Original Article

Solitary wave solutions of the MRLW equation using a spatial five-point stencil of finite difference approximation

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Abstract
This paper proposes a finite difference scheme with a three-level time and a five-point stencil in space to solve an initial boundary value problem for the MRLW equation. The scheme is shown to be marginally stable and convergent with a fourth-order convergence in the space direction and a second-order convergence in the time variable direction with regard to the maximum norm. The conservation properties of the proposed scheme are assessed using the three motion invariants for mass, momentum, and energy. To validate the theoretical results, numerical experiments are given for both single and interaction of two and three solitary waves.

Keywords Finite difference method · Modified regularized long wave equation · Stability analysis · Convergence rate · Solitary waves interaction

Mathematics Subject Classification 65M06 · 65M12 · 65M15

1 Introduction

Solitary waves, or solitons as they are also known, are nonlinear waves that have the ability to propagate through media over a prolonged period while maintaining key properties; for example, velocity and shape [1]. As solitons remain stable when they collide with other solitons, they have found widespread application in a range of areas including optics, fluid mechanics, finance, biology, physics, engineering sciences, and neuroscience. Nonlinear partial differential equations are used to model solitons. For example, the regularized long wave (RLW) equation, which was originally presented by Peregrine [2] and Benjamin et al. [3], is as follows:

$$u_t + u_x + \delta u u_x - \mu u_{xxt} = 0, \quad x \in \mathfrak{R}, \quad 0 \le t \le T, \tag{1}$$

where δ and μ are positive constants. It is considered with the homogeneous Dirichlet boundary conditions $u \to 0$ as $x \to \pm\infty$.

The RLW equation is particular case of the generalized long wave (GRLW) equation which has the form

$$u_t + u_x + \delta u^p u_x - \mu u_{xxt} = 0, \tag{2}$$

where p is a positive integer by setting $p = 1$. In our paper, we consider another particular case of the GRLW equation called the modified regularized long wave (MRLW) equation when $p = 2$ and is given by

$$u_t + u_x + \delta u^2 u_x - \mu u_{xxt} = 0. \tag{3}$$

The GRLW equation and its particulars cases: RLW equation and MRLW equation were initially suggested as a means of

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modeling phenomena that exhibited dispersion waves in combination with weak nonlinearity; for example, pressure waves in a liquid gas bubble mixture, ion-acoustic and magnetohydrodynamic waves in plasma, phonon packets in nonlinear crystals, and nonlinear transverse waves in shallow water. The analytical solution that underpins the MRLW equation is limited to single solitary waves under the restricted initial and boundary conditions. Scholars have yet to develop formulae for other cases, for example, within the context of the Maxwellian initial condition and situations involving the interaction of more than one soliton. As such, the development of accurate numerical approximations remains critical such as finite difference methods [4–9], finite element method [10–13], mixed finite element method [14, 15], collocation method [16–18], spectral method [19, 20] and Adomian decomposition method [21, 22].

Numerous finite difference schemes for the MRLW problem have been documented in the extant literature, with many scholars placing a specific focus on the finite difference methods and the MRLW equation. The MRLW problem, for instance, was solved using the finite difference approach by Khalifa et al. [6], who also investigated other aspects of the MRLW equation, including the interaction of solitary waves. Additionally, Fourier analysis was performed to demonstrate the stability of the scheme. The truncation error was also well controlled.

The generalized regularised long wave (GRLW) problem was solved by Akbari and Mokhtari using a new compact finite difference method (CFDM) [23]. The stability analysis of the energy method was explored, and an error estimate was presented. The method was validated using two solitary waves interaction and the propagation of single solitons. To ascertain the method conservation properties, three motion invariants were assessed.

A fully implicit finite difference technique for the numerical solution of the MRLW equation was presented by Inan and Bahadir [24]. The validity of the approach was tested using several MRLW equation examples. A comparison of the results with analytical and other numerical invariants demonstrated the accuracy and dependability of the outcomes achieved utilizing the fully implicit finite difference scheme.

