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Solitary waves in self-gravitating molecular clouds

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Abstract. Molecular clouds are self-gravitating fluids that support different waves and contain highly nonlinear clumps and filaments, for which explanations have been sought in terms of solitons. The present paper explores the possibility that several (neutral) species with different thermal speeds coexist, as in a molecular cloud consisting of gas and dust, or of a mixture of normal matter and dark matter. It is shown that this model can support soliton formation, both with humps or dips in the self-gravitational potential. The existence domain has been given in terms of the hot species Mach number and fractional mass density, in a gas-dynamic description which emphasizes the constraints coming from the sonic and neutral points, and from the limits due to infinite compression or total rarefaction. One species is compressed while the other is rarefied, allowing the system to reach a mass neutral point outside equilibrium. In this way, solitons are possible without invoking interaction with a weakly ionized cloud component or involving envelope solitons that are not really stationary structures.

Key words. gravitation – hydrodynamics – ISM: clouds – waves

1. Introduction

Molecular clouds are self-gravitating fluids that support many different waves, some of which have been invoked to explain the observed cloud structure. These clouds are not only interesting objects in themselves, but they also contain a sizeable fraction of the gas in a galaxy and are regions of star formation. Molecular clouds contain highly nonlinear structures, clumps and filaments, and explanations have been sought in terms of solitons. Strictly speaking, solitons are spatially localized, single-hump or dip entities which in addition exhibit a remarkable stability even when interacting with each other. However, for reasons of simplicity, many papers speak of solitons rather than of solitary waves, even though one cannot check that the large amplitude solitary waves exhibit the characteristic interaction property.

Attention has gradually shifted in recent years from linear mode studies and instability criteria, based on intricate modelling or on the simpler Jeans approach, to nonlinear stationary waves and solitons (Liang 1979; Adams & Fatuzzo 1993; Adams et al. 1993, 1994). It has been found that the intrinsic nonlinearity of the hydrodynamic equations, used to describe self-gravitating fluids, can be balanced by the dispersiveness of self-gravity to allow for non-breaking structures. However, these gravity-induced large amplitude periodic waves do not admit a solitary wave limit, i.e. there is no gravity-induced soliton. This conclusion has been reached for fluids with a general barotropic pressure law of the form $p = p(\rho)$ (Adams et al. 1994), where p represents the pressure and ρ the mass density.

When magnetic field effects were included, solitary waves have been shown to exist in molecular clouds which are weakly ionized (Adams & Fatuzzo 1993; Adams et al. 1994). In this model, most of the mass is assumed to reside in the neutral component but only the ionized component can be directly coupled to the magnetic field. The neutral component is influenced indirectly through ion-neutral collisions.

We should also mention that envelope solitons governed by the nonlinear Schrödinger equation have been found when studying the nonlinear amplitude modulation of linear acoustic modes (Kates 1986; Kates & Kaup 1988; Ono & Nakata 1994; Fujiwara & Soda 1996). However, these are not stationary structures and are thus not relevant to the present discussion, where we want to investigate the possibility that truly solitary waves can occur in (neutral) self-gravitating fluids.

Our paper explores the possibility that several neutral species with different thermal speeds coexist. For example, this could be a molecular cloud consisting of a hydrogen gas and a dust component or a mixture of normal matter and dark matter, as suggested by Kates & Kaup (1988). We show that this can lead to soliton formation, both of potential hill and potential dip type, provided there are at least two different species. This terminology refers to the humps or dips in the self-gravitational potential and is less confusing than the older labels compressive or rarefactive, which become ambiguous in multifluid compositions where some species are compressed while others are rarefied within the same solitary wave profile.

For the case with two species, one very cold and one hot, we will explicitly study the existence domains in the

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parameter space of the hot species Mach number M and cold species fractional mass density f. We follow a recent gas-dynamic approach (McKenzie 2002; Verheest et al. 2004; Cattaert et al. 2005) and study nonlinear structures in their own reference frame, emphasizing the gas-dynamical constraints on their existence. Such a description highlights the role of sonic and neutral points. It is also noted that there is a formal analogy with the study of plasmas with more than one electron species (Cattaert et al. 2005). Due to the sign of the potential, however, the physically interesting regions of parameter space and the conclusions these entail are radically different, as we will see.

In Sect. 2 we will develop a general theoretical framework for studying solitary waves in self-gravitating clouds consisting of several species. In Sect. 3 we look more specifically at the two-species case, and discuss an application in Sect. 4. Our conclusions are stated in Sect. 5.

2. General multispecies theory

We study a self-gravitating cloud consisting of several species j, to be specified later. The developments in this section will closely parallel those given in earlier treatments for electrostatic problems in multispecies plasmas (Verheest et al. 2004), and is included here in some detail to enhance the readability of the paper. We study stationary progressive waves of the form $f(x+Vt)$ in the wave frame. Species that in the absence of wave disturbances would be at rest in an inertial frame, now stream along the x axis in the wave frame, with a species' reference speed $v_{j0} = V$ at $x = -\infty$.

