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Soliton interaction for the extended Korteweg–de Vries equation

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Soliton interactions for the extended Korteweg–de Vries (KdV) equation are examined. It is shown that the extended KdV equation can be transformed (to its order of approximation) to a higher-order member of the KdV hierarchy of integrable equations. This transformation is used to derive the higher-order, two-soliton solution for the extended KdV equation. Hence it follows that the higher-order solitary-wave collisions are elastic, to the order of approximation of the extended KdV equation. In addition, the higher-order corrections to the phase shifts are found. To examine the exact nature of higher-order, solitary-wave collisions, numerical results for various special cases (including surface waves on shallow water) of the extended KdV equation are presented. The numerical results show evidence of inelastic behaviour well beyond the order of approximation of the extended KdV equation; after collision, a dispersive wavetrain of extremely small amplitude is found behind the smaller, higher-order solitary wave.

1. Introduction

The Korteweg–de Vries (KdV) equation is a generic model for the study of weakly nonlinear long waves, incorporating leading-order nonlinearity and dispersion. For example, it describes surface waves of long wavelength and small amplitude on shallow water (Whitham, 1974) and internal waves in a shallow density-stratified fluid (Benny, 1966). The solitary-wave solution of the KdV equation, thus named because it consists of a single humped wave, has a number of special properties. Zabusky & Kruskal (1965) numerically examined the nonlinear interaction of a large solitary-wave overtaking a smaller one. It was found that, after interaction, the solitary waves retained their original shapes (and hence conserved both mass and energy), the only effect of the collision being a phase shift. Due to this special property, amongst others, the solitary-wave solution of the KdV equation is termed a soliton. Gardner *et al.* (1967) showed that the KdV equation can be solved exactly using the inverse-scattering transform. Inverse scattering shows that the collision of KdV solitons is elastic, because the solitons retain their shape after an interaction and no dispersive radiation is generated as a result of a collision. The explicit solution for interacting KdV solitons was developed by Hirota (1972) using inverse scattering.

The KdV equation arises as an approximate equation governing weakly nonlinear long waves when terms up to the second order in the (small) wave amplitude are retained and when the weakly nonlinear and weakly dispersive terms are in balance (Whitham, 1974). If effects of higher order are of interest then retention of terms up to the third order in the (small) wave amplitude leads to the extended KdV equation

$$\eta_t + 6\eta\eta_x + \eta_{xxx} + \alpha c_1 \eta^2 \eta_x + \alpha c_2 \eta_x \eta_{xx} + \alpha c_3 \eta \eta_{xxx} + \alpha c_4 \eta_{xxxxx} = 0 \qquad (\alpha \ll 1), \quad (1.1)$$

where α is a nondimensional measure of the (small) wave amplitude. This equation describes the evolution of steeper waves with shorter wavelengths than in the KdV equation. If the terms involving α are dropped, the usual KdV equation results. The values of the parameters c_1, c_2, c_3, and c_4 depend on the physical context. In the special case of surface waves on shallow water these coefficients have the values

$$c_1 = -1, \quad c_2 = \tfrac{23}{6}, \quad c_3 = \tfrac{5}{3}, \quad c_4 = \tfrac{19}{60}. \qquad (1.2)$$

In the context of surface water waves (1.1) is also useful as a first step in understanding the progression from shallow-water waves to deep-water waves, such as occurs in a surf zone. The extended KdV equation of (1.1) with the coefficients given in (1.2) was derived by Marchant & Smyth (1990) to enable resonant flow of a fluid over topography to be modelled more accurately, and it was also derived by Byatt-Smith (1987a) to examine higher-order solitary-wave interactions. Chow (1989) derived (1.1) for surface waves in shallow water subject to a linear shear flow and obtained the coefficients given in (1.2) in the limit of no-shear. For each value of the Froude number F (which is a nondimensional measure of the magnitude of the shear flow) two wavespeeds are possible. One represents waves which propagate in the same direction as the shear flow, while the other represents waves propagating in opposition to the shear flow. Numerical values of the parameters c_1, c_2, c_3, and c_4 were presented for some special cases in which the waves propagate in the same direction as the shear flow. In addition, (1.1) describes internal waves of moderate amplitude in a shallow, density-stratified fluid. Gear & Grimshaw (1983) calculated second-order solitary-wave solutions for various density stratifications and shear flows; in the limit of no-shear and no stratification (1.2) would also be obtained.

The question arises as to whether the higher-order solitary-wave solutions of (1.1) are solitons or not, that is, whether or not they undergo elastic collisions. The special case of (1.1) with coefficients

$$c_1 = 1, \quad c_2 = \tfrac{2}{3}, \quad c_3 = \tfrac{1}{3}, \quad c_4 = \tfrac{1}{30} \qquad (1.3)$$

is a member of the KdV family of integrable equations (Newell, 1985). Hence it has N-soliton solutions. Also, the special case in which only the higher order nonlinear term $\alpha\eta^2\eta_x$ is retained in (1.1) is integrable. In this case (1.1) can be linearly transformed (Kivshar & Malomed, 1989) to the modified KdV equation

$$\eta_t + 6\eta^2\eta_x + \eta_{xxx} = 0, \qquad (1.4)$$

which is known to possess soliton solutions. This modified KdV equation can, in turn, be transformed to the KdV equation by the Miura transformation (Whitham, 1974). This linear transformation generates a soliton solution of (1.1) in which only c_1 is nonzero, having a sech profile, rather than a KdV-like sech^2 profile, on a nonzero background. Hence it is not known if an N-soliton solution for the KdV-like, higher-order solitary waves exists for this special case.

Fenton & Rienecker (1982) numerically solved the full Euler equations for water waves, and they examined the interaction of two long (but finite) waves travelling in the same and in opposite directions. For the case governed by (1.1), that of higher-order solitary waves travelling in the same direction, they found a small change in the wave amplitude after the collision (of magnitude $O(\alpha^2)$), in which the larger wave increased in amplitude and the smaller wave decreased in amplitude, but they were unable to find any evidence of dispersive radiation being shed as a result of the collision.

Mirie & Su (1982) developed perturbation and numerical solutions of a higher-order Boussinesq equation. A perturbation solution was developed for the head-on case, which showed that the solitary waves are unchanged at the third order after the collision except for a phase shift. This phase shift at the third order is not uniform however, which causes an asymmetry in the wave profile after the collision. This asymmetry leads to the shedding of secondary waves of small amplitude. Both weak (head-on) and strong (overtaking) solitary-wave collisions were considered numerically, with a small change in amplitude after the collision and the production of a dispersive wavetrain occuring in the head-on case. For the overtaking case, no firm conclusion on the nature of the collision was reached.