In the current study, we suggest a finite difference approach to solve the MRLW equation that has three levels in time and a five-point stencil in space. The stability of the Fourier analysis-based method is considered, and the accuracy of the convergence rate of $O(h^4 + k^2)$ is also proved. The remainder of this paper is structured as follows. The analytical solution of the MRLW equation and its conservative laws are reviewed in Sect. 2. The creation of the suggested scheme is the focus of Sect. 3. The stability and convergence rates of the scheme are examined in Sect. 4. To validate our theoretical findings, Sect. 5 presents various numerical experiments for single and interaction of solitons. Finally, our concluding remarks are contained in Sect. 6.

2 Analytical solution and conservation laws

The analytical solution of the MRLW equation Eq. (3) is given in the form [6]

$$u(x,t) = \sqrt{\frac{6c}{\delta}} sech\left(\sqrt{\frac{c}{\mu(c+1)}}(x-(c+1)t-x_0)\right), \tag{4}$$

where $\sqrt{\frac{6c}{\delta}}$ is the amplitude of the MRLW solitary wave. The solitary wave is initially centred at x_0 and its speed and its width are represented by c and $\sqrt{\frac{c}{\mu(c+1)}}$, respectively.

The validity of the numerical methods can be determined using the three invariants of the motion that the MRLW equation has; that is, the mass, momentum, and energy conservative laws given as [6, 25]:

$$I_1 = \int_{-\infty}^{\infty} u(x,t)dx, \tag{5}$$

$$I_2 = \int_{-\infty}^{\infty} (u^2 + \mu u_x^2)dx, \tag{6}$$

and

$$I_3 = \int_{-\infty}^{\infty} (u^4 - \frac{6\mu}{\delta} u_x^2)dx. \tag{7}$$

The use of these invariants can be particularly pertinent in situations for which there are no analytic solutions or during soliton interactions [25].

3 Construction of the finite difference scheme

To construct the finite difference scheme of the MRLW equation Eq. (3) with a three-level scheme in time and a five-point stencil in space, the following notations for the derivatives are used:

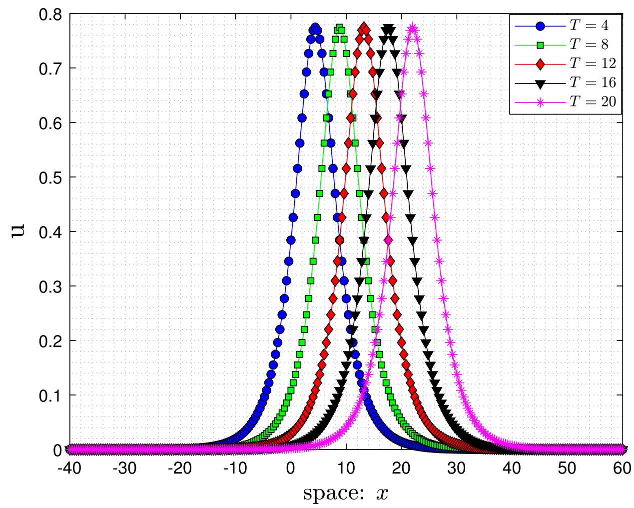

Fig. 1 Single solitary wave on $[-40, 60]$ with $h = 0.4, k = 0.05, c = 0.1$ at times: $T = 4, T = 8, T = 12, T = 16$ and $T = 20$

$$\begin{aligned}(u_j^n)_t &= \frac{u_j^{n+1} - u_j^{n-1}}{2k}\\ (u_j^n)_x &= \frac{-u_{j+2}^n + 8u_{j+1}^n - 8u_{j-1}^n + u_{j-2}^n}{12h}\\ (u_j^n)_{xxt} &= \frac{-u_{j+2}^{n+1} + 16u_{j+1}^{n+1} - 30u_j^{n+1} + 16u_{j-1}^{n+1} - u_{j-2}^{n+1}}{24kh^2}\\ &\quad + \frac{u_{j+2}^{n-1} - 16u_{j+1}^{n-1} + 30u_j^{n-1} - 16u_{j-1}^{n-1} + u_{j-2}^{n-1}}{24kh^2},\end{aligned} \tag{8}$$

where $k = \Delta t$ is the time step and $h = \Delta x$ represents the spatial step size. The superscript n denotes a quantity associated with time level t_n and subscript j denotes a quantity associated with space mesh point x_j. The grid points are $t_n = nk, n = 0, 1, 2, \dots, N$ for time and $x_j = jh, j = 0, 1, 2, \dots, M$ for space, where M and N are positive integers. The finite difference scheme therefore becomes