The entire treatment is one dimensional, and thus implicitly assumes, in common with many other papers in the literature, that the fluids extend to infinity in both perpendicular directions. Even though we know real astrophysical clouds to have a finite extent, plane wave solutions are admissible when the boundary conditions are not directly of great importance, for phenomena occurring in the bulk of the system. For this to be valid, the lateral directions should be larger than the soliton width. As will be seen in the illustrative example discussed in Sect. 4, this would not seem too great a restriction. Further to the same issue, gravitational forces tend to saturate in slab geometries, but increase faster with compression in higher dimensions. As a result, the plane parallel solutions obtained in our paper are likely to last longer than their higher dimensional counterparts, which should increase their chances of being observed.

The standard set of basic equations includes per species the continuity and momentum equations,

$$\frac{\partial \rho_j}{\partial t} + \frac{\partial}{\partial x}(\rho_j v_j) = 0,$$
$$\frac{\partial v_j}{\partial t} + v_j \frac{\partial v_j}{\partial x} + \frac{1}{\rho_j}\frac{\partial p_j}{\partial x} = -\frac{\partial \psi}{\partial x}, \tag{1}$$

and the system is then closed by Poisson's law,

$$\frac{\partial^2 \psi}{\partial x^2} = 4\pi G\left(\sum_j \rho_j - \rho_0\right). \tag{2}$$

Here ρ_j, v_j and p_j refer to the mass density, fluid velocity and pressure, respectively, of the different species j, while ψ is the self-gravitational potential. Note that we have subtracted the contribution from the background (equilibrium) mass density $\rho_0 = \sum_j \rho_{j0}$. This approach (Jeans 1928) has not only been found useful from a physical point of view (Binney & Tremaine 1987; Trigger et al. 2004) in giving the right order of magnitude for the dimensions of gravitationally (un)stable mass clouds, but has recently also been justified mathematically (Kiessling 2003).

Since in the wave frame there are no time derivatives, these equations can easily be integrated, starting with the continuity equation which expresses mass flux conservation,

$$\rho_j v_j = \rho_{j0} V. \tag{3}$$

This shows that $v_j \to 0$ or $+\infty$ induces $\rho_j \to +\infty$ or 0, respectively, and the limit of infinite compression or total rarefaction would be reached.

The pressures will be assumed polytropic, with index γ_j, so that $p_j \propto \rho_j^{\gamma_j} \propto v_j^{-\gamma_j}$. This includes the usual adiabatic flow for $\gamma_j = 3$, and the last proportionality comes from mass flux conservation. If necessary, the special case of isothermal pressures ($\gamma_j = 1$) can easily be included by adapting the general expressions. The equations of motion thus become

$$v_j\left(1 - \frac{V^{\gamma_j+1}}{v_j^{\gamma_j+1} M_j^2}\right)\frac{\mathrm{d}v_j}{\mathrm{d}x} = -\frac{\mathrm{d}\psi}{\mathrm{d}x} \tag{4}$$

where Eq. (3) has been used. The Mach number M_j and thermal velocity $c_{\mathrm{t}j}$ of each component are defined through

$$M_j = \frac{V}{c_{\mathrm{t}j}}, \qquad c_{\mathrm{t}j}^2 = \frac{\gamma_j p_{j0}}{\rho_{j0}}. \tag{5}$$

In the gas-dynamic description that we are following, the sonic points, defined as

$$\frac{v_{j\mathrm{s}}}{V} = \frac{1}{M_j^{2/(\gamma_j+1)}}, \tag{6}$$

correspond to where the species' local flow speeds match their local acoustic speeds $(\gamma_j p_j/\rho_j)^{1/2}$. Hence Eq. (4) indicates that these sonic points play a crucial role in limiting the amplitude of the wave, by choking the flow ($\mathrm{d}v_j/\mathrm{d}x \to +\infty$) before a possible equilibrium point can be reached from the initial values $v_j = V$, the beginning of the wave.

After integration of the equations of motion (4) a Bernoulli type integral is obtained per species,

$$\mathcal{E}_j \equiv \frac{1}{2}\left(v_j^2 - V^2\right) + \frac{c_{\mathrm{t}j}^2}{\gamma_j - 1}\left(\frac{V^{\gamma_j-1}}{v_j^{\gamma_j-1}} - 1\right) = -\psi. \tag{7}$$

Here the first term refers to changes in kinetic energy, whereas the second term gives changes in enthalpy. Equation (7) implies that all $\mathcal{E}_j$ are equal, which points to the possibility of expressing all velocities (and the potential) as a function of one of them.