Zou & Su (1986) considered higher-order interactions of solitary waves on shallow water by using a perturbation expansion of the Euler equations. At first-order, the solution was assumed to be the KdV two-soliton solution. A continuation of the perturbation procedure resulted in partial differential equations describing the solitary-wave collision at the second and third orders, which were then solved numerically. At second order (i.e. at $O(\alpha)$) the solitary-wave collision was found to be elastic, which means that there was no change in the solitary-wave amplitude and that no dispersive wavetrain was generated as a result of the collision. This result was shown to be consistent with the N-soliton solution, describing the interaction to $O(\alpha)$, found by Sachs (1984). This N-soliton solution was found by using the KdV inverse-scattering solution to analytically solve the partial differential equation describing the second-order effects. At third order, the numerical solutions again show no change in the solitary-wave amplitude after collision. However, a dispersive wavetrain of small amplitude occurs behind the smaller solitary wave after the interaction, which is taken as evidence that the interaction is inelastic.

Byatt-Smith (1987a) considered the interaction of two solitary waves governed by the version of (1.1) appropriate for surface waves on shallow water by using a perturbation method based on inverse scattering. It was found that the amplitudes of the waves underwent a change of $O(\alpha)$ due to the interaction; the larger wave had an increased amplitude, while the smaller one decreased in amplitude. This change in amplitude is larger than that found numerically by Fenton & Rienecker (1982) and by Zou & Su (1986) and that found theoretically by Sachs (1984). This anomalous result is explained in Section 3.2 (see p. 17). Byatt-Smith (1987b, 1988) considered the interaction of solitary-wave solutions of the Benjamin–Bona–Mahony (BBM) equation and the reflection of a solitary wave from a wall by the same perturbation technique based on inverse scattering.

Kichenassamy & Olver (1992) considered the extended KdV equation of (1.1) and showed that exact solitary-wave solutions (waves with sech^2 profiles) exist only for the special parameter combination

$$c_2 + c_3 = 30c_4, \qquad 3c_3 = c_1. \tag{1.5}$$

In particular, their work implies that the form of (1.1) appropriate for surface waves on shallow water does not have an exact solitary-wave solution. The series which represents the higher-order solitary-wave solution in this case is nonconvergent due to the presence of exponentially small terms.

Pomeau *et al.* (1988) and Grimshaw & Joshi (1995) considered the version of the extended KdV equation of (1.1) with only higher-order dispersive terms present (only c_4 is nonzero). They showed that the solitary-wave solution has the form

$$\eta = a \,\text{sech}^2\, \gamma\theta + \frac{b}{\epsilon^2} \exp\left(-\pi k/2\gamma\epsilon\right) \sin\left(k\,|\theta|/\epsilon - \delta\right) + \cdots, \qquad \epsilon^2 = \alpha c_4 \ll 1, \tag{1.6}$$

when $c_4 > 0$. Note that k is the wavenumber, the soliton ampltiude $a = 2\gamma^2$, and b is the amplitude of the nonlocal oscillatory tail, which is a function of the phase shift δ. Expression (1.6) is just the first term of the series for the higher-order solitary wave and the nonlocal oscillatory tail of exponentially small amplitude. Kichenassamy & Olver (1992) concluded that, except for the parameter combination of (1.5), exact, higher-order, solitary-wave solutions do not exist for the extended KdV equation of (1.1). However, series solutions for higher-order solitary waves which neglect exponentially small nonlocal tails (such as the tail in (1.6)) are good approximations to the exact solution of the extended KdV equation of (1.1) on all but the longest timescales.

Marchant & Smyth (1990) used the transformation

$$\eta = u + \tfrac{1}{12}\alpha c_1 (2u^2 + u_{xx}) \quad (\alpha \ll 1) \tag{1.7}$$

to transform (1.1), with only the higher-order coefficient c_1 nonzero, to the KdV equation (i.e. u satisfies the KdV equation when terms of $O(\alpha^2)$ are neglected). Transformation (1.7) was used to derive modulation equations, describing various wave properties—such as the amplitude, mean height, and wavenumber—for the extended KdV equation in this special case from the modulation equations for the KdV equation. The higher-order undular bore solution was then found as a centred simple wave solution of these modulation equations. This solution was used to model resonant fluid flow over topography more accurately. Transformation (1.7) also implies that higher-order, solitary-wave collisions for (1.1) with only c_1 nonzero are elastic to at least $O(\alpha)$; hence the change in amplitude can be no larger than the neglected terms, which are of $O(\alpha^2)$.

Kodama (1985) obtained an approximate Hamiltonian for the extended KdV equation of (1.1). This Hamiltonian is exact for the integrable version of the extended KdV equation, with coefficients given by (1.3), and it is accurate to $O(\alpha)$ in the general case. The Hamiltonian system is transformed, to $O(\alpha)$, to an integrable system, which implies that the extended KdV equation with arbitrary coefficients is approximately integrable. The asymptotic transformation used by Kodama (1985) includes the elements of transformation (1.7) and a nonlocal term. In this paper, transformation (1.7) is extended in a different manner to that used by Kodama (1985), to allow the extended KdV equation with arbitrary coefficients (1.1) to be transformed, to $O(\alpha)$, to the version of the extended KdV equation which is part of the KdV hierarchy of integrable equations ((1.1) with coefficients given by (1.3)).

In Section 2, the transformation, and the higher-order two-soliton solution of the integrable extended KdV equation with coefficients given by (1.3), is used to write down the higher-order two-soliton solution for (1.1), which implies that higher-order solitary waves of (1.1) are elastic to at least $O(\alpha)$. The $O(\alpha)$ corrections to the phase shifts of the higher-order solitary waves after collision are also found.

In Section 3, numerical results for higher-order solitary-wave collisions are presented for three cases. The first case is the integrable extended KdV equation with coefficients given by (1.3), which is known to possess soliton solutions. The numerical solution in this case allows error estimates for the numerical scheme to be determined. The other two cases are of a collision involving two higher-order solitary waves for versions of the extended KdV equation, equation (1.1), describing surface waves on shallow water with only coefficient c_3 nonzero. The conclusion is reached that higher-order solitary-wave collisions of (1.1) are elastic to at least $O(\alpha^2)$ with a change in the solitary-wave amplitudes no larger than $O(\alpha^3)$. Evidence of inelastic behaviour in these two cases is found; an oscillatory wavetrain of extremely small amplitude is found behind the smaller, higher-order solitary wave after the collision. In addition, there is good agreement between the higher-order phase shifts of the solitary waves, derived in Section 2, and the numerically obtained values. The Appendix details the numerical scheme used to solve (1.1).