$$(u_j^n)_t + (1 + \delta(u_j^n)^2)(u_j^n)_x - \mu(u_j^n)_{xxt} = 0. \tag{9}$$

Then substituting Eq. (8) into Eq. (9) yields

$$\begin{aligned}&\mu u_{j-2}^{n+1} - 16\mu u_{j-1}^{n+1} + (30\mu + 12h^2)u_j^{n+1} - 16\mu u_{j+1}^{n+1} + \mu u_{j+2}^{n+1}\\ &\quad = -Q_j^n u_{j-2}^n + 8Q_j^n u_{j-1}^n - 8Q_j^n u_{j+1}^n + Q_j^n u_{j+2}^n\\ &\qquad + \mu u_{j-2}^{n-1} - 16\mu u_{j-1}^{n-1} + (30\mu + 12h^2)u_j^{n-1} - 16\mu u_{j+1}^{n-1} + \mu u_{j+2}^{n-1},\end{aligned} \tag{10}$$

where $Q_j^n = kh(1 + \delta(u_j^n)^2)$. The scheme (10) is a tridiagonal system that can be easily simulated using MATLAB platform. From now in our work, we take $\delta = 1$.

4 Convergence and linear stability analysis

Lemma 1 *The finite difference scheme* (10) *is marginally stable.*

Proof In the case of applying the Von Neumann stability theory, the solution of Eq. (10) can be written as

$$u_j^n = \xi^n e^{ilx_j}, \; i = \sqrt{-1}, \tag{11}$$

where l is a mode number. Now, set

$$\xi^{n+1} = g\xi^n \Longrightarrow \xi^{n+1} = g^2\xi^{n-1}, \tag{12}$$

and then inserting Eqs. (11) and (12) into Eq. (10) yields

$$g^2 - (2i\sin\theta)g - 1 = 0, \tag{13}$$

where g is the growth factor and

$$\sin\theta = \frac{kh(1 + u^2)(\sin 2\phi - 8\sin\phi)}{2\mu\cos 2\phi - 32\mu\cos\phi + 30\mu + 12h^2}, \quad \phi = lh, \tag{14}$$

with assuming that $(u_j^n)^2$ in Eq. (10) is locally constant, and for simplicity, we write it as u^2. To verify Eq. (14), we must show that

$$\left|\frac{kh(1 + u^2)(\sin 2\phi - 8\sin\phi)}{2\mu\cos 2\phi - 32\mu\cos\phi + 30\mu + 12h^2}\right| \le 1.$$

To do this, we have

Table 1 Invariants and errors for single solitary wave on $[-40, 60]$ with $h = 0.4, k = 0.05, c = 0.1$

Time	L_∞-error	L_2-error	I_1	I_2	I_3
4	$2.81713E-05$	$6.40273E-05$	8.0708808	4.1005468	0.8683974
8	$5.45539E-05$	$1.26491E-04$	8.0708795	4.1005468	0.8683974
12	$7.81422E-05$	$1.85598E-04$	8.0708701	4.1005468	0.8683974
16	$9.93777E-05$	$2.41313E-04$	8.07084851	4.1005468	0.8683974
20	$1.182044E-04$	$2.93952E-04$	8.0707907	4.1005468	0.8683974

Table 2 Invariants and errors for single solitary wave on $[-80, 120]$ with $h = 0.4, k = 0.05, c = 0.03$

Time	L_∞-error	L_2-error	I_1	I_2	I_3
4	$4.11986E-06$	$1.21516E-05$	7.8098623	2.1298427	0.1305175
8	$8.40787E-06$	$2.42824E-05$	7.8098546	2.1298427	0.1305175
12	$1.26799E-05$	$3.61161E-05$	7.8098476	2.1298427	0.1305175
16	$1.66021E-05$	$4.77528E-05$	7.8098409	2.1298427	0.1305175
20	$2.09146E-05$	$5.92125E-05$	7.8098325	2.1298427	0.1305175

$$\left|\frac{kh(1+u^2)(\sin 2\phi - 8\sin\phi)}{2\mu\cos 2\phi - 32\mu\cos\phi + 30\mu + 12h^2}\right| = \left|\frac{kh(1+u^2)\sin\phi(\cos\phi - 4)}{16\mu(1-\cos\phi) - 2\mu\sin^2\phi + 6h^2}\right|.$$

