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Solitons and collapses: two evolution scenarios of nonlinear wave systems

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Abstract. Two alternative scenarios pertaining to the evolution of nonlinear wave systems are considered: solitons and wave collapses. For the former, it suffices that the Hamiltonian be bounded from below (or above), and then the soliton realizing its minimum (or maximum) is Lyapunov stable. The extremum is approached via the radiation of small-amplitude waves, a process absent in systems with finitely many degrees of freedom. The framework of the nonlinear Schrödinger equation and the three-wave system is used to show how the boundedness of the Hamiltonian — and hence the stability of the soliton minimizing it — can be proved rigorously using the integral estimate method based on the Sobolev embedding theorems. Wave systems with the Hamiltonians unbounded from below must evolve to a collapse, which can be considered as the fall of a particle in an unbounded potential. The radiation of small-amplitude waves promotes collapse in this case.

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1. Introduction

The main goal of this review is to demonstrate, based on the Hamiltonian approach, the difference between two main nonlinear wave phenomena: solitons, or solitary waves, and wave collapse — the process in which a wave field becomes singular in a finite time. The singularity type depends on the physical model. For example, the breaking of acoustic waves is accompanied by the formation of sharp gradients (the gradient catastrophe) or, in mathematical parlance, a fold [1, 2]. In light self-focusing, which is just another example of wave collapse, the light intensity becomes anomalously high as the focus is approached. For water waves, singularities in surface elevation have the form of wedges. In this case, the second derivative of surface elevation becomes infinite.

Collapses play a significant role in various branches of physics, not only in nonlinear optics and fluid dynamics but also in plasma physics, physics of the atmosphere and ocean, and solid state physics. For many physical systems, collapse, as a process of singularity formation in finite time, is one of the most effective mechanisms of wave energy conversion to heat. For example, the collapse of plasma waves determines the efficiency of various collective methods of plasma heating in problems of controlled thermonuclear fusion. In nonlinear optics, this process is stopped because of multi-photon absorption for moderate intensities and as a result of atom ionization for large amplitudes. Collapse in nonlinear optics can therefore be utilized to explore the interaction between light and matter over a wide range of intensities, including the processes of ionization by strong electromagnetic fields and the formation of beams of relativistic particles. One more possibility in optical systems — exploiting collapses to create ultrashort pulses — has long been discussed, but still awaits its experimental implementation.

In speaking about solitons, we suppose, first, that their lifetime, set by dissipation, is sufficiently large, and hence their dynamics, for example, as they interact with each other, can be treated as Hamiltonian; this is assumed virtually everywhere in the review. Second, the existence of solitons implies their stability. We consider two approaches to the analysis of soliton stability: by exploring stability at the infinitesimal level, i.e., based on linearized equations and the subsequent analysis of linear spectral problems, and by invoking the Lyapunov method, which is the most versatile method that allows assessing the stability under not only sufficiently small but also finite perturbations. According to the Lyapunov theorem, a soliton is stable if it realizes a minimum (or maximum) of the Hamiltonian.

The problem of soliton stability in its most complete form was discussed for the first time by us together with Rubenchik in review [3] published already more than twenty-five years ago. The research on soliton stability has seen dramatic changes since that time, especially in what concerns the use of integral majorizing inequalities that follow from the Sobolev embedding theorems. Earlier reviews did not touch at sufficient length on the role of unstable solitons in the process of singularity formation — the wave collapse. We demonstrate in what follows that the wave collapse unfolds in systems with unbounded Hamiltonians and can be interpreted as a particle fall in an unbounded potential. In that case, the soliton represents a saddle point that the system crosses when passing to the wave collapse mode.

This review mainly deals with two models: the nonlinear Schrödinger (NLS) equation and the three-wave system. These models have numerous applications in optics, plasma physics, fluid dynamics, etc. (see, e.g., review [4]). The NLS equation belongs to the class of universal models, it can be shown to describe the propagation of wave packets in a weakly nonlinear medium when the interaction with low-frequency (LF) acoustic waves is not essential. For example, it is well known [5, 6] that the propagation of optical solitons in optical fibers can be described by a one-dimensional NLS equation to a good accuracy. A two-dimensional NLS equation describes stationary light self-focusing in media with the Kerr nonlinearity. The universality of the NLS equation is manifested, in particular, in the universality of the methods used to explore the stability of solitons (see, e.g., review [3]).

The three-wave system describes solitons in media with quadratic nonlinearity, referred to as $\chi^{(2)}$-media in optics. This system is written for the amplitudes of three wave packets coupled via the quadratic nonlinearity. In a particular case, this system describes the interaction between the first and second harmonics. If the difference between the group velocities of three packets is sufficiently large, this system reduces to the Blombergen equations [7], which can be integrated with the help of the inverse scattering method [8]. For close group velocities, as is commonly the case in nonlinear optics, both the dispersion and diffraction terms have to be taken into account [9–11]. The model then represents a vector system of the NLS type, but with a quadratic nonlinearity.

Solitons in both systems exist as a result of balance between nonlinear interaction and dispersion. The soliton solutions for these two systems correspond to stationary points of their Hamiltonians for other integrals of motion, such as the momentum, the particle number, or the Manley–Rowe integrals, kept fixed. In other words, solitons are associated with conditional extrema. This is a very essential point. Had it not been the case, for example, had the solitons been associated only with a stationary point of the Hamiltonian, such a stationary point would be unstable for systems with Hamiltonians unbounded from below (i.e., in the absence of a vacuum). The last statement is in essence the main part of the Derrick arguments [12]. Luckily, solitons viewed as stationary points of Hamiltonians correspond to *conditional* extrema, which allows exploring their stability by resorting to the Lyapunov theorem. In this case, for proving the soliton stability, it suffices to show that the Hamiltonian is bounded (other motion integrals being fixed). Obviously, a soliton realizing a minimum (or maximum) of a given Hamiltonian is Lyapunov stable. Such an approach was first applied to Korteweg–de Vries (KdV) solitons by Benjamin (1972) [13], and later, to three-dimensional ion-acoustic solitons in strongly magnetized plasma by us [14]. It is now one of the most powerful methods of exploring soliton stability.