2. The higher-order two-soliton solution

If the transformation

$$\begin{gathered}\eta = u + \alpha\left(\frac{3c_2}{2} - \frac{c_1}{6} - 10c_4\right)u^2 + \alpha\left(\frac{c_2}{12} + \frac{2c_3}{3} - \frac{c_1}{12} - \frac{35c_4}{6}\right)u_{xx}, \\ \tau = t + \alpha\left(\frac{5c_4}{3} - \frac{c_3}{6}\right)x, \qquad \xi = x, \qquad \alpha \ll 1,\end{gathered} \tag{2.1}$$

is substituted into (1.1), then $u(\xi, \tau)$ is a solution of the extended KdV equation

$$u_\tau + 6uu_\xi + u_{\xi\xi\xi} + \alpha' u^2 u_\xi + \tfrac{2}{3}\alpha' u_\xi u_{\xi\xi} + \tfrac{1}{3}\alpha' u u_{\xi\xi\xi} + \tfrac{1}{30}\alpha' u_{\xi\xi\xi\xi\xi} = 0, \qquad (2.2)$$

where $\alpha' = \alpha 15(c_3 - 8c_4) \ll 1$, and terms of $O(\alpha^2)$ are neglected. Since (2.2) is a member of the family of integrable KdV equations, it has an N-soliton solution. Transformation (2.1) shall be applied to the two-soliton solution of the extended KdV equation, equation (2.2), to obtain the corresponding higher-order two-soliton solution for the extended KdV equation, equation (1.1), with arbitrary coefficients. Note in the special case when $c_3 = 8c_4$ that $\alpha = 0$ and that transformation (2.1) reduces the extended KdV equation, equation (1.1), to the KdV equation. If the nonlocal term in the transformation of Kodama (1985) is added to (2.1) then an asymptotic transformation is obtained which reduces the extended KdV equation with arbitrary coefficients to the KdV equation.

The two-soliton-solution of the extended KdV equation, equation (2.2), is

$$\tfrac{1}{8}u = \frac{(k_1^*)^2 f_1 + (k_2^*)^2 f_2 + (k_2^* - k_1^*) f_1 f_2 + m[(k_2^*)^2 f_1^2 f_2 + (k_1^*)^2 f_1 f_2^2]}{(1 + f_1 + f_2 + m f_1 f_2)^2}, \qquad (2.3)$$

where

$$f_i = \exp 2k_i^*(V_i^*\tau - \xi + s_i) \quad (i = 1, 2),$$

$$V_i^* = 2A_i^* + \alpha' \tfrac{2}{15}(A_i^*)^2 \quad \text{and} \quad m = \left(\frac{k_2^* - k_1^*}{k_2^* + k_1^*}\right)^2$$

(Newell, 1985). The velocity of the ith soliton is V_i^*, and its position is $s_i + V_i^*\tau$. The two-soliton solution given by (2.3) is the same as the KdV two-soliton solution except for the correction at $O(\alpha')$ to the velocity. Well before interaction (as $\tau \to -\infty$) the two-soliton solution of (2.3) is

$$u = A_1^* \operatorname{sech}^2 \theta_1 + A_2^* \operatorname{sech}^2 \theta_2, \qquad (2.4)$$

where

$$\theta_i = k_i^*(\xi - s_i - V_i^*\tau),$$

$$V_i^* = 2A_i^* + \alpha' \tfrac{2}{15}(A_i^*)^2 \quad (i = 1, 2).$$

Expression (2.4) is just the sum of two solitons. The one-soliton solution ($A_2^* = 0$) can be found directly from results given by Kichenassamy & Olver (1992) or by Marchant & Smyth (1990). Expression (2.4) satisfies the extended KdV equation of (2.2) because the solitons are a long distance apart (hence the interaction terms, such as $f_1 f_2$, in (2.3), are all negligible). Also we choose $A_1^* > A_2^*$ and $s_1 < s_2$; this means that the larger soliton is behind the smaller one initially. Since the first soliton is larger, it travels faster; hence it will interact with the second soliton as it overtakes it. To see the result of the collision, the solution is considered well after the interaction (as $\tau \to \infty$). After the interaction the solution is again given by (2.4), but with the phase shifts

$$+\frac{1}{k_1^*}\log\left(\frac{k_1^* + k_2^*}{k_1^* - k_2^*}\right) \quad \text{and} \quad -\frac{1}{k_2^*}\log\left(\frac{k_1^* + k_2^*}{k_1^* - k_2^*}\right) \qquad (2.5)$$

for the larger and smaller solitons, respectively. These are the phase shifts of KdV solitons; solitons corresponding to higher-order equations (such as (2.2)) in the KdV hierarchy have the same phase shifts. Hence the soliton collision is elastic, with no change in the solitons' shapes (hence mass and energy are conserved for each soliton) and no dispersive radiation is produced as a result of the interaction. The only effect of the collision is a forward phase shift for the larger soliton and a backward phase shift for the smaller soliton.

The higher-order two-soliton solution, to $O(\alpha)$, describing the interaction of two higher-order solitary waves governed by the extended KdV equation of (1.1) with arbitrary coefficients is just the extended KdV two-soliton solution of (2.3) transformed by using (2.1). Due to the complicated form of (2.3) the explicit higher-order two-soliton solution will not be calculated; the nature of the collision can be found by considering the solution well before and after interaction. Expression (2.4) describes the higher-order two-soliton solution of (2.3) before and after interaction; substitution of (2.4) into transformation (2.1) gives

$$\eta = A_1 \operatorname{sech}^2 \theta_1 + A_2 \operatorname{sech}^2 \theta_2 + \alpha A_1^2 c_5 \operatorname{sech}^2 \theta_1 + \alpha A_2^2 c_5 \operatorname{sech}^2 \theta_2 + \alpha A_1^2 c_6 \operatorname{sech}^4 \theta_1 + \alpha A_2^2 c_6 \operatorname{sech}^4 \theta_2 + \cdots,$$

where

$$c_5 = (\tfrac{2}{3}c_3 - \tfrac{1}{6}c_1 + \tfrac{1}{6}c_2 - 5c_4), \qquad c_6 = (\tfrac{15}{2}c_4 - \tfrac{1}{2}c_3 + \tfrac{1}{12}c_1 - \tfrac{1}{4}c_2), \tag{2.6}$$

$$\theta_i = k_i\{x - s_i[1 + \alpha(\tfrac{10}{3}c_4 - \tfrac{1}{3}c_3)A_i] - V_i t\},$$

$$V_i = 2A_i + \alpha 4c_4 A_i^2 + \cdots \quad (i = 1, 2),$$

where the cross terms (such as $\operatorname{sech}^2 \theta_1 \operatorname{sech}^2 \theta_2$) are negligible because of the large distance between the solitons.

Marchant & Smyth (1990) derived the higher-order cnoidal wave solution for (1.1). The solitary-wave limit of this solution (their (2.25)) is the same as (2.6) in the special case of one solitary wave ($A_2 = 0$). It should be noted that the transformation has shifted the higher-order solitary wave slightly, from $\xi = s_1$ at $\tau = 0$ to $x = s_1[1 + \alpha(10c_4 - c_3)A_1/3]$ at $t = 0$. Hence it can be seen that transformation (2.1) gives the appropriate expression for the case of a single, higher-order solitary wave.