When we now compute

$$\frac{\mathrm{d}\mathcal{E}_j}{\mathrm{d}v_j} = v_j\left(1 - \frac{V^{\gamma_j+1}}{v_j^{\gamma_j+1} M_j^2}\right), \tag{8}$$

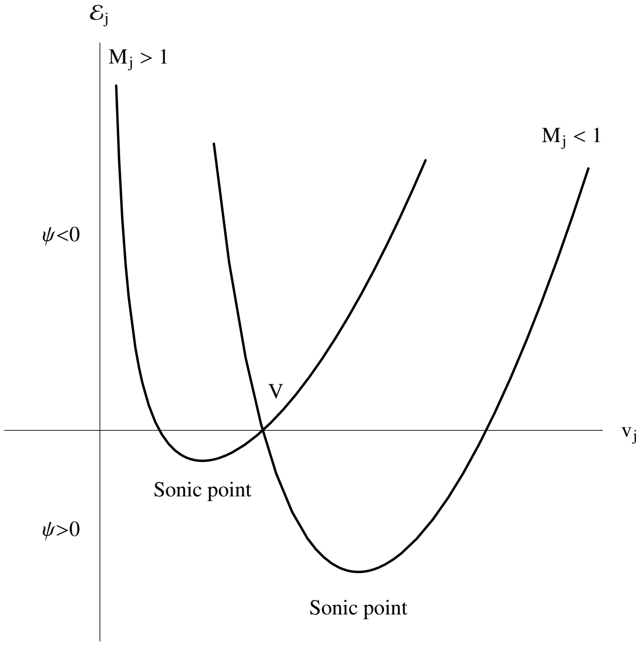

Fig. 1. Bernoulli integrals for supersonic (cool) and subsonic (hot) species, having a minimum at the respective sonic points.

we first of all note that at the sonic point given by (6) $\mathcal{E}_j$ reaches a minimum, negative value. Initially $\mathcal{E}_j$ is zero, and its derivative is then given by

$$\left.\frac{\mathrm{d}\mathcal{E}_j}{\mathrm{d}v_j}\right|_{v_j=V} = V\left(1 - \frac{1}{M_j^2}\right). \tag{9}$$

Thus $\mathcal{E}_j$ has a slope at the initial point which is positive if the incoming flow is supersonic ($M_j > 1$) and negative for subsonic flows ($M_j < 1$). We learn from the behaviour of the Bernoulli relations that in a potential hill ($\psi > 0$) all species are driven towards their sonic points. For supersonic, cold species (with $M_j < 1$) this means deceleration ($v_j < V$) and compression, whereas for subsonic, hotter positive species (with $M_j < 1$) we have acceleration ($v_j > V$) and rarefaction. In a potential dip ($\psi < 0$), the conclusions are reversed and all species are driven away from their sonic point. This is illustrated in Fig. 1.

We finally multiply the equations of motion (4) by ρ_j, sum over all species, use mass conservation (3) and the Poisson Eq. (2), and integrate. This yields a global pressure invariant

$$\mathcal{R} \equiv \sum_j \rho_{j0}\left(\mathcal{P}_j - \mathcal{E}_j\right) = -\frac{1}{8\pi G}\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2, \tag{10}$$

where the (normalized) particle pressure functions $\mathcal{P}_j$ for all species are given by

$$\mathcal{P}_j \equiv V\left(v_j - V\right) + \frac{c_{tj}^2}{\gamma_j}\left(\frac{V^{\gamma_j}}{v_j^{\gamma_j}} - 1\right). \tag{11}$$

The first term relates to changes in dynamic pressure, and the second term to changes in thermal pressure.

Now we address some of the necessary conditions for the existence of a solitary wave. The location of the zeros of $\mathcal{R}$ is assisted by computing with the help of Eq. (4) its derivative in the form

$$\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}v_\ell} = v_\ell\left(1 - \frac{V^{\gamma_\ell+1}}{v_\ell^{\gamma_\ell+1}M_\ell^2}\right)\left(\sum_j \rho_j - \rho_0\right). \tag{12}$$

Here it is understood that all velocities (or densities) have formally been expressed as functions of a chosen v_ℓ with the help of mass flux conservation (3) and the Bernoulli relations (7). Hence the extrema of $\mathcal{R}$ are the sonic point in question and the mass neutral points, defined by $\sum_j \rho_j = \rho_0$, of which one occurs at the initial point. Thus we know that $\mathcal{R}$ has a double root at $v_\ell = V$. For a solitary wave to be formed we need $\mathcal{R}$ to be negative and a necessary soliton condition therefore is

$$\left.\frac{\mathrm{d}^2\mathcal{R}}{\mathrm{d}v_\ell^2}\right|_{v_\ell=V} = \left(1 - \frac{1}{M_\ell^2}\right)^2 \sum_j \frac{\rho_{j0}}{1/M_j^2 - 1} < 0. \tag{13}$$

This will determine different regimes for the speed V of the nonlinear structure.

The second condition for a solitary pulse to exist comes from finding the maximum or minimum $v_{\ell,m}$ of the pulse, where $\mathrm{d}v_\ell/\mathrm{d}x = 0$. This corresponds to finding a root for $\mathcal{R}(v_{\ell,m}) = 0$ before one of the sonic points, or infinite compression or total rarefaction is reached. Since the roots of a function are intertwined with the roots of its derivative, there should be at least one value for v_ℓ between V and $v_{\ell,m}$ where $\mathrm{d}\mathcal{R}/\mathrm{d}v_\ell = 0$, i.e. a mass neutral point. Unfortunately, the existence of mass neutral points outside equilibrium is a necessary, but not a sufficient condition to encounter solitary wave structures.