We note that it is often easier to prove the Lyapunov stability than to solve the linear stability problem. In the latter case, the completeness of all eigenfunctions of the linearized problem must be proved, which is an extremely difficult task. In this review, we pay special attention to embedding theorems and show how they lead to integral estimates, and then use them to prove the boundedness of Hamiltonians.

When a Hamiltonian is bounded and a soliton realizes its extremum, we can speak about the energy principle. A soliton realizing a minimum shows up as an attractor. In particular, under a collision of two such solitons, the formation of a single, more powerful soliton with lower energy is favorable from the standpoint of energy. However, in this coalescence, not only energy but also other motion integrals, for example, the number of particles, must also be preserved. This is possible only in very special cases, for example, for integrable models. Nonintegrable systems are characterized by inelastic scattering and the formation of a soliton with a larger amplitude as a result. This process is accompanied by the radiation of waves in the nonsoliton sector, which evolve into linear waves far from the scattering region owing to dispersion. Here, radiation plays the role of friction, leading to the formation of a more powerful soliton. Large-amplitude solitons survive in such systems in the presence of multiple scattering, being in equilibrium with radiation (the linear waves). Solitons in this case behave like a peculiar kind of drop, called statistical attractors by Yan'kov [15, 16], for which radiation plays the role of vapor.

If a Hamiltonian is unbounded, its unboundedness indicates that solitons must correspond to saddle points of the Hamiltonian and be unstable entities. The system behavior cannot be stationary in this case. Two variants are possible: either the system tends to completely spread out because of dispersion, in which case nonlinear interaction becomes insignificant and waves become linear at large times, or the system collapses, and a field singularity forms as a result. The latter process can be viewed as an analog of the fall of a particle on a center in an unbounded potential. Based on the fact that the Hamiltonian is unbounded, we can ascertain the role of radiation in the process of collapse. As was first shown by Zakharov (1972) using the collapse of Langmuir waves as an example [17], the radiation promotes collapse. This turns out to be a general property, inherent in many collapsing wave systems.

However, a conclusion regarding the finiteness of the collapse time cannot be drawn from this reasoning. Only rigorous theorems, like the Vlasov–Petrishchev–Talanov theorem [18], allow formulating a sufficient criterion for the collapse as a process through which a singularity develops in a finite time.

The method based on the Lyapunov theorem is difficult to apply when exploring the stability of local stationary points. In this case, a linear stability analysis seems to be most efficient. In this review, following the work by Vakhitov and Kolokolov [19], we derive the stability criterion for NLS solitons. The main point of this derivation relies on the oscillation theorem for the Schrödinger operator, which establishes a one-to-one correspondence between the level number and the number of zero crossings of the respective wave function. Importantly, this theorem is only valid for a scalar Schrödinger operator and cannot be used for a vector Schrödinger operator. This indicates that the Vakhitov–Kolokolov criteria are typically only sufficient for the soliton instability and cannot be used as a necessary condition of stability. As we show below, the three-wave system belongs to just that category.

For the three-wave system, the linearized operator is the product of two 3×3 matrix Schrödinger operators, for which the oscillation theorem is inapplicable. We discuss this situation in detail for solitons describing a coupled state of the first and second harmonics, and show with this example how the two approaches — the Lyapunov method and the linear stability analysis — work. In particular, we discuss how the stability of solitons is affected by phase detuning.

2. Main equations

We begin with the main equations, the nonlinear Schrödinger equation and equations for the three-wave system.

2.1 Nonlinear Schrödinger equation

In dimensionless variables, the NLS equation is written in the canonical form

$$\mathrm{i}\psi_t + \frac{1}{2}\,\Delta\psi + |\psi|^2\psi = 0\,. \tag{1}$$

In the context of nonlinear optics, ψ in Eqn (1) stands for the amplitude of the envelope of the electric field with a certain (for example, linear) polarization, and the time t is a coordinate in the direction of the wave packet propagation. The second term in Eqn (1) describes both diffraction and positive dispersion of group velocity, which is realized in optics in the anomalous dispersion range. In the case of normal dispersion, the operator Δ is replaced by the hyperbolic operator $\Delta_\perp - \partial_z^2$. The nonlinear term $|\psi|^2\psi$ in Eqn (1) corresponds to the Kerr contribution to the refractive index.

We note that the NLS equation, which models a broad range of nonlinear wave phenomena, plays a central role in the theory of wave collapse. This has become especially clear after the prediction of light self-focusing [20] and the development of the related theory [21–25] (also see [26]). The applications of this model of wave collapse are not exhausted by light self-focusing: the NLS equation finds diverse applications in many other branches of physics (see, e.g., reviews [4, 27–29] and the references therein). Commonly, the NLS equation and its modifications result from a reduction of the equations of motion of a nonlinear medium to equations for the envelope of a quasimonochromatic wave, which involves averaging the original equations over fast oscillations in time and space. Accordingly, the NLS equation is very frequently regarded as an equation for envelopes. We mention monograph [30], which is dedicated exclusively to the NLS equation; it touches on its numerous mathematical aspects, in addition to physical issues.