Expression (2.6) is just two higher-order solitary waves as required. As for the integrable case, the higher-order solitary waves are unchanged in shape after the collision. The transformation modifies the phase shifts of the extended KdV solitons given in (2.5). After interaction, the phase shifts for the higher-order solitary waves, corresponding to (1.1), are

$$+\frac{1}{k_1}\log\left(\frac{k_1 + k_2\Delta}{k_1 - k_2\Delta}\right) \quad \text{and} \quad -\frac{1}{k_2}\log\left(\frac{k_1 + k_2\Delta}{k_1 - k_2\Delta}\right), \tag{2.7}$$

where

$$\Delta = 1 + \tfrac{1}{3}\alpha(10c_4 - c_3)(A_2 - A_1),$$

for the larger and smaller higher-order solitary waves, respectively. Note that in the special case $c_3 = 10c_4$ the phase shifts are unchanged from the KdV case.

In summary, the interaction of two higher-order solitary waves, governed by the extended KdV equation (1.1), is elastic to $O(\alpha)$, which is the order of approximation of the extended KdV equation and of transformation (2.1). The higher-order solitary waves are solitons; they maintain their shape (and hence their mass and energy) after interaction, the only effect of the collision being the phase shifts (2.7).

In the special case of surface waves on shallow water, governed by (1.1) with the coefficients given by (1.2), the phase shifts are, after expanding (2.7) in a Taylor series,

$$+\frac{1}{k_1}\log\left(\frac{k_1+k_2}{k_1-k_2}\right)-2\alpha k_2 \quad \text{and} \quad -\frac{1}{k_2}\log\left(\frac{k_1+k_2}{k_1-k_2}\right)+2\alpha k_1, \tag{2.8}$$

for the larger and smaller waves, respectively. The phase shifts given in (2.8) are, after appropriate scalings of space, time, and the amplitude α, the same as those obtained by Zou & Su (1986) directly from the Euler water-wave equations.

3. Numerical results

In Section 2 it was found that the higher-order solitary waves of (1.1) interact in an elastic manner to $O(\alpha)$. In this section, the interaction of higher-order solitary waves is examined numerically (the Appendix gives details of the numerical scheme) to determine the exact nature (at $O(\alpha^2)$ and higher) of the interaction. Of particular interest is the interaction of surface waves on shallow water, which corresponds to (1.1) with the parameters given by (1.2).

3.1 *Accuracy of the numerical method*

For an elastic collision, both the masses and the energies of the individual solitons are unchanged by the collision. If the energy of the higher-order solitary waves changes due to the collision, then the collision is not elastic and a dispersive wavetrain will be generated behind the solitary waves. Hence we examine both the amplitude and energy of each higher-order solitary wave before and after interaction to determine if the collision is elastic. Also of interest is the amplitude of any dispersive wavetrain generated by the collision.

The mass and energy of the wavetrain are

$$M=\int_{-\infty}^{+\infty}\eta\,\mathrm{d}x \quad \text{and} \quad E=\tfrac{1}{2}\int_{-\infty}^{+\infty}\eta^2\,\mathrm{d}x, \tag{3.1}$$

respectively. Substituting the expression for a single higher-order solitary wave ((2.6) with $A_2=0$) into the second expression of (3.1) gives

$$E=\frac{2}{3k}A^2[1+\tfrac{1}{30}\alpha A(210c_4+2c_3-3c_1-7c_2)], \tag{3.2}$$

for the energy, to $O(\alpha)$, of a higher-order solitary wave of first-order amplitude

A. The first-order amplitude A is related to the numerically obtained second-order amplitude

$$A[1 + \tfrac{1}{12}\alpha(30c_4 + 2c_3 - c_1 - c_2)A]. \tag{3.3}$$

The energy of each higher-order solitary wave can be calculated using (3.2), (3.3), and the numerically obtained second-order wave amplitude. Hence it is assumed that after interaction the numerical solutions have regained the form (3.1).

When the higher-order solitary wave of (2.6) (with $A_2 = 0$) propagates alone, it evolves to a numerically exact solitary-wave solution of (1.1). This evolution, of magnitude $O(\alpha^2)$, changes the amplitude of the solitary waves and causes mass and energy to be shed, which results in the production of a dispersive wavetrain behind the solitary wave. Hence, if the form (2.6) is used for the initial higher-order solitary waves, any effects due to the interaction will be obscured by the change in amplitude and by the dispersive wavetrain generated by the numerical evolution of these higher-order solitary waves. To eliminate this effect numerically, exact higher-order solitary waves are generated by propagation on the numerical grid of a single higher-order solitary wave ((2.6) with $A_2 = 0$). The numerically exact higher-order solitary waves, trimmed to a finite width on which the surface profile is greater than 1×10^{-6}, are copied to a new numerical grid. The initial condition is then composed of the larger numerically exact higher-order solitary wave located at $x = 42$ with the smaller wave located at $x = 78$. The wave amplitudes are chosen so that they are approximately equal to 1 and $\frac{1}{3}$ so the waves interact at a time $t \approx 54$ and the calculation is continued up to a time $t = 95$ to allow the higher-order solitary waves to separate and to allow any dispersive wavetrain to completely form behind the smaller wave. The numerical scheme uses damped boundaries (see (A.4)) which absorb any dispersive radiation. This prevents small-amplitude high-frequency waves being reflected back into the solution domain, which can also obscure any effects due to the interaction.

Fundamental quantities which should be conserved during any motion governed by the extended KdV equation of (1.1) include the total mass and energy of the fluid. Table 1 presents these two constants of the fluid motion, giving the

TABLE 1
Mass and energy conservation

	Before numerical integration		After numerical integration		Error ($\times 10^4$)	
Example	Mass	Energy	Mass	Energy	Mass	Energy
1	2·828 093	0·942 821	2·828 092	0·942 821	2	0·5
	4·452 788	1·121 550	4·452 786	1·121 550	3	0·1
2	2·880 601	0·942 513	2·880 596	0·942 517	5	0·1
	4·517 588	1·122 084	4·517 557	1·122 089	5	0·1
3	3·075 146	1·029 151	3·075 149	1·029 151	2	0·5
	4·751 265	1·214 155	4·751 282	1·214 155	2	0·6

values for the larger higher-order solitary wave propagating alone (first entry) and of the full interacting system (second entry) both before and after the numerical integration of the extended KdV equation of (1.1) for the three examples. The quantities are calculated using Simpson's method. For examples 1 and 2, a spatial step of $\Delta x = 0{\cdot}15$ was used, while a spatial step of $\Delta x = 0{\cdot}075$ was chosen for example 3. The discretization error in the quantities of $O(\Delta x^2)$, was estimated by using Richardson extrapolation between the presented results and those obtained with a larger spatial step. The discretization error in the mass and energy was found to be no larger than 0·01%, while the change in mass and energy before and after the numerical runs was much less than the magnitude of the discretization error. Hence it seems that most of the error, as predicted by Richardson extrapolation, is consistent between time steps. However, in summary, the error is very small, allowing the nature of the collision to be examined to $O(\alpha^2)$.

Table 2 shows the numerical amplitude and energy of the individual higher-order solitary waves both before and after interaction for the three examples. To improve the accuracy of the numerical results a quadratic curve was fitted to the three grid points surrounding the solitary-wave's peak. The position and amplitude of the solitary wave was then assumed to occur at the turning point of the quadratic curve.

