^2}\left(\left(1+\frac{21}{4}e^2\right) + \left(3+\frac{3}{4}e^2\right)\cos 2i\right)\\ &\approx 1.15185 \times 10^{-4} n J_2,\end{aligned} \tag{43}$$

where, n_M is a correction to Kepler's third law (29). Owing to the small J_2, the correction (43) is negligible.

3.7 Long-periodic mean-motion disturbance

From (21), the unique long-periodic effect of the solar oblateness, that is, the mean-motion disturbance, has the form

$$\Delta M = \frac{-3na_{\mathrm{s}}^2 J_2}{64a^2 n_\omega} e^2 \sin^2 i \sin 2\omega. \tag{44}$$

Taking (22) into account, it follows that

$$\begin{aligned} \Delta M &= \frac{-e^2 \sin^2 i}{4\sqrt{1-e^2}(3+5\cos 2i)} \sin 2\omega \\ &\approx -2.04629 \times 10^{-5} \sin 2\omega \text{ (rad)} \\ &= -4.22078 \sin 2\omega \text{ (arcsec)}. \end{aligned} \tag{45}$$

A magnitude of −4.22078 arcsec is a notable mean-motion effect.

4 SUMMARY

The analytic solutions of a planetary orbit disturbed by the solar gravitational oblateness are derived, including the short- and long-periodic terms. The secular terms are then used to study the perihelion precessions of Mercury, Venus and Mars. It is notable that the amplitudes of the short-periodic terms are much larger than the secular (linear) ones, which indicates that care has to be taken in the determination of J_2 through planetary orbit observation, and the solutions derived in this paper have to be used to fit the data. Comparison shows that the difference between the solar equatorial plane and the ecliptic plane is not negligible.

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