It is also clear that in order to obtain a mass neutral point outside the initial conditions, one needs at least two species, otherwise the lone species is merely compressed or rarefied. From the Bernoulli integrals (7) it follows further that if all species are supersonic ($M_j > 1$), they are all decelerated and compressed in a potential hill ($\psi > 0$), so that a mass neutral point outside the initial point cannot be reached before one of the species reaches its sonic point, where its flow is choked. In a potential dip ($\psi < 0$), all species are accelerated and rarefied, so that also there a mass neutral point cannot be attained. An analogous reasoning can be made when all species would be subsonic ($M_j < 1$). Thus for the proper existence of a solitary wave at least one supersonic and one subsonic species is necessary.

3. One cool and one hot species

We now look at a cloud consisting of a cold species (j = c) and a hot species (j = h). For notational simplicity we denote the hot species Mach number by $M^2 = M_\mathrm{h}^2 < 1$, and introduce the temperature ratio $\tau = c_\mathrm{tc}^2/c_\mathrm{th}^2$, such that the cold species Mach number is given by $M_\mathrm{c}^2 = M^2/\tau > 1$. With the help of the fractional densities $f = \rho_{\mathrm{c}0}/\rho_0$ and $1 - f = \rho_{\mathrm{h}0}/\rho_0$ the soliton condition (13) reads

$$\frac{f}{\tau/M^2 - 1} + \frac{1 - f}{1/M^2 - 1} < 0. \tag{14}$$

This leads to allowed Mach numbers $\tau < M^2 < f + (1 - f)\tau$ or, in equivalent terms, $c_\mathrm{tc}^2 < V^2 < f c_\mathrm{th}^2 + (1 - f)c_\mathrm{tc}^2$. Furthermore, we introduce dimensionless velocities $u_j = v_j/V$ and a dimensionless coordinate $\xi = x\omega_J/V$, where ω_J is the total Jeans frequency defined through $\omega_J^2 = 4\pi G\rho_0$.

We express also the energy and pressure functions in dimensionless variables, i.e. we take

$$\begin{aligned} E_c &= \frac{1}{2}\left(u_c^2 - 1\right) + \frac{\tau}{(\gamma_c - 1)M^2}\left(\frac{1}{u_c^{\gamma_c-1}} - 1\right),\\ E_h &= \frac{1}{2}\left(u_h^2 - 1\right) + \frac{1}{(\gamma_h - 1)M^2}\left(\frac{1}{u_h^{\gamma_h-1}} - 1\right),\\ P_c &= u_c - 1 + \frac{\tau}{\gamma_c M^2}\left(\frac{1}{u_c^{\gamma_c}} - 1\right),\\ P_h &= u_h - 1 + \frac{1}{\gamma_h M^2}\left(\frac{1}{u_h^{\gamma_h}} - 1\right). \end{aligned} \tag{15}$$

One then has to solve (numerically) for u_c as function of u_h (or vice versa) from the energy hodograph $E_c = E_h$ and finally (numerically) study the behaviour of the dimensionless global pressure function $R = \mathcal{R}/\rho_0 V^2$, or

$$\begin{aligned} R(u_h) &= f P_c + (1 - f)P_h - E_h\\ &= -\frac{u_h^2}{2}\left(1 - \frac{1}{u_h^{\gamma_h+1}M^2}\right)^2\left(\frac{du_h}{d\xi}\right)^2. \end{aligned} \tag{16}$$

To facilitate our expose and get a first idea, we take in the following discussions $\tau = 0$, which means that we neglect the cold thermal pressure, and write for simplicity $\gamma = \gamma_h$. In other words, the cooler species is treated as very cold ($c_{tc} \to 0$ or $M_c \to \infty$), in the sense that its enthalpy is neglected compared to its kinetic energy and its sonic point has been shifted to zero. The necessary soliton condition reduces thus to $0 < M^2 < f$, and the energy hodograph can now be solved analytically to yield

$$u_c^2 = u_h^2 + \frac{2}{(\gamma - 1)M^2}\left(\frac{1}{u_h^{\gamma-1}} - 1\right). \tag{17}$$

The global pressure function is given by

$$\begin{aligned} R(u_h) = f&\left[\sqrt{u_h^2 + \frac{2}{(\gamma - 1)M^2}\left(\frac{1}{u_h^{\gamma-1}} - 1\right)} - 1\right]\\ &+ (1 - f)\left[u_h - 1 + \frac{1}{\gamma M^2}\left(\frac{1}{u_h^{\gamma}} - 1\right)\right]\\ &- \left[\frac{1}{2}\left(u_h^2 - 1\right) + \frac{1}{(\gamma - 1)M^2}\left(\frac{1}{u_h^{\gamma-1}} - 1\right)\right]. \end{aligned} \tag{18}$$