Example 1 is the extended KdV equation with coefficients given by (1.3), which is a member of the KdV hierarchy of integrable equations. The parameters were $\alpha = 0{\cdot}4$, $\Delta x = 0{\cdot}15$, and $\Delta t = 10^{-3}$. For this case a two-soliton solution is known to exist, so the interaction is elastic. Figure 1 shows a perspective plot of the higher-order solitons of example 1 up to $t = 50$. Examination of the numerical solution allows the size of the numerical errors to be determined. The amplitudes and energies of the two higher-order solitons before ($t = 0{\cdot}2$) and after interaction ($t = 95$) were found. Firstly, as a check, numerical runs were performed in which each higher-order soliton propagated alone (without interaction with the other wave). Table 2 shows that the change in wave amplitude for both higher-order solitons, both with and without interaction, was much less than the discretization error, which was about 0·01% of the amplitude. Examination of the free surface behind the smaller higher order solitary wave after interaction (at $t = 95$) showed the presence of a spurious dispersive wavetrain. However, the amplitude of this wavetrain was extremely small, with the largest peak in the wavetrain having an amplitude of about 2×10^{-6}.

Herman & Knickerbocker (1993) considered the truncation error associated with a scheme used by Zabusky & Kruskal (1965) for the KdV equation. In particular, they showed that the $O(\Delta x^2)$ error in the soliton's velocity introduced by the numerical scheme caused a large error in the soliton's position at long times. For example 1, the larger soliton was, without interaction, at $x = 236{\cdot}29$ at $t = 95$. The soliton's velocity (the last equation of (2.6)), however, places the wave at $x = 237{\cdot}40$ after $t = 95$. The discrepancy in the position, then, due to the discretization error of the numerical scheme, is about 0·5% of the distance travelled at $t = 95$; compare this error with the $O(\alpha)$ difference in the distance

TABLE 2
Numerical data for the solitary-wave interactions

Example	Before		After, no interaction		Error in amplitude ($\times 10^4$)	After, with interaction		Phase shift		
	Amplitude	Energy	Amplitude	Energy		Amplitude	Energy	KdV theory	eKdV theory[a]	Numerical
1	1·001 79	0·945 34	1·001 78	0·945 34	1	1·001 76	0·945 30	1·874	1·847	1·859
	0·330 23	0·178 92	0·330 23	0·178 92	0·003	0·330 23	0·178 92	−3·217	−3·217	−3·221
2	0·985 17	0·918 28	0·985 17	0·918 28	7	0·985 16	0·918 26	2·040	1·962	1·957
	0·329 62	0·179 16	0·329 62	0·179 16	2	0·329 63	0·179 16	−3·427	−3·925	−3·283
3	0·993 42	1·009 02	0·993 42	1·009 02	4	0·993 42	1·009 02	1·989	2·212	2·176
	0·329 82	0·184 61	0·329 82	0·184 61	0·6	0·329 82	0·184 61	−3·372	−3·753	−3·701

[a] Extended kdV theory.

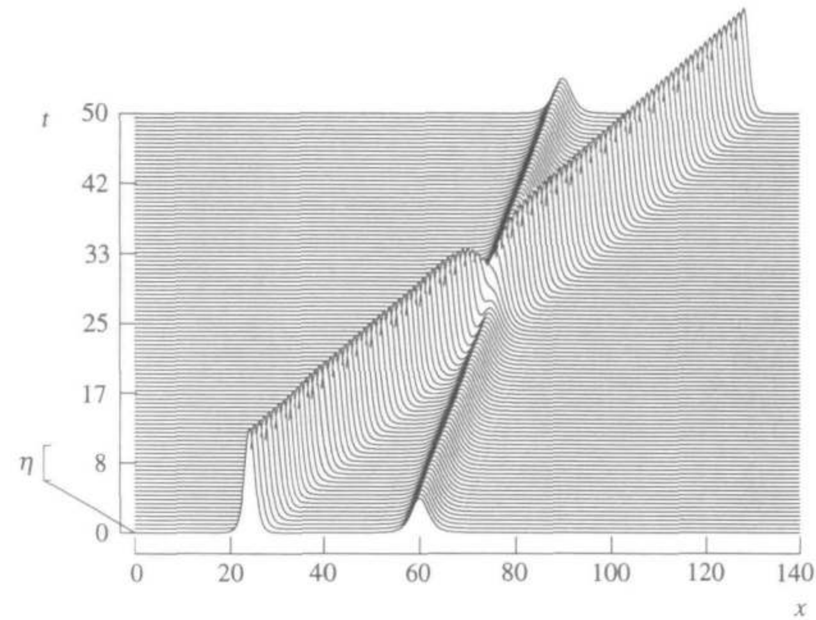

FIG. 1. A perspective plot of the interaction of two higher-order solitons up to $t = 50$. The parameters were $\alpha = 0{\cdot}4$, $\Delta x = 0{\cdot}15$, and $\Delta t = 10^{-3}$, and the coefficients are given by (1.3).

travelled of about 2·5% between a KdV soliton and its higher-order counterpart in this case. Hence the discretization error here is about $O(\alpha^2)$.

The numerical phase shift of each higher-order soliton, which is clearly visible in Fig. 1, was taken to be the difference in the positions of each higher-order soliton at $t = 95$ for numerical runs involving interaction and in which the higher-order soliton propagates alone. This eliminates from the calculation the error in the soliton's position due to the discretization because this error is the same for both numerical runs. The phase shifts of the higher-order solitons obtained numerically, together with the theoretical predictions for the KdV equation, as given by (2.5), are presented in Table 2. The correspondence between the numerically obtained phase shifts and the exact phase shifts is excellent, with an error of about 0·5% between the two results, due to the finite grid size.

In summary, the numerical scheme is capable of determining the higher-order-soliton amplitudes to about 0·01% (which is generally less than any $O(\alpha^2)$ changes), and it produces a spurious dispersive wavetrain after collision of no more than 10^{-4}% of the amplitude of the larger soliton. In addition, the phase shifts of the higher-order solitary waves after interaction can be found to about 0·5%. These error bounds will be used as benchmarks when examining other special cases of the extended KdV equation of (1.1) for which the exact nature of the solitary-wave interactions is not known.

3.2 *Inelastic examples*

Example 2 is for the case $\alpha = 0{\cdot}1$, $\Delta x = 0{\cdot}15$, and $\Delta t = 10^{-4}$ and with the parameters given by (1.2). Example 2 corresponds to surface waves on shallow water. Table 2 shows the amplitudes and energies of the two higher-order solitary waves before ($t = 0{\cdot}2$) and after interaction ($t = 95$), as well as for test runs up to $t = 95$ with no interaction. Comparison of the amplitudes show that any changes are much smaller than the discretization error, which is about 0·07% both with and without interaction. For example 2 the $O(\alpha)$ differences in the amplitude and energy between the larger KdV soliton and its higher-order counterpart are about 10% and 12%, respectively. Hence it can be concluded that the changes in amplitudes after interaction are smaller than $O(\alpha^2)$.