Nonlinear Schrödinger equation (1) is often called the Gross–Pitaevskii equation [31, 32], which very accurately describes long-wave oscillations of the condensate of a weakly nonideal Bose gas with a negative scattering length. Presently, Eqn (1) is the main model used in the research on nonlinear dynamics of Bose condensates (see, e.g., Refs [33–38]). In that case, Eqn (1) is the Schrödinger equation and ψ is the wave function. Accordingly, Eqn (1) describes the motion of a quantum mechanical particle in the self-consistent attracting potential $U = -|\psi|^2$. Just the attraction is the cause of the singularity occurrence. From the quantum mechanical standpoint, collapse in the NLS framework can be interpreted as the fall of a particle on the center (the point of collapse) [39].

Collapse in NLS equation (1) is, however, possible, not for all space dimensions D but only for $D \geqslant 2$. In the one-dimensional (1D) case, as shown by Zakharov and Shabat [40] in 1971, Eqn (1) is exactly integrable with the help of the inverse scattering method. This result has demonstrated that solitons, being stationary localized objects, play an essential role in the dynamics of nonlinear waves described by the NLS equation. The solitons of the one-dimensional case turned out to be given by structurally stable formations, i.e., those stable with respect to not only small but also finite perturbations, like those a soliton encounters when scattering on other solitons. Just this very attractive property of solitons underlies the applications of optical solitons in optical fibers [5]. This idea is now realized in practice (see, e.g., Ref. [41]). But solitons play a different role in higher dimensions.

2.2 Hamiltonian structure

It is well known (see, e.g., Ref. [11]) that NLS equation (1) is a Hamiltonian equation and can be written in the form

$$\mathrm{i}\psi_t = \frac{\delta H}{\delta\psi^*} \tag{2}$$

with the Hamiltonian

$$H = \frac{1}{2}\left(\int |\nabla\psi|^2\,\mathrm{d}\mathbf{r} - \int |\psi|^4\,\mathrm{d}\mathbf{r}\right) \equiv \frac{1}{2}\,(I_1 - I_2)\,. \tag{3}$$

In addition to H, this equation has two more simple integrals of motion: the number of particles

$$N = \int |\psi|^2\,\mathrm{d}\mathbf{r} \tag{4}$$

(which coincides, up to a constant factor, with the energy of the wave packet) and the momentum

$$\mathbf{P} = \frac{\mathrm{i}}{2}\int(\psi\nabla\psi^* - \psi^*\nabla\psi)\,\mathrm{d}\mathbf{r}\,. \tag{5}$$

The conservation of N follows from the gauge symmetry $\psi \to \psi\exp(\mathrm{i}\alpha)$, and the conservation of $\mathbf{P}$ is conditionally related to the symmetry under translations. These two symmetries ensure the Galilean invariance of Eqn (1). In particular, a simple solution of Eqn (1),

$$\psi_{\mathrm{s}} = \psi_0(\mathbf{r})\exp\frac{\mathrm{i}\lambda^2 t}{2}\,, \tag{6}$$

with $\psi_0(\mathbf{r}) \to 0$ as $r \to \infty$, corresponds to a soliton at rest. A solution for a moving soliton follows by applying a Galilean transformation to Eqn (6).

It can also be easily established that soliton solution (6) represents a stationary point of the Hamiltonian H for a fixed particle number (cf. Ref. [3]):

$$\delta(H + \lambda^2 N) = 0\,, \tag{7}$$

where λ^2 is the Lagrange multiplier. Solving variational problem (7) is equivalent to finding a solution of the stationary NLS equation

$$-\lambda^2\psi + \Delta\psi + |\psi|^2\psi = 0\,.$$

This implies the dependence of N on λ for soliton solution (6),

$$N_{\mathrm{s}} = \lambda^{2-D} N_0\,, \qquad N_0 = \int |f(\xi)|^2\,\mathrm{d}\xi\,, \tag{8}$$

where f satisfies the equation

$$-f + \frac{1}{2}\,\Delta f + |f|^2 f = 0\,.$$

Dependence (8) proves to be crucial for the criterion of soliton linear stability, the subject of Section 3.

2.3 Three-wave system and its reductions

We next consider the three-waves system, which can be written in the form (see, e.g., Refs [10, 11])

$$\mathrm{i}\,\frac{\partial\psi_1}{\partial t} - \omega_1\psi_1 + \mathrm{i}(\mathbf{v}_1\nabla)\psi_1 + \frac{1}{2}\,\omega_1^{\alpha\beta}\partial^2_{\alpha\beta}\psi_1 = V\psi_2\psi_3\,, \tag{9}$$

$$\mathrm{i}\,\frac{\partial\psi_2}{\partial t} - \omega_2\psi_2 + \mathrm{i}(\mathbf{v}_2\nabla)\psi_2 + \frac{1}{2}\,\omega_2^{\alpha\beta}\partial^2_{\alpha\beta}\psi_2 = V\psi_1\psi_3^*\,, \tag{10}$$

$$\mathrm{i}\,\frac{\partial\psi_3}{\partial t} - \omega_3\psi_3 + \mathrm{i}(\mathbf{v}_3\nabla)\psi_3 + \frac{1}{2}\,\omega_3^{\alpha\beta}\partial^2_{\alpha\beta}\psi_3 = V\psi_1\psi_2^*\,, \tag{11}$$

where the amplitudes of three wave packets $\psi_l(\mathbf{x}, t)$ $(l = 1, 2, 3)$ are slowly varying functions of $\mathbf{x}$ $(k_l L_l \gg 1$, where $\mathbf{k}_l$ is the carrier wave vector of the lth packet and L_l is its characteristic length), $\mathbf{v}_l = \partial\omega_l(\mathbf{k}_l)/\partial\mathbf{k}_l$ are the group velocities of packets,

$$\omega_l^{\alpha\beta} = \frac{\partial^2\omega_l(\mathbf{k}_l)}{\partial k_{l\alpha}\,\partial k_{l\beta}}$$

is the dispersion tensor, and V is the three-wave matrix element, which can be considered real valued without the loss of generality. It is assumed that the carrier frequencies $\omega_l = \omega_l(\mathbf{k}_l)$ are close to the resonance

$$\omega_1(\mathbf{k}_1) = \omega_2(\mathbf{k}_2) + \omega_3(\mathbf{k}_3)\,, \tag{12}$$