Further analytical progress can be made for weakly nonlinear solitons, for which $R(u_h)$ can be Taylor expanded around the initial condition $u_h = 1$. This gives

$$\frac{1}{2}\left(\frac{du_h}{d\xi}\right)^2 = \frac{f - M^2}{2(1 - M^2)}(u_h - 1)^2 - A(u_h - 1)^3, \tag{19}$$

where

$$A = \frac{6M^6 - (10f + f\gamma + 2 - 4\gamma)M^4 + 3f(3 - \gamma)M^2 - 3f}{6M^2(1 - M^2)^2}. \tag{20}$$

In order to obtain a solution, both terms in the expansion must contribute equally, hence we take $M^2 = f - \delta^2$, with δ^2 of the order of $|u_h - 1|$. This leads to

$$\frac{1}{2}\left(\frac{du_h}{d\xi}\right)^2 = \frac{\delta^2}{2(1 - f)}(u_h - 1)^2 + \frac{3 - (\gamma + 4)f}{6(1 - f)}(u_h - 1)^3, \tag{21}$$

which has the stationary Korteweg-de Vries (KdV) soliton solution

$$u_h = 1 + \frac{3\delta^2}{(\gamma + 4)f - 3}\,\mathrm{sech}^2\left(\frac{\delta}{2\sqrt{1 - f}}\xi\right). \tag{22}$$

Hence there is a critical fraction $f_C = 3/(\gamma + 4)$, at which the solitary wave changes character from hill to dip. Note that these results could also have been obtained by using reductive perturbation techniques.

At the critical cold fraction the expansion of (18) yields

$$\frac{1}{2}\left(\frac{du_h}{d\xi}\right)^2 = \frac{\delta^2(\gamma + 4)}{2(\gamma + 1)}(u_h - 1)^2 - \frac{\gamma + 10}{24}(u_h - 1)^4, \tag{23}$$

which allows stationary modified KdV (mKdV) soliton solutions

$$u_h = 1 \pm 2\delta\sqrt{\frac{3(\gamma + 4)}{(\gamma + 1)(\gamma + 10)}}\,\mathrm{sech}\left(\delta\sqrt{\frac{\gamma + 4}{\gamma + 1}}\xi\right) \tag{24}$$

of both kinds. For near critical cold fractions, one has to retain all three terms in the expansion. This leads to a mixed KdV-mKdV equation which also has weak solitary wave solutions of both types. Remark that no (weak) double layers are possible.

Now we address the large amplitude treatment. For $\psi < 0$, going away from the sonic points towards $u_h \to 0$, we have $u_c \to \infty$ going as $u_h^{(1-\gamma)/2}$ and $R(u_h) \to +\infty$ as $u_h^{-\gamma}$. This means there has to be a zero of R for some $u_h < 1$. Hence there is always a potential dip ($\psi < 0$) solitary wave, as long as the soliton condition $M^2 < f$ is fulfilled. From weak amplitude theory we know that these are small near the diagonal segment OC in Fig. 2. To get a first idea what happens elsewhere in the parameter space shown in Fig. 2, we study the behaviour of the solution of $R(u_h) = 0$ for $M^2 \to 0$, which yields

$$f = \frac{(\gamma - 1)(1 - v^\gamma) - \gamma v(1 - v^{\gamma-1})}{(\gamma - 1)(1 - v^\gamma)}. \tag{25}$$

Close to the line $M^2 = 0$ one starts in O for $u_h = 1$ and goes for decreasing u_h towards A where $u_h = 0$. For $f \to 1$, the amplitude remains maximal along AB, where $u_h = 0$ is the only non-trivial solution of $du_h/d\xi = 0$, but of course the limit itself is unphysical. Near the diagonal $M^2 = f$, the soliton amplitude decreases from B where $u_h = 0$ to C where $u_h = 1$.

Turning to the potential hill ($\psi > 0$) solitons, there are in principle three special curves limiting soliton amplitudes, i.e. limiting curves due to encountering the cold or hot sonic point, or else to finding a double layer. However, it turns out both numerically and analytically (Cattaert et al. 2005) that the cold sonic point limiting curve and the double layer curve are entirely located in the forbidden region where $M^2 > f$, hence only the hot sonic point limiting curve is important for the present discussion. The limits due to hot sonic points are found from