Now, we set $y = 1 - \cos\phi$ and so $\sin\phi = \sqrt{y(2-y)}$. Then we have

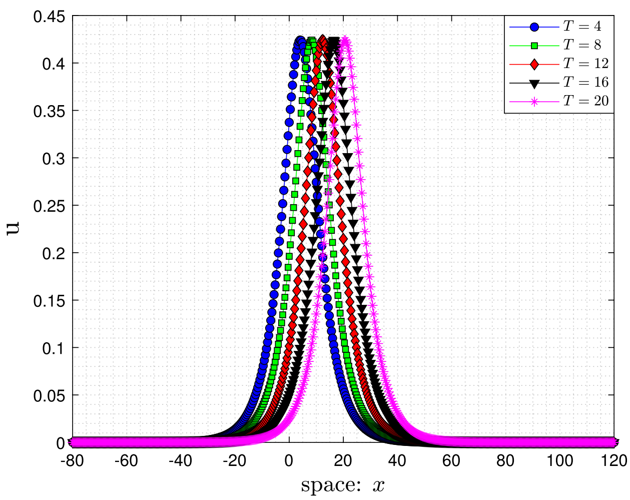

Fig. 2 Single solitary wave on $[-80, 120]$ with $h = 0.4, k = 0.05, c = 0.03$ at times: $T = 4, T = 8, T = 12, T = 16$ and $T = 20$

$$\sup_{\forall\phi}\frac{k^2h^2(1+u^2)^2\sin^2\phi(\cos\phi-4)^2}{(16\mu(1-\cos\phi)-2\mu\sin^2\phi+6h^2)^2} = \sup_{0\le y\le 2}\frac{k^2h^2(1+u^2)^2y(2-y)(3+y)^2}{(16\mu y-2\mu(y(2-y))+6h^2)^2} = \frac{16k^2h^2(1+u^2)^2}{(14\mu+6h^2)^2} \le 1,$$

for small spatial step size h and small time step k, and μ is practically taken to be unity.

Now, Eq. (13) yields that $g_1 = -e^{i\theta}$ and $g_2 = e^{-i\theta}$, which implies that $\|g_1\| = \|g_2\| = 1$, and therefore the finite difference scheme (10) is marginally stable. □

Lemma 2. *If $u(x,t)$ is smooth enough, then the local truncation error of finite difference scheme (10) is $O(h^4+k^2)$.*

Proof Let $v_j^n = v(x_j, t_n)$ represents the exact solution for the Eq. (3) with independent variables x and t. The local truncation error of Eq. (9) is thus as follows:

$$T_j^n = (v_j^n)_t + (1+(v_j^n)^2)(v_j^n)_x - \mu(v_j^n)_{xxt}. \quad (15)$$

Now, using Tylor expansion, it is easily shown that T_j^n at the point (x_j, t_n) can be written as

$$\begin{aligned} T_j^n =& [v_t + (1+(v)^2)v_x - \mu v_{xxt}]\|_{(x_j,t_n)} + \frac{k^2}{6}v_{ttt}\|_{(x_j,t_n)} \\ & - \frac{h^4}{30}[(1+v^2)v_{xxxxx}]\|_{(x_j,t_n)} + \cdots, \end{aligned} \quad (16)$$

and hence we have

$$T_j^n = O(h^4+k^2). \quad (17)$$

□

Table 3 Comparison of our results with some of recent results using maximum error ($L_\infty \times 10^3$) for single solitary wave on $[0, 100]$ with $h = 0.2$, $k = 0.025, c = 1, x_0 = 40$ and $\delta = 6$

Time	Proposed method	Conservative FDM [9]	Subdomain FEM [12]	Petrov Galerkin method [13]	Conservative linerized FDM [8]
2	0.507914	1.07385	1.1080	1.190456	0.50356
4	0.688228	2.03322	2.077	1.222450	0.82373
6	0.846081	2.97460	3.045	1.198936	1.13199
8	0.924369	3.91054	4.009	1.150862	1.43811
10	1.116198	4.84457	4.971	1.079686	1.64849