It was found that the amplitude of the largest wave in the dispersive wavetrain generated by the interaction was 4×10^{-5}. Richardson extrapolation indicates that this numerical estimate of the dispersive-wave amplitude is about 50% larger than the converged amplitude as Δx, $\Delta t \to 0$. The converged amplitude is an order of magnitude larger than the spurious dispersive wavetrain, of amplitude $O(10^{-6})$, generated behind the solitons in Fig. 1. Also, note that no discernible dispersive wavetrain is generated by the propagation of single, numerically exact, higher-order solitary waves. Hence the dispersive wavetrain can be taken as evidence that the interaction of higher-order, solitary, water waves is inelastic, even though no change in the solitary-wave amplitude could be detected after the interaction.

Numerical calculations show that the mass in the dispersive wavetrain is negative and of $O(10^{-5})$, whilst the energy is negligible. As the pair of higher-order solitary water waves have suffered a net loss in mass and no change in energy after collision, conservation of mass and energy implies that the larger, higher-order solitary wave has increased in amplitude and that the smaller, higher-order solitary wave has decreased in amplitude, with the change in amplitudes being $O(10^{-5})$.

Numerical calculations for steeper, solitary water waves confirm these predictions. For the case $\alpha = 0{\cdot}2$, the largest wave in the dispersive wavetrain has a converged amplitude of $1{\cdot}2 \times 10^{-4}$, again with the amplitude of the larger wave increasing and that of the smaller wave decreasing after interaction. Figure 2 shows a perspective plot of the dispersive wavetrain generated by the interaction of these higher-order solitary water waves. The parameters were $\alpha = 0{\cdot}2$, $\Delta x = 0{\cdot}3$, and $\Delta t = 1 \times 10^{-3}$, and the coefficients are given by (1.2). So that the dispersive wavetrain is visible, the higher-order solitary waves are cut off at a height of 1×10^{-3}.

Table 2 shows comparisons of the numerically obtained phase shifts of example 2 with the KdV theory of (2.5) and with the extended KdV (eKdV) theory of (2.8). In this case the $O(\alpha)$ correction to the KdV phase shift is about 4%. The difference between the numerically obtained phase shifts and the extended KdV theory is less than 0·4%, which is the size of the numerical error in example 1.

Example 3 is for the case $\alpha = 0{\cdot}4$, $c_1 = c_2 = c_4 = 0$, $c_3 = 1$, $\Delta x = 0{\cdot}075$, and $\Delta t = 2{\cdot}5 \times 10^{-4}$. This example was chosen because its numerical scheme is stable

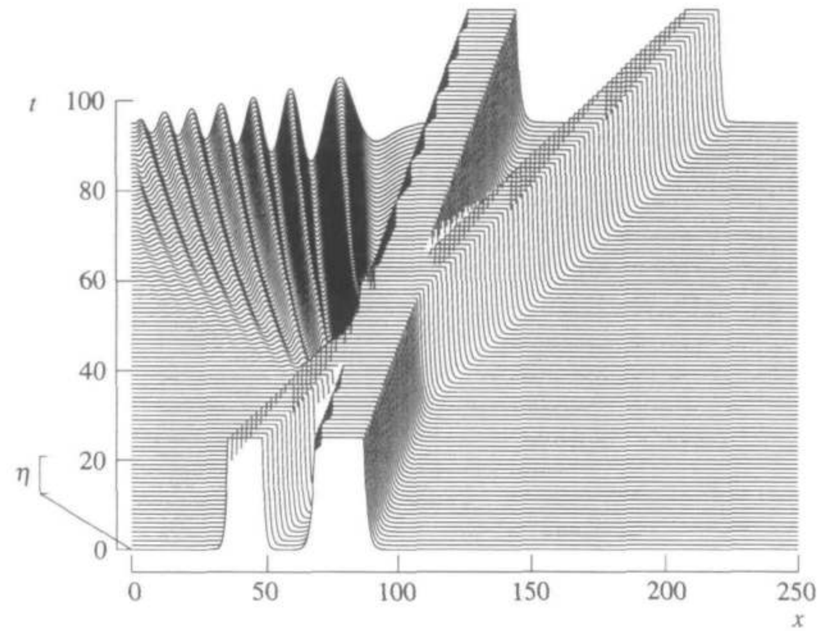

FIG. 2. A perspective plot of the dispersive wavetrain generated by the interaction of two higher-order solitary water waves up to $t = 95$. The solitary waves are cut off at a height of 1×10^{-3}. The parameters were $\alpha = 0{\cdot}2$, $\Delta x = 0{\cdot}3$, and $\Delta t = 10^{-3}$, and the coefficients are given by (1.2).

for much steeper waves (larger α) than for the version of the extended KdV equation considered in example 2. Table 2 shows the amplitudes and energies of the two higher-order solitary waves before ($t = 0.2$) and after interaction ($t = 95$), as well as for test runs up to $t = 95$ with no interaction. A comparison of the amplitudes and energies shows that no change was greater than the discretization error, which was about 0.04% in this case. Here the $O(\alpha)$ differences in the amplitude and energy between the larger KdV soliton and its higher-order counterpart are about 6·5% and 20%, respectively. Hence, as in example 2, it can be concluded that the changes in higher-order solitary-wave amplitudes after interaction are smaller than $O(\alpha^2)$.

Figure 3 shows the dispersive wavetrain generated by the interaction of the higher-order solitary waves, in time intervals of 10, for the parameters of example 3. So that the dispersive wavetrain is visible, the higher-order solitary waves are cut off at a height of 6×10^{-5}. The amplitude of the largest wave, $2{\cdot}8 \times 10^{-5}$, is an order of magnitude larger than the spurious dispersive wavetrain generated by the numerical scheme in example 1. Richardson extrapolation indicates that the error in this numerical estimate of the amplitude is about 10%. Here the error is much smaller than the corresponding error in the amplitude of the dispersive wavetrain of example 2 since a much smaller spatial step has been used. In this case the mass in the dispersive wavetrain is positive and of $O(10^{-6})$, while the energy is negligible. This implies that the larger wave increases in amplitude and the smaller wave decreases after interaction, with the changes in