$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3\,. \tag{13}$$

Like the NLS equation, system of equations (9)–(11) is Hamiltonian:

$$\mathrm{i}\,\frac{\partial\psi_l}{\partial t} = \frac{\delta H}{\delta\psi_l^*}\,, \tag{14}$$

where

$$H = H_0 + H_1\,,$$

$$H_0 = \sum_{l=1}^{3}\left(\int\omega_l|\psi_l|^2\,\mathrm{d}\mathbf{r} - \mathrm{i}\int\psi_l^*(\mathbf{v}_l\nabla)\psi_l\,\mathrm{d}\mathbf{r} + \frac{1}{2}\int\nabla_\alpha\psi_l^*\omega_l^{\alpha\beta}\nabla_\beta\psi_l\,\mathrm{d}\mathbf{r}\right), \tag{15}$$

$$H_1 = V\int(\psi_1\psi_2^*\psi_3^* + \psi_1^*\psi_2\psi_3)\,\mathrm{d}\mathbf{r}\,. \tag{16}$$

Here, H_0 corresponds to linear waves: the first term in the right-hand side of Eqn (15) makes a dominant contribution to the energy, the second term corresponds to the motion of packets with the group velocity, and the third term is responsible for dispersive broadening of packets. It is noteworthy that the dispersive term is small in the parameter $\Delta k_l/k_l$ compared to the second term. The third term in H_0 should therefore be retained only when the differences between the group velocities are small. For instance, such a situation can be realized in optics.

If the carrier ω_l and $\mathbf{k}_l$ exactly satisfy resonance conditions (12), then the terms $\sim \omega_l$ can be eliminated by the transformation $\psi_l \to \psi_l\exp(-\mathrm{i}\omega_l t)$.

We note that three-wave system (9)–(11) allows a physically important reduction that corresponds to the interaction between the first (ψ_1) and second (ψ_2) harmonics. In this case, the Hamiltonian becomes

$$H = \sum_{l=1}^{2}\left(\int\omega_l|\psi_l|^2\,\mathrm{d}\mathbf{r} - \mathrm{i}\int\psi_l^*(\mathbf{v}_l\nabla)\psi_l\,\mathrm{d}\mathbf{r} + \frac{1}{2}\int\partial_\alpha\psi_l^*\omega_l^{\alpha\beta}\partial_\beta\psi_l\,\mathrm{d}\mathbf{r}\right) + V\int(\psi_2^*\psi_1^2 + \mathrm{c.c.})\,\mathrm{d}\mathbf{r}\,,$$

and the respective equations of motion are written as

$$\mathrm{i}\,\frac{\partial\psi_1}{\partial t} - \omega_1\psi_1 + \mathrm{i}(\mathbf{v}_1\nabla)\psi_1 + \frac{1}{2}\,\omega_1^{\alpha\beta}\partial^2_{\alpha\beta}\psi_1 = 2V\psi_2\psi_1^*\,, \tag{17}$$

$$\mathrm{i}\,\frac{\partial\psi_2}{\partial t} - \omega_2\psi_2 + \mathrm{i}(\mathbf{v}_2\nabla)\psi_2 + \frac{1}{2}\,\omega_2^{\alpha\beta}\partial^2_{\alpha\beta}\psi_2 = V\psi_1^2\,, \tag{18}$$

where $\omega_2 \simeq 2\omega_1$. If the resonance is exact, the terms proportional to ω_l can be eliminated by the transformations $\psi_1 \to \psi_1\exp(-\mathrm{i}\omega_1 t)$, $\psi_2 \to \psi_2\exp(-\mathrm{i}2\omega_1 t)$. In the one-dimensional case, the three-wave system (9)–(11) allows simplifications. By introducing new variables

$$\psi_l = \tilde{\psi}_l(x - vt, t)\exp(\mathrm{i}\kappa_l x)\,, \qquad \kappa_1 = \kappa_2 + \kappa_3\,, \tag{19}$$

it is possible to eliminate the first derivatives if the velocity v and wave numbers κ_l are chosen as

$$v = \frac{v_1 d_1 - v_2 d_2 - v_3 d_3}{d_1 - d_2 - d_3}, \quad \kappa_l = d_l (v - v_l), \quad d_l = \frac{1}{\omega_l''}. \quad (20)$$

(We note that the equations turn out to be Galilei invariant for $d_1 = d_2 + d_3$.) As a result, system (9)–(11) in the one-dimensional case takes the form (the tilde over ψ_l is omitted below and the matrix element of the three-wave coupling V is set to -1)

$$\mathrm{i}\,\frac{\partial \psi_1}{\partial t} - \Omega_1 \psi_1 + \frac{1}{2}\,\omega_1'' \psi_{1xx} = -\psi_2 \psi_3, \quad (21)$$

$$\mathrm{i}\,\frac{\partial \psi_2}{\partial t} - \Omega_2 \psi_2 + \frac{1}{2}\,\omega_2'' \psi_{2xx} = -\psi_1 \psi_3^*, \quad (22)$$

$$\mathrm{i}\,\frac{\partial \psi_3}{\partial t} - \Omega_3 \psi_3 + \frac{1}{2}\,\omega_3'' \psi_{3xx} = -\psi_1 \psi_2^*, \quad (23)$$

where

$$\Omega_l = \omega_l + \kappa_l v_l + \frac{\omega_l'' \kappa_l^2}{2}. \quad (24)$$

Here, as previously, it is assumed that the new frequencies are close to the resonance,

$$\Omega_1 \approx \Omega_2 + \Omega_3,$$

or, in other words, the frequency detuning is small.