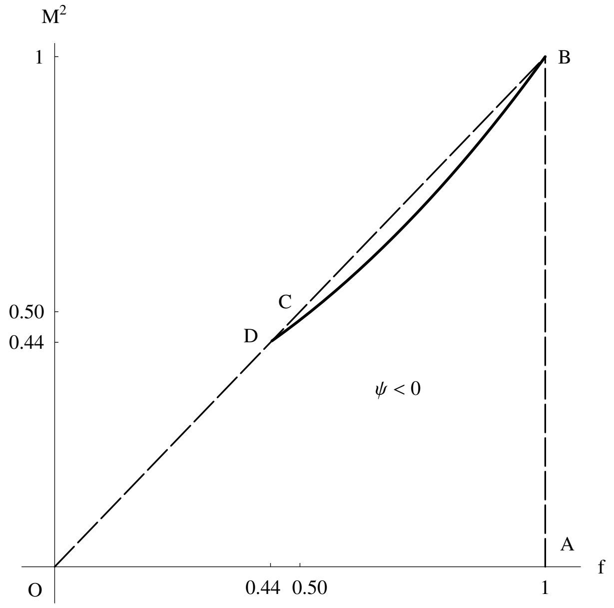

Fig. 2. Hot sonic point bounding curve for $\gamma = 2$. There are potential dip solutions in the whole triangle OAB. Both potential hill and dip solutions occur in the region bounded by the curve BD and the dashed line BCD.

putting $u_{\rm hs} = M^{-2/\gamma+1}$ into $R(u_{\rm hs}) = 0$ which then can be solved for f, leading to

$$f = \frac{-\frac{1}{2}(1-u_{\rm h})^2 + \frac{u_{\rm h}(1-u_{\rm h}^{\gamma})}{\gamma} - \frac{u_{\rm h}^2(1-u_{\rm h}^{\gamma-1})}{\gamma-1}}{u_{\rm h}\left(1 + \frac{1-u_{\rm h}^{\gamma}}{\gamma} - \sqrt{1 + \frac{2(1-u_{\rm h}^{\gamma-1})}{\gamma-1}}\right)}. \tag{26}$$

Together with $M^2 = u_{\rm h}^{1/(\gamma+1)}$, this yields a parameter representation for the hot sonic point limiting curve. The case $\gamma = 1$ can be deduced from the general expression by taking the appropriate limit $\gamma \to 1$ (Cattaert et al. 2005).

Pursuing our discussion of Fig. 2, for $u_{\rm h} = 1$, this curve starts in B, and goes for growing $u_{\rm h} > 1$ to the left, hereby staying under the diagonal line $M^2 = f$, and finally crosses this in a point D, left of C, where it reaches maximum amplitude. Hence the potential hill solitons ($\psi > 0$) are limited to the region between the diagonal line $M^2 = f$ and the hot sonic point limiting curve BD. We know from small amplitude theory that soliton amplitudes are small near the diagonal segment BC. These then grow with decreasing $M^2 < f$ until they meet the hot sonic limiting curve. Left of C, there are no small amplitude solutions. Near the diagonal segment CD, amplitudes are growing from C where $u_{\rm h} = 1$ towards D.

To illustrate these results, we give in Figs. 3–6 some examples for $\gamma = 2$ of structure functions R for different values of f and M^2. Results for other values of γ (not shown here), like $\gamma = 3$ (adiabatic) and $\gamma = 1$ (isothermal), show a similar behaviour, as can also be inferred from Cattaert et al. (2005). For reasons of continuity, intermediate γ values will not qualitatively change these conclusions following from Eq. (26), an expression which is in principle valid for all $\gamma \geq 1$.

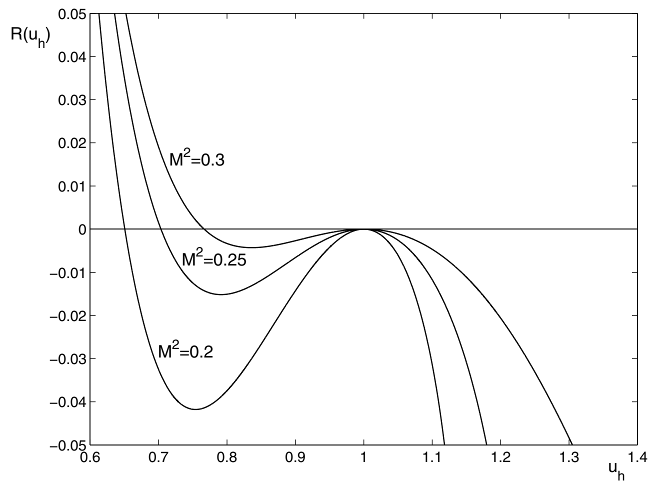

Fig. 3. Structure function for $f = 0.4$ and $\gamma = 2$. There are only potential dip solitons. These are small for $M^2 \simeq 0.5$ and grow with decreasing M^2.