Table 4 The convergence rates in space and maximum errors at $T = 20$ for single solitary wave of MRLW equation on $[-40, 60]$ with $c = 0.1$ and $x_0 = 0$

h	k	L_∞-error	Convergence rate
0.8	0.8	0.0704949	
$\frac{h}{2}$	$\frac{k}{4}$	0.0037656	4.22658
$\frac{h}{4}$	$\frac{k}{16}$	0.0002348	4.00336
$\frac{h}{8}$	$\frac{k}{64}$	0.0000147	3.99398

Table 5 The convergence rates in time and maximum errors at $T = 20$ for single solitary wave of MRLW equation on $[-40, 60]$ with $c = 0.1$ and $x_0 = 0$

h	k	L_∞-error	Convergence rate
0.8	0.8	0.0704949	
$\frac{h}{4}$	$\frac{k}{2}$	0.0160614	2.13392
$\frac{h}{16}$	$\frac{k}{4}$	0.0039098	2.03842
$\frac{h}{64}$	$\frac{k}{8}$	0.0009716	2.00863

5 Numerical experiments

In this section, we present some numerical experiments to verify our theoretical results obtained in the previous section. The accuracy of the proposed scheme is measured using the L_∞ and L_2 errors at $t = t_N$ that are approximated by

$$L_\infty = \max_{1\le j\le M} \|v(x_j, t_N) - u(x_j, t_N)\|, \tag{18}$$

$$L_2 = \sqrt{h\sum_{j=1}^{M}(v(x_j, t_N) - u(x_j, t_N))^2}. \tag{19}$$

In addition, the invariants of mass, momentum, and energy for the MRLW equation are calculated for a single soliton and during the interaction of two and three solitary waves to measure the conservation properties of the proposed scheme.

5.1 Motion of single solitary waves

With $h = 0.4$ and $k = 0.05$ fixed, two experiments were performed to demonstrate the viability of our scheme in the situation of a single soliton motion. Calculations up to $T = 20$ were performed. In line with the work of [23], the model parameters were selected as $x_0 = 0, c = 0.1$ with range $[-40, 60]$ for the first example and $x_0 = 0, c = 0.03$ with range $[-80, 120]$ for the second example. Tables 1 and 2 show the values of L_∞ and L_2 errors and the invariants I_1, I_2, and I_3. The invariant I_1 is changed by less than 10^{-5} in both circumstances, whilst the changes for the invariants I_2 and I_3 approach zero, demonstrating the reasonable conservatism of our proposed scheme. At $T = 20$, the errors for the first example are reasonably minimal, at 1.182044×10^{-4} for L_∞ error and 2.93952×10^{-4} for L_2 error. Given that our approach is highly accurate, similar results are obtained for the second example as 2.09146×10^{-5} for L_∞ error and 5.92125×10^{-5} for L_2 error. In Fig. 1 for the first example, the motion of the single wave is plotted at various time levels with an amplitude of 0.3, and in Fig. 2 for the second case, with an amplitude of 0.17. The fact that the waves at $t = 16$ and $t = 20$ adequately agree with those at $t = 4$ demonstrates the reliability and accuracy of our scheme.

Furthermore, we compare our results in terms of maximum errors with the obtained results in [8, 9, 12, 13] to examine the validity of our scheme. For the purpose of comparisons, the parameters are chosen as $h = 0.2$, $k = 0.025$, $c = 1$, $x_0 = 40$, and $\delta = 6$, with a range [0, 100]. The computations are performed up to $T = 10$ and are listed in Table 3. It is clearly observed that L_∞-errors obtained by our scheme are marginally smaller than those obtained by others, indicating that our scheme is more accurate.