At the next step, Ω_2 and Ω_3 can be eliminated with the help of transformations

$$\psi_1(x,t) \to \psi_1(x,t) \exp\left[-\mathrm{i}(\Omega_2 + \Omega_3)t\right],$$
$$\psi_2(x,t) \to \psi_2(x,t) \exp(-\mathrm{i}\Omega_2 t),$$
$$\psi_3(x,t) \to \psi_3(x,t) \exp(-\mathrm{i}\Omega_3 t).$$

As a result, Eqns (21)–(23) become

$$\mathrm{i}\,\frac{\partial \psi_1}{\partial t} - \Omega \psi_1 + \frac{1}{2}\,\omega_1'' \psi_{1xx} = -\psi_2 \psi_3, \quad (25)$$

$$\mathrm{i}\,\frac{\partial \psi_2}{\partial t} + \frac{1}{2}\,\omega_2'' \psi_{2xx} = -\psi_1 \psi_3^*, \quad (26)$$

$$\mathrm{i}\,\frac{\partial \psi_3}{\partial t} + \frac{1}{2}\,\omega_3'' \psi_{3xx} = -\psi_1 \psi_2^*, \quad (27)$$

where $\Omega = \Omega_1 - \Omega_2 - \Omega_3$ is the frequency detuning characterizing how far the carrier frequencies are from the resonance Eqn (12).

System of equations (25)–(27) preserves its canonical structure with the Hamiltonian

$$H = \int \Omega |\psi_1|^2\, \mathrm{d}x + \sum_l \int \frac{1}{2}\,\omega_l'' |\psi_{lx}|^2\, \mathrm{d}x - \int (\psi_1^* \psi_2 \psi_3 + \text{c.c.})\, \mathrm{d}x. \quad (28)$$

In addition to H, these equations (as well as the nontransformed equations) have two additional integrals of motion, the so-called Manley–Rowe integrals

$$N_1 = \int \left(|\psi_1|^2 + |\psi_2|^2\right) \mathrm{d}x, \qquad N_2 = \int \left(|\psi_1|^2 + |\psi_3|^2\right) \mathrm{d}x. \quad (29)$$

The invariants N_1 and N_2 arise as a result of the averaging procedure that removes all nonresonant terms except the one pertaining to the three-wave coupling.

The one-dimensional system (17), (18) describing the interaction of the first and second harmonics is transformed similarly to Eqn (19):

$$\mathrm{i}\,\frac{\partial \psi_1}{\partial t} + \frac{1}{2}\,\omega_1'' \psi_{1xx} = -2\psi_2 \psi_1^*, \quad (30)$$

$$\mathrm{i}\,\frac{\partial \psi_2}{\partial t} - \Omega \psi_2 + \frac{1}{2}\,\omega_2'' \psi_{2xx} = -\psi_1^2, \quad (31)$$

where the frequencies Ω_l are defined by Eqns (24) with κ_l given by (20). Only one Manley–Rowe integral for system (30), (31) exists:

$$N = \int \left(|\psi_1|^2 + 2|\psi_2|^2\right) \mathrm{d}x. \quad (32)$$

Similar transformations for the three-wave system in Eqns (21)–(23) [as well as for the coupling of the first and second harmonics (17), (18)] can be carried out in the case of many dimensions. For this, it is necessary to make the change $\omega_l'' \psi_{lxx} \to \omega_l^{\alpha\beta} \partial^2_{\alpha\beta} \psi_l$ in all integrals of motion and to replace d_l with the matrix inverse to $\omega_l^{\alpha\beta}$, with the velocities v_l considered vector quantities. It is pertinent to note that for the three-wave system, the difference between one-dimensional and multi-dimensional cases for $D \leqslant 3$ is not so essential, in contrast, for example, to the case of the NLS equation. This can be seen most vividly when exploring the stability of solitons — the coupled states of three-wave packets.

To conclude this section, we note one more important reduction of system (9)–(11), which corresponds to stationary waves when the time derivative in these equations is absent. In this case, the system of equations, which is also Hamiltonian, describes spatial solitons — the distributions localized in one direction or in a plane (see, e.g., Refs [42–44] for more details).

3. Lyapunov stability for scalar models

3.1 Stability of NLS solitons

We begin by exploring the stability of solitons of the one-dimensional NLS equation,

$$\mathrm{i}\psi_t = \frac{\delta H}{\delta \psi^*} = -\frac{1}{2}\,\psi_{xx} - |\psi|^2 \psi, \quad (33)$$

where the Hamiltonian H is defined in (3) with the integration performed only with respect to the coordinate x. As mentioned in Section 2, from the quantum mechanical standpoint, Eqn (33) describes the motion of a particle in the self-consistent attracting potential $U = -|\psi|^2$. But in the one-dimensional case, nonlinear attraction cannot overcome dispersion ($\sim \psi_{xx}$), the cause of smearing. The exact balance of these two opposing effects leads to the existence of a soliton — a stationary solution of Eqn (33) of the form $\psi(x,t) = \psi_0(x) \exp(\mathrm{i}\lambda^2 t/2)$, where

$$\psi_0(x) = \lambda\, \mathrm{sech}\,(\lambda x) \quad (34)$$

satisfies the stationary NLS equation

$$-\lambda^2 \psi_0 + \psi_{0xx} + 2|\psi_0|^2 \psi_0 = 0. \quad (35)$$

The simplest solution (34) describes a soliton at rest. Moving solitons are obtained from it by performing Galilean transformations.