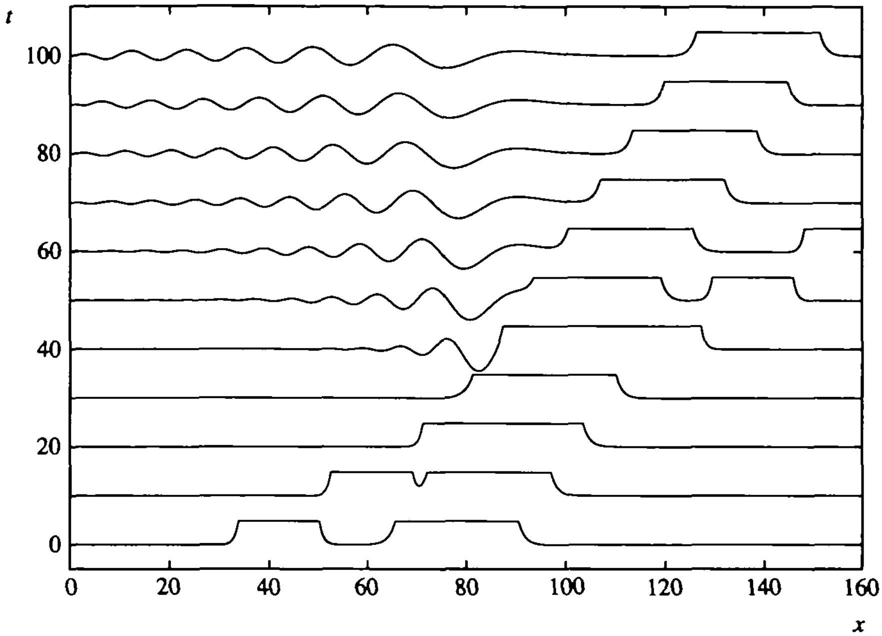

FIG. 3. The dispersive wavetrain produced by the interaction of two higher-order solitary waves at time steps up to $t = 100$ in intervals of 10. The solitary waves are cut off at a height of 6×10^{-5}. The parameters were $\alpha = 0{\cdot}4$, $c_1 = c_2 = c_4 = 0$, $c_3 = 1$, $\Delta x = 0{\cdot}075$, and $\Delta t = 2{\cdot}5 \times 10^{-4}$.

amplitude of $O(10^{-6})$. The conclusions are similar to those for example 2; the collision is inelastic, but no change in the solitary-wave amplitude could be found after collision.

Figure 4 shows the location of the higher-order solitary wave-crests in time for example 3. Both before and after the interaction, the trajectories of the higher-order solitary waves are straight lines. The constant velocities represented by these straight lines are 1·870 and 0·645 before the collision and 1·868 and 0·645 after the collision. Hence there is a difference in the respective solitary-wave velocities of at most 0·1%. In this case the wavespeed of KdV solitons and their higher-order counterparts is the same, but the magnitude of the $O(\alpha)$ difference in the amplitude indicates that the change in the wavespeed after interaction is less than $O(\alpha^2)$. Hence the higher-order solitary-wave trajectories, as shown on Fig. 4, are parallel, to $O(\alpha^2)$, before and after interaction.

Figure 4 shows that, during the interaction, the crests initially merge, and only one peak can be identified. However, the peaks then separate, and at the interaction centre two peaks form close together with a shallow trough between them. After this, the peaks merge again into one before finally separating into two higher-order solitary waves after the collision. According to the classification given by Lax (1968), this means the collision is of type b. It should be noted that the dynamics of the interaction itself has little effect on the waves after the interaction. Table 2 gives comparisons of the phase shifts. In this case the $O(\alpha)$

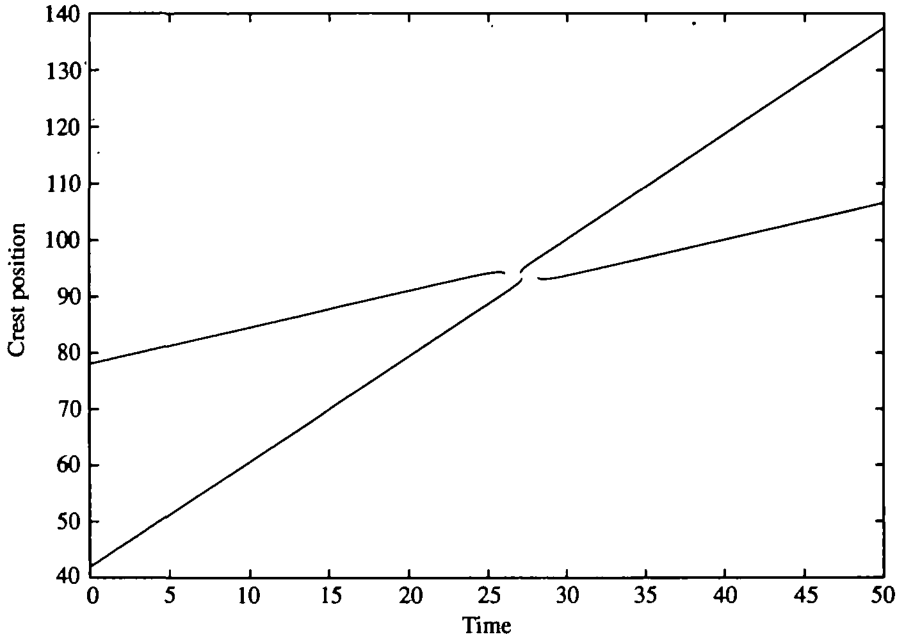

FIG. 4. The position of the two higher-order solitary-wave crests for times up to $t = 50$. The parameters were $\alpha = 0{\cdot}4$, $c_1 = c_2 = c_4 = 0$, $c_3 = 1$, $\Delta x = 0{\cdot}075$, and $\Delta t = 2{\cdot}5 \times 10^{-4}$.

correction to the KdV phase shift is about 10%. It can be seen the difference between the numerically obtained phase shifts and the phase shifts obtained using the extended KdV theory, which is presumably an $O(\alpha^2)$ effect, is less than 2%.

Previous numerical work on higher-order solitary-water-wave interaction has been done by Fenton & Rienecker (1982), by Mirie & Su (1982), and by Zou & Su (1986), while theoretical studies have been made by Sachs (1984) and by Byatt-Smith (1987a). Fenton & Rienecker (1982) solved the full Euler equations numerically, and they considered an example of higher-order solitary waves interacting whilst travelling in the same direction. Their example corresponds to $\alpha \approx 0{\cdot}45$ in our notation, a case for which the numerical scheme used here is unstable. They found that the larger solitary wave increased in (unscaled) amplitude from 0·3252 by 0·00364, whilst the smaller solitary wave decreased in amplitude from 0·1035 by 0·0018. Hence the change in scaled amplitude is of $O(\alpha^2)$. These results are qualitatively consistent with example 2, which also predicts an increase in amplitude for the larger wave and a decrease for the smaller wave after interaction.

Numerical solutions of a higher-order Boussinesq equation were found by Mirie & Su (1982). The equation considered was not consistent with the perturbation expansion of the full Euler equations to the appropriate order because terms involving fifth-order derivatives were neglected. The neglect of these terms makes the numerical scheme much easier to implement, whilst the solutions are qualitatively representative of solutions of the higher-order Boussinesq equation appropriate for surface waves on shallow water. Both weak (head-on) and strong (overtaking) solitary-wave collisions were considered. A head-on collision of solitary waves of dimensional amplitude 0·5 results in a change in amplitude of about 2% after collision coupled with the creation of an oscillatory wavetrain of small amplitude. A collision between a solitary wave of dimensional amplitude 0·6 overtaking a solitary wave of dimensional amplitude 0·2 shows that the waves recover 99% of their original amplitude. While an oscillatory wavetrain is produced, its amplitude is compared with the numerical error. Hence no conclusion about the elasticity of solitary-wave interactions could be deduced in the case of a solitary wave overtaking another, except that the change in wave amplitude was less than $O(\alpha)$.