5.2 Convergence rate

To calculate the convergence rates in space and in time, we use the following formula [23]

$$Order = \log_2\left(\frac{L_\infty(h,k)}{L_\infty(\frac{h}{2}, \frac{k}{4})}\right)$$

for the convergence rate in space, and

$$Order = \log_2\left(\frac{L_\infty(h,k)}{L_\infty(\frac{h}{4}, \frac{k}{2})}\right)$$

Table 6 Invariants for interaction of two solitary waves on [0, 300] with $h = 0.4, k = 0.05, c_1 = \frac{4}{21}, c_2 = \frac{9}{91}$

Time	I_1	I_2	I_3
0	16.4507739	10.2223550	3.5691135
40	16.4803713	10.2229476	3.5658059
80	16.4800934	10.2223531	3.5608808
120	16.4798642	10.2234672	3.5613810
160	16.4796309	10.2228295	3.5667504
200	16.4793775	10.2224818	3.5697989

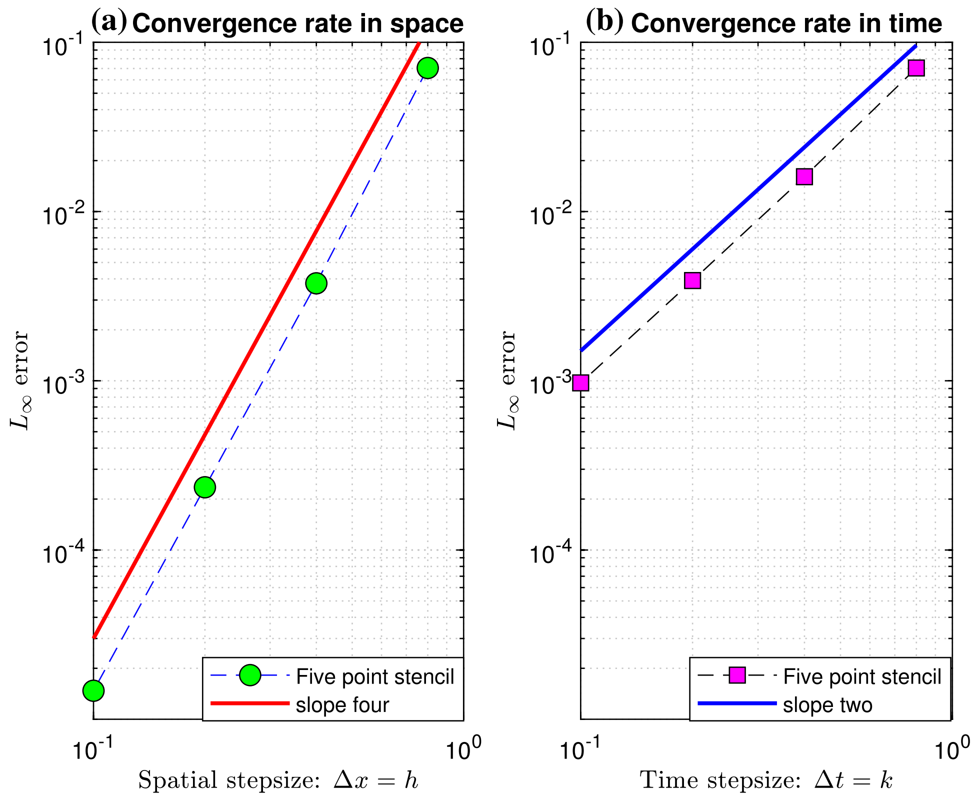

Fig. 3 **a** The L_∞ errors with respect to h at time $T = 20$ with parameters chosen as in Table 4. **b** The L_∞ errors with respect to k at time $T = 20$ with parameters chosen as in Table 5