It can be easily verified that solution (34) corresponds to a stationary point of the Hamiltonian for a fixed number of particles $N = \int |\psi|^2 \, dx$, because stationary NLS equation (35) directly follows from the variational problem

$$\delta\left(H + \frac{\lambda^2 N}{2}\right) = 0 \,. \tag{36}$$

It is also easy to find that the number of particles increases linearly with λ, $N_s = 2\lambda$, in these solutions. Here and henceforth, the index s indicates that a functional (in this case, N) is computed at the soliton solution.

According to the Lyapunov theorem, to prove the stability of soliton (34), it suffices to show that the soliton realizes a minimum of the Hamiltonian. (The unboundedness of H from above is obvious: for a fixed N, which is essentially the normalization of the wave function, the Hamiltonian can become arbitrarily large because of the kinetic energy I_1 in the class of strongly varying functions ψ.)

We begin with simple, essentially, dimensional, estimates that indicate that H is bounded from below. Let A be a characteristic amplitude of the soliton and l be its characteristic size. Then the particle number is approximately given by $N \approx A^2 l$; hence, $A^2 \approx N l^{-1}$. For integrals I_1 and I_2, we have the estimates

$$I_1 \approx \frac{A^2}{l} \approx \frac{N}{l^2} \,, \qquad I_2 \approx A^4 l \approx \frac{N^2}{l} \,,$$

which give the Hamiltonian

$$H \approx \frac{1}{2}\left(\frac{N}{l^2} - \frac{N^2}{l}\right).$$

The last expression immediately implies that H, as a function of the soliton size l, is positive for small l (it grows as l^{-2}) and is negative and tends to zero from below for large l. It is obviously bounded from below. Its minimum is achieved for a soliton with the size $l_s \approx 2/N$.

This same conclusion can be drawn by considering scale transformations that preserve the particle number:

$$\psi(x) \to a^{-1/2}\psi\left(\frac{x}{a}\right). \tag{37}$$

As a result of transformation (37), the Hamiltonian becomes a function of the scaling parameter a,

$$H(a) = \frac{1}{2}\left(\frac{I_1}{a^2} - \frac{I_2}{a}\right). \tag{38}$$

The function $H(a)$ reaches a minimum at the point $a = 1$, which corresponds to a soliton (Fig. 1):

$$\left.\frac{\partial H}{\partial a}\right|_{a=1} = 0 \Rightarrow 2I_{1s} = I_{2s} = \frac{4\lambda^3}{3} \,, \qquad H_s = -\frac{1}{2} I_{1s} = -\frac{\lambda^3}{3} \,. \tag{39}$$

We can additionally verify that the soliton realizes a minimum of H with respect to phase transformations

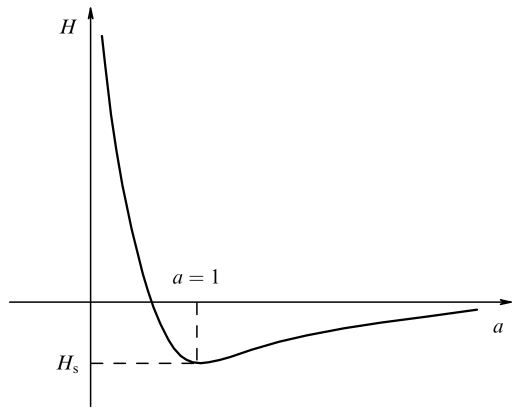

Figure 1. The profile of $H(a)$.

$\psi_0(x) \to \psi_0(x) \exp(i\chi(x))$, which also preserve N:

$$H = H_s + \frac{1}{2}\int\left(\frac{\partial\chi}{\partial x}\right)^2 \psi_0^2 \, dx \,.$$

The two simple transformations defined by Eqns (37) and (38) therefore give a minimum of H, which hints at the soliton stability, but, strictly speaking, does not prove it.

We offer a rigorous proof. The main element of this proof relies on Sobolev integral estimates, which follow from the general Sobolev embedding theorems. The Sobolev theorem states that the space L_p can be embedded in the Sobolev space W_2^1 if the dimension of the space R^D over which the integration is performed satisfies the inequality

$$D < \frac{2}{p}\,(p+4) \,.$$

This implies that the norms

$$\|u\|_p = \left[\int |u|^p \, d^D x\right]^{1/p} , \qquad p > 0 \,,$$

$$\|u\|_{W_2^1} = \left[\int \left(\mu^2 |u|^2 + |\nabla u|^2\right) d^D x\right]^{1/2} , \qquad \mu^2 > 0 \,,$$

are related by the inequality (see, e.g., Ref. [45])

$$\|u\|_p \leqslant M \|u\|_{W_2^1} \,, \tag{40}$$

where M is a positive constant. Notably, for $D = 1$ and $p = 4$, inequality (40) can be written as

$$\int_{-\infty}^{\infty} |\psi|^4 \, dx \leqslant M_1 \left(\int_{-\infty}^{\infty} \left(\mu^2 |\psi|^2 + |\psi_x|^2\right) dx\right)^2 . \tag{41}$$

Using this, we can derive a multiplicative variant of the Sobolev inequality, the so-called Gagliardo–Nirenberg inequality [46] (see also Refs [29, 45, 47]). With this aim, we perform scale transformations $x \to \alpha x$ in inequality (41), after which the right-hand side of (41) becomes dependent on the parameter α,

$$\int_{-\infty}^{\infty} |\psi|^4 \, dx \leqslant M_1 \left(\alpha\mu^2 \int_{-\infty}^{\infty} |\psi|^2 \, dx + \frac{1}{\alpha}\int_{-\infty}^{\infty} |\psi_x|^2 \, dx\right)^2 .$$

This inequality is valid for any positive α, including the values for which the right-hand side of the inequality is minimum, which gives the Gagliardo–Nirenberg inequality

$$I_2 \leqslant C N^{3/2} I_1^{1/2}\,, \tag{42}$$

where I_1 and I_2 are the integrals defined by Eqn (3) for $D = 1$, and C is a new constant.

