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Soliton Creation-Annihilation Operators as the Canonical Transformation

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Recently, one of the authors (Y.K.) and Wadati have shown that the Bäcklund transformation for the sine-Gordon equation

$$\partial_t^2\phi-\partial_x^2\phi+\sin\phi=0 \tag{1}$$

is a canonical transformation which keeps the Hamiltonian form invariant.[1] In the present paper, this result is investigated in view of quantum theory. The canonical transformation $T_a:(\phi,\pi)\to(\phi',\pi')$ for Eq. (1) is given by the formulas

$$\pi(x,t)=\frac{\delta W_a}{\delta\phi}=\partial_x\phi'+a\sin\frac{1}{2}(\phi+\phi')-\frac{1}{a}\sin\frac{1}{2}(\phi-\phi'), \tag{2·a}$$

$$\pi'(x,t)=-\frac{\delta W_a}{\delta\phi'}=\partial_x\phi-a\sin\frac{1}{2}(\phi+\phi')-\frac{1}{a}\sin\frac{1}{2}(\phi-\phi'), \tag{2·b}$$

where $W_a[\phi,\phi';t]$ is the generating functional expressed as

$$W_a[\phi,\phi';t]=\int_{-\infty}^{\infty}dx\left\{\phi\partial_x\phi'-2a\cos\frac{1}{2}(\phi+\phi')+\frac{2}{a}\cos\frac{1}{2}(\phi-\phi')\right\}-Et. \tag{3}$$

Here E is a constant (energy integral) determined by the boundary conditions of the flows (ϕ,π) and (ϕ',π'). Particularly, in the case of the transformation (with positive a or negative a) which transforms a trivial solution $\phi^{(0)}=0$ into a solution $\phi^{(\pm1)}$ with the boundary condition $\phi^{(\pm1)}\to(\mp)2\pi$ as $x\to-\infty$, $\phi^{(\pm1)}\to0$ as $x\to\infty$, we see that $\phi^{(\pm1)}$ is a one-soliton, or antisoliton, solution with the velocity $u=(1-a^2)/(1+a^2)$ and energy $E=8/\sqrt{1-u^2}$. Classically, this transformation may be interpreted as the creation-annihilation of soliton. Also, in relation to the quantum theoretic version of soliton, Ichikawa et al. have shown that a soliton solution to the Korteweg-de Vries equation can be interpreted as a coherent state of interacting phonons in a one-dimensional anharmonic lattice.[2] We construct a transformation operator generating the transformation (2) by coherent representation.

In quantum field theory, the Lagrangian for Eq. (1) is defined by

$$\hat{L}=:\int dx\{\tfrac{1}{2}(\partial_t\hat{\phi})^2-\tfrac{1}{2}(\partial_x\hat{\phi})^2+(\cos\hat{\phi}-1)\}:. \tag{4}$$

The operators $\hat{\phi}(x,0)$ and $\hat{\pi}(x,0)$ are expressed as

$$\hat{\phi}(x,0)=\int\frac{dk}{\sqrt{2\omega_k}}(\hat{a}_k+\hat{a}^{+}_{-k})\frac{e^{ikx}}{\sqrt{2\pi}},$$
$$\pi(x,0)=-i\int dk\sqrt{\frac{\omega_k}{2}}(\hat{a}_k-\hat{a}^{+}_{-k})\frac{e^{ikx}}{\sqrt{2\pi}} \tag{5}$$

in the Schrödinger representation. Here $\hat{a}_k$ and $\hat{a}_k{}^+$ hold the commutation relations of boson.

Consider the following operator:

$$\hat{A}[\phi]=\exp\left[-i\int_{-\infty}^{\infty}dx\{\phi(x,t)\hat{\pi}(x,0)-\pi(x,t)\hat{\phi}(x,0)\}\right], \tag{6}$$

which transforms the vacuum state $|0\rangle$ $(a_k|0\rangle=0)$ into one state $|\phi;t\rangle$, i.e.,

$$|\phi;t\rangle=\hat{A}[\phi]|0\rangle, \tag{7}$$

where $\phi(x,t)$ and $\pi(x,t)=\partial_t\phi(x,t)$ are arbitrary functions. It is easily shown that the expectation value $\langle\phi|\hat{H}|\phi\rangle$ of the

Hamiltonian $\hat{H}$ is the energy of the *classical* sine-Gordon system, and that the invariant variational problem to $\langle\phi|\hat{H}|\phi\rangle$ yields Eq. (1) for $\phi(x,t)$.