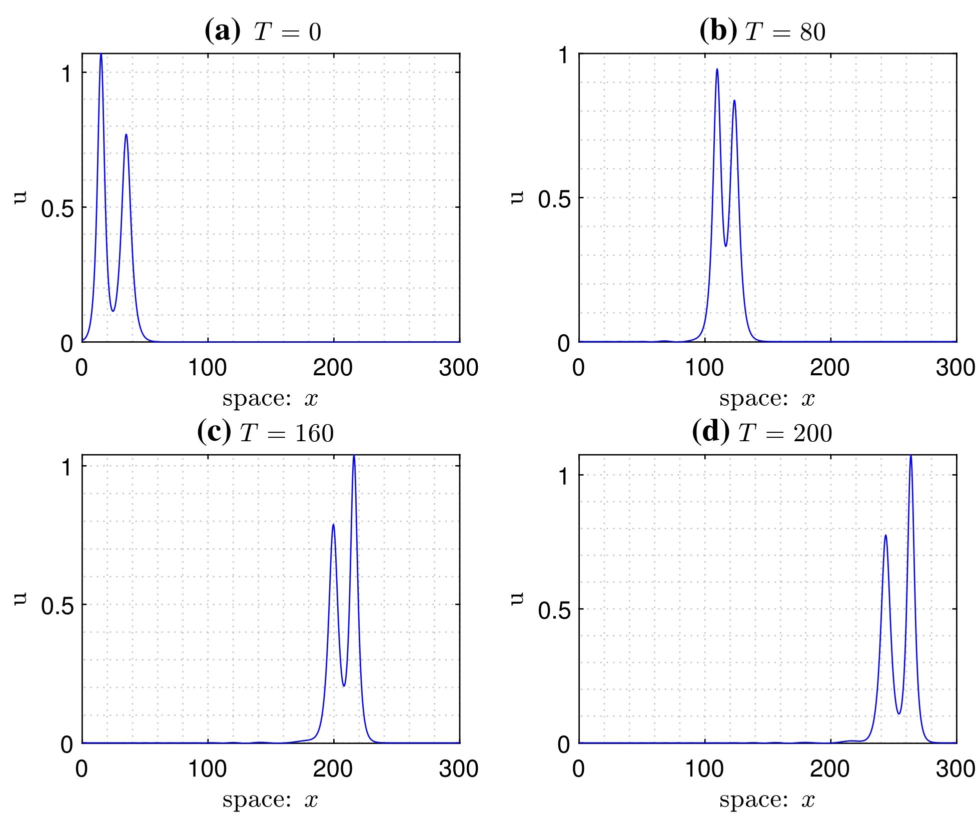

Fig. 4 Interaction of two solitary waves on [0, 300] with $h = 0.4, k = 0.05$ at times: $T = 0, T = 80, T = 160, T = 200$

Table 7 Invariants for interaction of three solitary waves on [0, 300] with $h = 0.4, k = 0.05, c_1 = \frac{4}{21}, c_2 = \frac{9}{91}, c_3 = \frac{15}{301}$

Time	I_1	I_2	I_3
0	24.3354551	13.4330475	4.1523353
40	24.3652457	13.4340675	4.1466276
80	24.3648005	13.4350644	4.1384570
120	24.3645399	13.4356560	4.1336719
160	24.3642525	13.4354500	4.1353112
200	24.3612567	13.4346203	4.1420136

for the convergence rate in time, with respect to the maximum norm errors. With the other parameters remaining the same as in Table 1, we started with $h = k = 0.8$, and reduced the spatial and temporal variables by 2 and 4, respectively, to calculate the spatial convergence rates. The resultant L_∞ errors and the corresponding convergence rates of our scheme are recorded in Table 4, which reveals that the fourth order of convergence in the spatial direction was achieved. Table 5, in contrast, displays the temporal convergence rates, as we first set $h = k = 0.8$, then scaled them back by a factor of 2 for k and 4 for h. Other parameters are chosen as in Table 1. Based on the proposed scheme, the accuracy of order 2 in the temporal direction is obtained. These spatial and temporal rates are aligned with the theoretical conclusions in Lemma 2.

Additionally, the log-log scale depicted in Fig. 3. shows the resultant L_∞ errors on the solitary wave solutions with regard to spatial variable h and temporal variable k. In Fig. 3a and b, lines of slope 4 and 2 are added as references. As can be observed in these figures, our scheme achieved an $O(h^4 + k^2)$ accuracy, coinciding with the theoretical predictions.