Inequality (42) can be refined if the minimum value of the constant C is known. To find it, we need to consider all extrema of the functional

$$J\{\psi\} = \frac{I_2}{N^{3/2} I_1^{1/2}}\,, \tag{43}$$

which are found from the variational problem $\delta J = 0$. It can be readily seen that this amounts to finding solutions of stationary NLS equation (35),

$$-\lambda^2 \psi + \psi_{xx} + 2|\psi|^2 \psi = 0\,.$$

The solution of the last equation is unique (up to a constant phase factor) and represents the one-dimensional soliton of NLS equation (34),

$$\psi = \frac{\lambda}{\cosh \lambda x}\,.$$

We therefore conclude that the best constant C_{best} equals the value of $J\{\psi\}$ at soliton solution (34):

$$C_{\mathrm{best}} = \frac{I_{2\mathrm{s}}}{N_{\mathrm{s}}^{3/2} I_{1\mathrm{s}}^{1/2}} = \frac{2 I_{1\mathrm{s}}^{1/2}}{N_{\mathrm{s}}^{3/2}}\,. \tag{44}$$

Inequality (42) with $C = C_{\mathrm{best}}$ immediately implies the stability proof for the one-dimensional soliton.

The substitution of Eqn (42) with $C = C_{\mathrm{best}}$ in Eqn (3) for $D = 1$ gives the desired estimate for the lower bound of the Hamiltonian:

$$H \geqslant \frac{1}{2}\left(I_1 - C_{\mathrm{best}} I_1^{1/2} N^{3/2}\right) = H_{\mathrm{s}} + \frac{1}{2}\left(I_1^{1/2} - I_{1\mathrm{s}}^{1/2}\right)^2. \tag{45}$$

Inequality (45) becomes exact at the soliton solution, which proves that the NLS soliton is Lyapunov stable not only against small but also against finite perturbations.

Similarly, we can prove the stability of the 'ground-state' soliton (a radially symmetric solution with no zeros) for a multidimensional NLS equation with a power-like nonlinearity

$$\mathrm{i}\psi_t + \frac{1}{2}\,\Delta\psi + \frac{\sigma}{2}\,|\psi|^{2\sigma-2}\psi = 0\,, \tag{46}$$

where $\sigma > 1$.

The Hamiltonian for Eqn (46) can be written in the form

$$H = \frac{1}{2}\int\left(|\nabla\psi|^2 - |\psi|^{2\sigma}\right)\mathrm{d}^D x \equiv I_1 - I_\sigma\,, \tag{47}$$

and the ground-state soliton solution

$$\psi_{\mathrm{s}} = \exp\left(\frac{\mathrm{i}\lambda^2 t}{2}\right)\lambda^{1/(\sigma-2)} g(\lambda r)\,,$$

with the function $g(\xi)$ satisfying the equation

$$-g + \nabla_\xi^2 g + \sigma |g|^{2\sigma-2} g = 0\,,$$

corresponds to a stationary point of H for a fixed particle number N,

$$\delta\left(H + \frac{\lambda^2}{2}\,N\right) = 0\,.$$

As previously, performing the scaling transformation

$$\psi_{\mathrm{s}}(r) \to a^{-D/2}\psi_{\mathrm{s}}\left(\frac{r}{a}\right),$$

which preserves the total number of particles, we find that the corresponding Hamilton function

$$H(a) = \frac{1}{2}\left(\frac{I_1}{a^2} - \frac{I_\sigma}{a^{(\sigma-1)/D}}\right)$$

is bounded from below and attains a minimum only if

$$\frac{\sigma - 1}{D} < 2\,.$$

This minimum (for $a = 1$) corresponds to the ground-state soliton solution, which hints at its stability. To rigorously prove the stability, we have to resort to the related multiplicative inequality for I_σ, which follows from Eqn (40),

$$I_\sigma = \int |\psi|^{2\sigma}\,\mathrm{d}^D x \leqslant M_\sigma\left[\int |\psi|^2\,\mathrm{d}^D x\right]^\alpha\left[\int |\nabla\psi|^2\,\mathrm{d}^D x\right]^\beta, \tag{48}$$

where $\alpha = D/2 + \sigma(1 - D/2)$ and $\beta = (\sigma - 1)D/2$. Then, as in the case with the one-dimensional cubic NLS equation (33), we need to find the best constant M_σ as a minimum of the functional $N^\alpha I_1^\beta / I_\sigma$. It is easy to verify that this minimum is provided by the ground-state soliton solution (46).

Substituting inequality (48) with the best constant M_{best} in Hamiltonian (47),

$$H \geqslant \frac{1}{2}\left(I_1 - M_{\mathrm{best}} N^\alpha I_1^\beta\right),$$

we find that the right-hand side of the last inequality as a function of I_1 is bounded from below and reaches a minimum at the ground-state soliton solution, which just proves the Lyapunov stability of this solution.