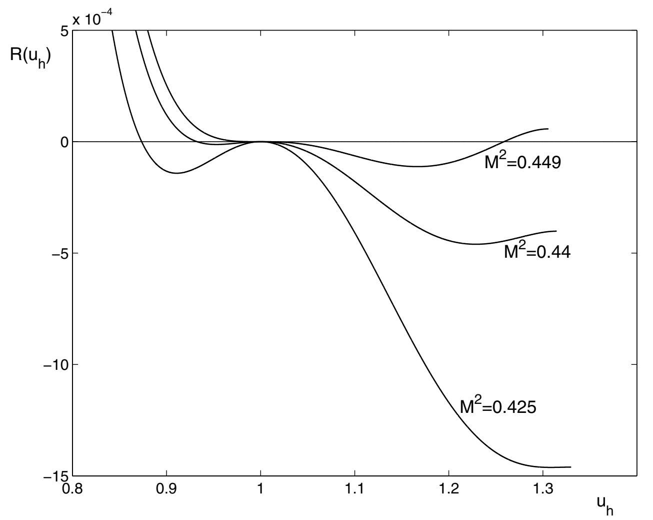

Fig. 4. Structure function for $f = 0.45$ and $\gamma = 2$. For Mach numbers close to $M^2 \simeq 0.5$, there are small amplitude potential dip solitons and large amplitude potential hill solitons. For decreasing M^2 the potential dip solitons grow and the potential hill solitons disappear.

At the end of this section we would like to reiterate that for reasons of simplicity and following the usage of many papers dealing with similar problems, we have sometimes used the term soliton rather than solitary wave, even though we cannot check that the large amplitude solitary waves we have discussed exhibit the characteristic interaction property that is required of "true" solitons. However, the weaker nonlinear modes (22) or (24) that are obtained from the large amplitude ones obey the KdV or mKdV equations, respectively, and are thus true solitons, giving some justification for the terminology, even if we have not directly used the reductive perturbation analysis to derive these equations explicitly here.

Because of the stationarity assumption where we work in a co-moving frame, thus fixing V, the interaction properties between several such nonlinear structures cannot be investigated. To discuss the full stability of one of these large amplitude

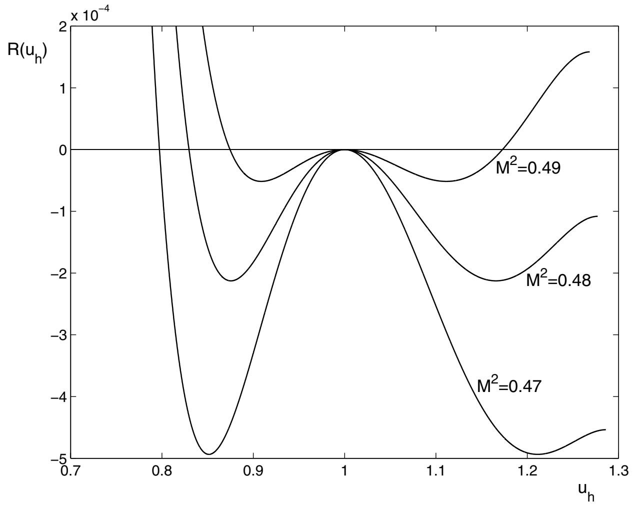

Fig. 5. Structure function for $f = 0.5$ and $\gamma = 2$. For Mach numbers close to $M^2 \simeq 0.5$, there are small amplitude potential dip solitons and small amplitude potential hill solitons. For decreasing M^2 both grow but the potential hill solitons disappear when encountering the cold sonic point limitation.

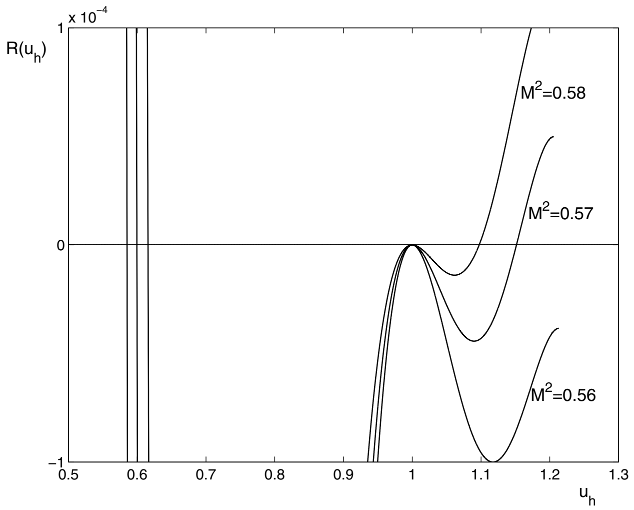

Fig. 6. Structure function for $f = 0.6$ and $\gamma = 2$. Now the potential dip solitons are always big. The potential hill solitons are small for Mach numbers close to $M^2 \simeq 0.5$, grow with decreasing M^2 and eventually disappear.

solitary waves against small perturbations would require an analytical expression for the solutions, which we do not obtain from the present formalism, which focuses on the existence conditions in parameter space. Such discussion are also lacking in many Sagdeev type treatments of electrostatic large amplitude waves in plasmas, precisely for the same reasons. It is only for simple pseudopotentials or structure functions that analytical expressions can formally be obtained. Otherwise the stability problem has to be studied numerically, which is outside the scope of our analysis.