5.3 Interaction of solitary waves

The interaction of two and three solitary waves travelling in the same direction is discussed in this section. The initial conditions in these scenarios can be described by a linear sum of two and three well-separated solitary waves of different amplitudes, as follows:

$$u(x,0) = \sum_{i=1}^{p} \sqrt{6c_i} sech\left(\sqrt{\frac{c_i}{\mu(c_i+1)}}(x - x_i)\right), \tag{20}$$

where $p = 2$ and $p = 3$ for the interaction of two and three solitary waves, respectively. We performed the simulations up to $T = 200$ and on the range [0, 300], with fixed $h = 0.4$ and $k = 0.05$ to enable the interaction to take place. Two solitary waves with $c_1 = \frac{4}{21}$, $c_2 = \frac{9}{91}$, $x_1 = 15$ and $x_2 = 35$ interacted at various time levels, as shown

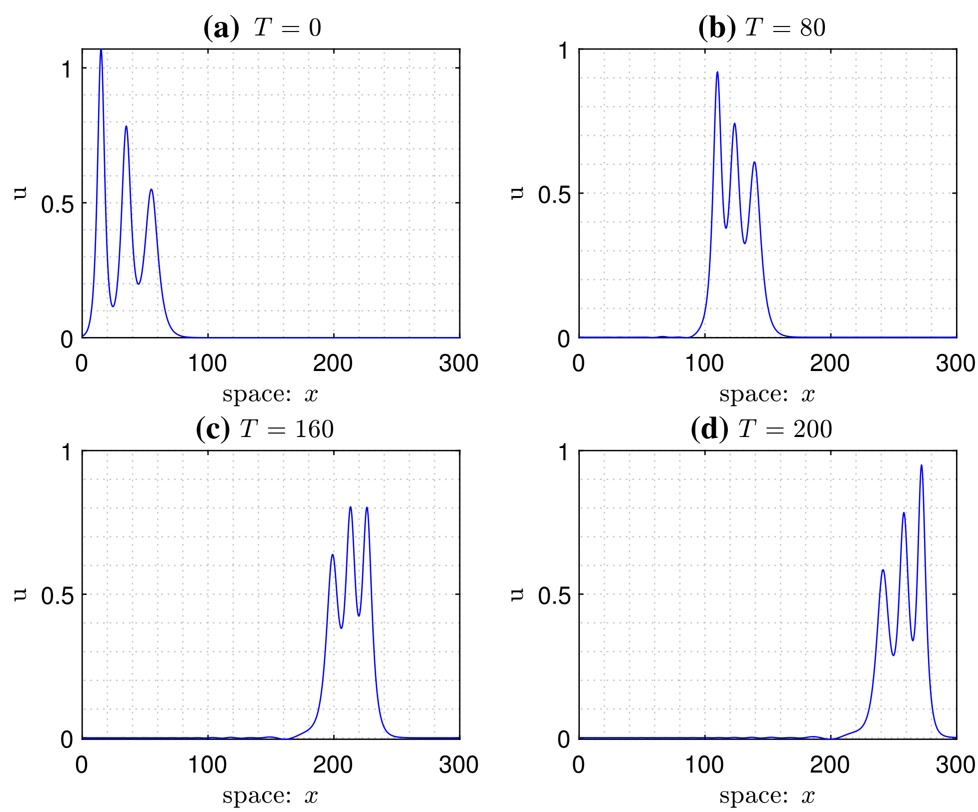

Fig. 5 Interaction of three solitary waves on [0, 300] with $h = 0.4, k = 0.05$ at times: $T = 0, T = 80, T = 160, T = 200$

in Fig 4. Table 6 lists the outcomes of the conservation of three laws. Fig 5 presents a plot of the interaction of three solitons at various time levels, where $c_1 = \frac{4}{21}$, $c_2 = \frac{9}{91}$, $c_3 = \frac{15}{301}$, $x_1 = 15$, $x_2 = 35$ and $x_3 = 55$. Table 7 lists the three invariants for this case. The outputs of the experiments reveal that the three invariants held relatively steady throughout the interaction process. Our approach effectively maintains the soliton properties because the waves interact and maintain their shape.

6 Conclusion

In this paper, we described the use of a finite difference method based on a three-level temporal scheme and a five-point space stencil to solve the initial boundary-value problem for the MRLW equation. Based on the maximum norm, the scheme is shown to be marginally stable and convergent with second-order accuracy in time and fourth-order accuracy in space. In order to demonstrate the effectiveness and accuracy of the proposed method, numerical experiments using single and solitary waves interaction were described. Investigations into the conservation quantities for mass, momentum, and energy were also conducted, and the results were deemed satisfactory. Comparisons with other previous results are given to show the accuracy and the efficiency of the proposed scheme.

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose. The authors declare no conflict of interest.

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