Zou & Su (1986) considered a perturbation expansion of the Euler water-wave equations. The partial differential equations describing the effects at $O(\alpha)$ and $O(\alpha^2)$ were solved numerically. At $O(\alpha)$ the interaction was found to be elastic, while at $O(\alpha^2)$ the solution consisted of a correction to the phase shifts and of a dispersive wavetrain which occurred behind the smaller solitary wave after collision; hence the collision was inelastic. In the example presented of solitary waves with amplitudes in the ratio of four to one, the oscillatory wavetrain has an amplitude of about 10% of the correction to the phase shift at $O(\alpha^2)$. There is no change in the solitary-wave amplitude, which is presumed to be an $O(\alpha^3)$ effect. These conclusions are consistent with the results presented in example 2 for surface waves on shallow water.

Byatt-Smith (1987a) used a perturbation method based on the method of inverse scattering to examine the higher-order solitary-wave interaction for the extended KdV equation. The perturbation equations (see his (4.10)) which describe the evolution of the scattering data were considered for the special case of the two-soliton solution of the KdV equation. This gives the change in amplitude of the solitary waves due to the higher-order terms, at $O(\alpha)$, of the extended KdV equation. The change in the solitary-wave amplitude was found in terms of a number of integrals which were simplified to a single equivalent integral, and the change in the solitary-wave amplitude was shown to be of $O(\alpha)$. For example 2, his theory predicted that the larger solitary wave increases in amplitude by about 4% and that the smaller solitary wave decreases in amplitude by about 20%. This change in amplitude is much larger than that found numerically in Section 3, and it also contradicts the theoretical results of Section 2 and those given by Sachs (1984). A numerical integration of the perturbation equations given by Byatt-Smith (1987a; his (4.10)) by the present authors showed no change in the solitary-wave amplitude after interaction. The present authors believe that there are algebraic errors in Appendix B given by Byatt-Smith (1987a), which simplifies the various integrals of his (4.15) to his equivalent integral (4.19).

Numerical evidence has been presented for surface water waves of moderate steepness ($\alpha = 0{\cdot}1$ and $\alpha = 0{\cdot}2$) and for much steeper higher-order solitary waves

($\alpha = 0{\cdot}4$) governed by another version of the extended KdV equation. This has confirmed the theoretical prediction of Section 2 that higher-order solitary waves governed by (1.1) interact in an elastic manner to $O(\alpha)$ and that the phase shifts of higher-order solitary waves are given by (2.7) to $O(\alpha)$.

In addition, these examples strongly suggest that the interaction of solitary waves on shallow water, and indeed the interaction of higher-order solitary waves governed by the complete family of equations (1.1), is elastic to at least $O(\alpha^2)$. No change in wave amplitude can be detected after collision, but a dispersive wavetrain of very small amplitude is generated, which shows that the higher-order solitary-wave collisions are inelastic. The dispersive wavetrain allows an estimate of the change in wave amplitudes after collision to be found indirectly via mass and energy conservation. For surface waves on shallow water the larger wave is increased while the smaller wave is decreased in amplitude after collision.

Appendix: The numerical scheme

The numerical solutions of the extended KdV equation of (1.1) were obtained by using centred finite differences in the spatial coordinate x and a fourth-order Runge–Kutta method for the temporal coordinate t. This method was chosen over straight finite-difference methods (such as an extension to the finite-difference scheme used by the KdV equation by Zabusky & Kruskal, 1965) because of its stability. The hybrid fourth-order Runge–Kutta finite-difference method for (1.1) detailed below is bound to be stable for moderate values of α when reasonable choices are used for the space step Δx and for the time step Δt, in contrast to various straight finite-difference methods considered by the authors which were found to be unstable for nearly all nonzero values of α.

Given that the solution at the time t_i is

$$\eta_{i,j} = \eta(t_i = i\Delta t, x_j = j\Delta x) \quad (j = 1, \ldots, N), \tag{A.1}$$

then the solution at time t_{i+1} is given by

$$\eta_{i+1,j} = \eta_{i,j} + \tfrac{1}{6}(a_{i,j} + 2b_{i,j} + 2c_{i,j} + d_{i,j}) + \gamma(x_j) \quad (j = 1, \ldots, N),$$

where

$$\begin{aligned} a_{i,j} &= \Delta t f(\eta_{i,j}), & b_{i,j} &= \Delta t f(\eta_{i,j} + \tfrac{1}{2}a_{i,j}), \\ c_{i,j} &= \Delta t f(\eta_{i,j} + \tfrac{1}{2}b_{i,j}), & d_{i,j} &= \Delta t f(\eta_{i,j} + c_{i,j}), \end{aligned} \tag{A.2}$$

and $\gamma(x)$ is the damping function used at the boundaries. The function f is the finite-differenced form of all the terms in (1.1) involving spatial derivatives,

$$
\begin{aligned}
f(p_{i,j}) = & -\frac{3p_{i,j}}{\Delta x}\left(1+\frac{\alpha c_1}{6}p_{i,j}\right)(p_{i,j+1}-p_{i,j-1}) \\
& -\frac{1}{2\Delta x^3}(1+\alpha c_3 p_{i,j})(p_{i,j+2}+2p_{i,j+1}+2p_{i,j-1}-p_{i,j-2}) \\
& -\frac{\alpha c_2}{18\Delta x^3}(p_{i,j+1}-p_{i,j-1})(2p_{i,j+2}+p_{i,j+1}-6p_{i,j}+p_{i,j-1}-2p_{i,j-2}) \\
& -\frac{\alpha c_4}{8\Delta x^5}(p_{i,j+4}-2p_{i,j+3}-2p_{i,j+2}+6p_{i,j+1} \\
& \qquad -6p_{i,j-1}+2p_{i,j-2}+2p_{i,j-3}-p_{i,j-4}). \qquad \text{(A.3)}
\end{aligned}
$$

The damping function

$$\gamma(x_j) = 20[\operatorname{sech}^2(j\Delta x)] + \operatorname{sech}^2[(j-N)\Delta x] \qquad \text{(A.4)}$$

is used to absorb the small-amplitude dispersive radiation at the boundaries of the solution domain. If *et al.* (1987) describe its application to the nonlinear Schrödinger equation. The boundary conditions used are

$$\eta_{i,j} = 0 \quad (i = -3, \ldots, 0; j = N+1, \ldots, N+4). \qquad \text{(A.5)}$$

This boundary condition maintains the surface elevation at zero far ahead and behind the solitary waves. The accuracy of the numerical method at each time step is $O(\Delta t^4, \Delta x^2)$. Hence, for an integration to a time $t = M\Delta t$, the accuracy of the method is $O(\Delta t^3, \Delta x^2)$.

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