4. Application

We now want to illustrate the above results for the example of a molecular cloud, where the (hydrogen) gas is the light and hot species, and a dust component serves as the cooler and heavier species. We recall that the cold fluid density fraction

$$f = \frac{\rho_{c0}}{\rho_0} = \frac{\rho_{c0}}{\rho_{c0} + \rho_{h0}} = \frac{n_{c0} m_c}{n_{c0} m_c + n_{h0} m_h} \tag{27}$$

is needed in terms of the mass densities, not the number densities. To get large density increases in the hot species density, say a factor of order 10, one needs higher f values like 0.85, and because of the mass conservation invariant (3), a density compression corresponds to a decrease in u_h, for which a potential dip solution is needed. For the cold species this means that when $f \simeq 0.85$, we see that

$$\rho_{c0} = \frac{f}{1-f} \rho_{h0} \simeq 5.7 \rho_{h0}, \tag{28}$$

in other words, the cold species need not have a high number density (compared to that of the hot species) but the m_c/m_h ratio should be large enough to compensate for the disparities in number densities. Applied to our model molecular clouds this should be easy enough to satisfy, but when dark matter plays the role of the cooler species, nothing is known with great certainty, and guesses for its mass density will have to do. There is also no problem in taking dust or dark matter very cold, unlike for a hydrogen-helium mixture, where finite τ effects are expected to play a role.

Next, there are several parameters that are difficult to estimate directly, because in the dimensionless discussion we used it is the dimensionless ratios that are of importance. As a simple example, the hot species Mach number has to obey $M^2 = V^2/c_{th}^2 < f < 1$ when its definition is combined with the soliton condition. Moreover, we have expressed all our lengths in terms of the scale $L = c_{th}/\omega_J$, which we can compare to the Jeans lengths $\lambda_{Jh} = c_{th}/\omega_{Jh}$ of the hot species through

$$L^2 = \frac{c_{th}^2}{\omega_J^2} = (1-f)\lambda_{Jh}^2. \tag{29}$$

Taking temperatures T_h of the order of 10^3 K gives $c_{th} \simeq 2.9$ km s^{-1} for molecular hydrogen, and with typically $n_{h0} \simeq 10^9$ m^{-3} we find $\omega_{Jh} \simeq 5.3\times10^{-14}$ s^{-1}. Thus $\lambda_{Jh} \simeq 5.4\times10^{13}$ km, and $L \simeq 2.1 \times 10^{13}$ km $\simeq 0.7$ pc. Of course, these figures are only indicative, given the rather large uncertainties and margins in the modelling used. The form of the soliton is then given in Fig. 7.

We would thus expect large amplitude solitons to have widths typically of the order of some pc, which could be relevant for larger molecular clouds. However, their velocities would be of the order of some km s^{-1}, and thus it would take many thousands of years to get through a distance of 1 pc. Hence we would think that such solitons would be observed as almost stationary, and our mechanism might be viewed as a possible candidate to generate rather large scale structures. Presumably these would then slowly dissipate away, but including such losses is beyond the scope of the present model.

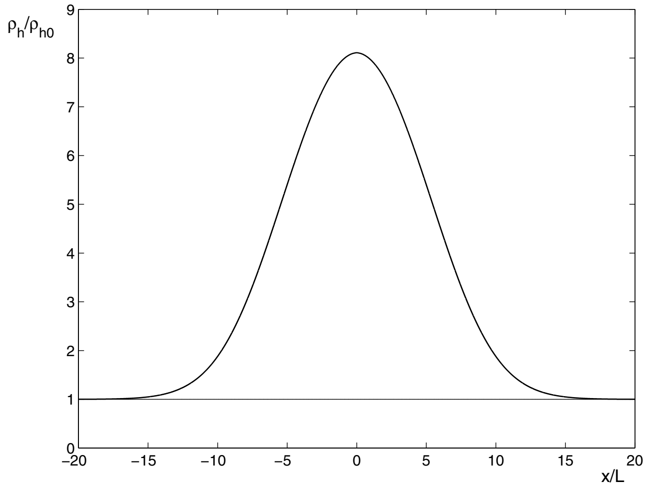

Fig. 7. Form of the soliton for $f = 0.85$, $\gamma = 2$ and $M = 0.8$. Plotted is the compression in density ρ_h/ρ_{h0}, whereas lengths are in terms of $L = V/\omega_J$. It is seen that the half width at half maximum is of order 10 or less in these units.

5. Conclusions

It has been shown that at least two fluids with different thermal velocities are needed for soliton formation in self-gravitating systems. We found explicitly the regions in parameter space where potential hill and potential dip solitary waves exist in a two-species molecular cloud consisting of one very cold and one hot species. Although, for reasons of analytical tractability, we have only considered the limiting case $\tau \to 0$, there is no reason to believe that finite τ would materially alter our significant results, viz. the existence of positive and negative potential solitons and the absence of double layers, and this for arbitrary values of the polytropic index γ.

Our work has implications on the structure of molecular clouds, where, as suggested elsewhere (Adams et al. 1994), stationary nonlinear waves might have been observed. Moreover, our model does not involve collisional coupling between neutral and ionized material or other plasma effects, but presents a mechanism to generate large amplitude solitary waves in a neutral mass system with two species at distinct temperatures, in particular where such structures would be observed as almost stationary.

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