It can be also shown that $|\phi;t\rangle$ is a coherent state, so that $|\phi;t\rangle$ is an eigenstate of an annihilation operator $\hat{a}_k$,[3]

$$\hat{a}_k|\phi;t\rangle=\alpha_k(t)|\phi;t\rangle, \tag{8}$$

where $\alpha_k(t)$ is given by

$$\alpha_k(t)=\sqrt{\frac{\omega_k}{2}}\phi_k+i\frac{\pi_k}{\sqrt{2\omega_k}},$$

$$\phi_k=\frac{1}{\sqrt{2\pi}}\int dx\phi(x,t)e^{-ikx}. \tag{9}$$

Thus we see that a soliton solution $\phi(x,t)$ to Eq. (1) is interpreted as a coherent state $|\phi;t\rangle$ of self-interacting bosons. The relation between N-soliton coherent state $|\phi^{(N)};t\rangle$ and classical N-soliton solution $\phi^{(N)}(x,t)$ $(N=0,\pm1,\pm2,\cdots)$ is given by[4]

$$\langle\phi^{(N)};t|\hat{\phi}(x,0)|\phi^{(N)};t\rangle$$
$$=\langle 0|\hat{\phi}^{(N)}(x,t)|0\rangle=\phi^{(N)}(x,t). \tag{10}$$

The canonical transformation (2), T: $\phi^{(N)}\to\phi'$, can be expressed as an operator transformation relation,

$$\hat{T}^{-1}\hat{\phi}^{(N)}\hat{T}=\hat{\phi}',\ \hat{T}^{-1}\hat{\pi}^{(N)}\hat{T}=\hat{\pi}', \tag{11}$$

where the transformation $\hat{T}$ is given by

$$\hat{\pi}^{(N)}=\partial_x\hat{\phi}'+a\!:\sin\tfrac{1}{2}(\hat{\phi}^{(N)}+\hat{\phi}')\!:$$
$$-a^{-1}\!:\sin\tfrac{1}{2}(\hat{\phi}^{(N)}-\hat{\phi}')\!:, \tag{12·a}$$

$$\hat{\pi}'=\partial_x\hat{\phi}^{(N)}-a\!:\sin\tfrac{1}{2}(\hat{\phi}^{(N)}+\hat{\phi}')\!:$$
$$-a^{-1}\!:\sin\tfrac{1}{2}(\hat{\phi}^{(N)}-\hat{\phi}')\!:. \tag{12·b}$$

Calculating the expectation value of this transformation in the state $|0\rangle$, we obtain the classical transformation (2), *if and only if* $\hat{T}_{N\to N\pm1}=\hat{A}^{-1}[\phi^{(N)}]\hat{A}[\phi'=\phi^{(N\pm1)}]$. This result is directly connected with the well-known property of the Bäcklund transformation. Note that the transformation $\hat{T}_{N\to N\pm1}$ can be rewritten as

$$\hat{T}_{N\to N\pm1}=e^{iC}\hat{A}[\phi^{(N\pm1)}]\hat{A}^{-1}[\phi^{(N)}], \tag{13}$$

where C is a *real* c-number constant given by $C=\int dx\{\phi^{(N)}\pi^{(N\pm1)}-\phi^{(N\pm1)}\pi^{(N)}\}$. Thus, except for the phase factor e^{iC}, we have

$$|\phi^{(N\pm1)}\rangle=\hat{T}_{N\to N\pm1}|\phi^{(N)}\rangle. \tag{14}$$

Therefore, we conclude that the creation-annihilation operators of soliton is given by the quantum theoretic version, $\hat{T}_{N\to N\pm1}$, of the canonical transformation (2).

To close this paper, we give some remarks. Considering the quantum fluctuations around the state $|\phi^{(N)}\rangle$, we can construct the Fock space $H_F^{(N)}$,

$$H_F^{(N)}=\{|\phi^{(N)}\rangle\equiv A[\phi^{(N)}]|0\rangle,$$
$$\hat{a}_k^{(N)+}|\phi^{(N)}\rangle,\ \hat{a}_k^{(N)+}\hat{a}_q^{(N)+}|\phi^{(N)}\rangle,\ \cdots\}, \tag{15}$$

where $\hat{a}_k^{(N)+}$ is defined by the following expansion:

$$\hat{\phi}^{(N)}(x,0)=\sum_k\{\hat{a}_k^{(N)}u_k(x)$$
$$+\hat{a}_k^{(N)+}u_k^*(x)\} \tag{16}$$

in proper eigenfunctions $u_k(x)\in L^1\cap L^2$, and $\hat{a}_{k_1}^{(N)+}\cdots\hat{a}_{k_n}^{(N)+}|\phi^{(N)}\rangle$ may be interpreted as N-soliton state on which n-bosons are excited. We note that N-soliton operator $\hat{\phi}^{(N)}$ in (10) can be written by

$$\hat{\phi}^{(N)}(x,t)=\hat{\phi}(x,0)+\phi^{(N)}(x,t) \tag{17}$$

which is quite analogous to the *Bogoliubov prescription* in Bose condensation. It can be shown that the transformation $\hat{T}$ in the case $\phi\notin L^2$ is non-unitary just as the Bogoliubov transformation. This fact implies that the Fock spaces $H_F^{(N)}$ are *non-equivalent* representations of the Hilbert space for the sine-Gordon system.

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