Stability criterion (45) and its generalization for $(\sigma - 1)/D < 2$ can be interpreted as the energy principle. If we ignore the integrability of the one-dimensional NLS equation (with $\sigma = 2$) [40], the coalescence of two solitons should be favorable in terms of energy. This implies that a new soliton appearing as a result of the interaction of two solitons must have a lower energy (the energy of the soliton being negative!) than the energy of the interacting solitons. In the general case, the interaction between solitons is accompanied by radiation of small-amplitude waves. Why does the radiation favor the emergence of a deeper soliton?

We consider a domain G where the Hamiltonian is negative, $H_{\mathrm{G}} < 0$. The following obvious estimate is then valid for the integral $I_\sigma = \int_{\mathrm{G}} |\psi|^{2\sigma}\,\mathrm{d}x$:

$$\int_{\mathrm{G}} |\psi|^{2\sigma}\,\mathrm{d}x \leqslant \left(\max_x |\psi|^2\right)^{2\sigma-2}\int_{\mathrm{G}} |\psi|^2\,\mathrm{d}x\,.$$

Hence, it follows that

$$\left(\max_{\mathrm{G}} |\psi|^2\right)^{2\sigma-2} \geqslant \frac{|H_{\mathrm{G}}|}{N_{\mathrm{G}}}\,. \tag{49}$$

We note that inequality (49) is valid for an arbitrary physical dimension D.

If small-amplitude waves are radiated from the domain G, they carry away positive energy (because the nonlinearity is not essential for them). As a result, because of the conservation of the Hamiltonian and the number of particles, $|H_G|$ increases, and the number of waves N_G decreases. Their ratio therefore increases owing to the radiation, and $\max|\psi|^2$ increases accordingly. This process continues until a soliton — a stationary stable state realizing the minimum of H_G for a fixed N_G — is formed in the domain G.

It is worth noting that in some, rather special cases, the radiation can be absent. This is so if the energy and particle number conservation laws hold:

$$E(N) = E(N_1) + E(N_2)\,,$$
$$N = N_1 + N_2\,,$$

where N_1 (N_2) is the total number of particles for the first (second) soliton. These equations cannot be satisfied for all functions $E(N)$. For example, if $E(N) \sim N^{\alpha}$ ($\alpha > 0$), then the system of equations does not have any solutions.

In the opposite case where $(\sigma - 1)/D > 2$, the function $H(a)$ is unbounded from below for $a \to 0$ and, instead of a minimum for $(\sigma - 1)/D < 2$, a maximum is achieved, which points to the instability of the soliton solution.

3.2 Lyapunov stability for the anisotropic Korteweg–de Vries equation

The next example to be considered here is the anisotropic KdV equation [14] derived by us in 1974:

$$u_t + \frac{\partial}{\partial z}\,\Delta u + 6uu_z = 0\,, \tag{50}$$

where Δ is the three-dimensional Laplace operator. In the one-dimensional case, Eqn (50) reduces to the classic KdV equation.

Equation (50) describes three-dimensional ion–acoustic solitons $u = u_s(z - Vt, r_\perp)$ propagating along the magnetic field vector (parallel to the z axis) in a strongly magnetized plasma, where the plasma thermal pressure nT (with the density n and temperature T) is small compared to the magnetic field pressure $B^2/8\pi$, i.e., $\beta = 8\pi nT/B^2 \ll 1$. Soliton solutions are sought by integrating the equation

$$-Vu_s + \Delta u_s + 3u_s^2 = 0\,.$$

It can be readily seen that localized solutions ($u_s \to 0$ as $r \to \infty$) are only possible if the velocity V is positive. The soliton solution can be written in analytic form only in the one-dimensional case, and it is then just the classic KdV soliton

$$u = \frac{2\kappa^2}{\cosh^2 \kappa(x - 4\kappa^2 t)}\,, \qquad V = 4\kappa^2\,.$$

In the case of many dimensions, localized solutions can be found only numerically (see Ref. [14]).

For Eqn (50), as for the NLS equation, solitons correspond to stationary points of the Hamiltonian

$$H = \int\left(\frac{1}{2}(\nabla u)^2 - u^3\right)\mathrm{d}\mathbf{r} \equiv \frac{1}{2}\,I_1 - I_2 \tag{51}$$

for the fixed momentum $P = (1/2)\int u^2\,\mathrm{d}\mathbf{r}$; in other words, solitons are solutions of the variational problem $\delta(H - VP) = 0$, with the velocity V entering as a Lagrange multiplier.

The scaling transformations

$$u(\mathbf{r}) \to \frac{1}{a^{D/2}}\,u\left(\frac{\mathbf{r}}{a}\right), \tag{52}$$

which preserve P, lead to approximately the same dependence

$$H(a) = \frac{1}{2}\frac{I_1}{a^2} - \frac{I_2}{a^{D/2}}\,, \qquad D \leqslant 3\,,$$

as in the case of the one-dimensional NLS equation (compare this with the dependence in Fig. 1).

Accordingly, in analogy with the NLS equation, for the soliton solution (at the point $a = 1$, where $\partial H/\partial a = 0$), we have

$$I_{2s} = \frac{2}{D}\,I_{1s}\,, \qquad H_s = \frac{D-4}{2D}\,I_{1s} \equiv \frac{D-4}{6-D}\,VP_s\,,$$

and hence $H_s < 0$ for $D \leqslant 3$.

The multiplicative variant of the Sobolev inequality in this case becomes

$$I_2 \leqslant CP^{1/(4-D)}I_1^{\beta}\,,$$

where $\beta = (1/2)(10 - 3D)/(4 - D)$. The best value $C = C_{\mathrm{best}}$ is defined by the ground-state soliton (the one without zeros). Substitution of the last inequality with the best constant in (51) shows the boundedness of H for all space dimensions $D = 1, 2, 3$:

$$H \geqslant \frac{1}{2}\,I_1 - C_{\mathrm{best}}P^{1/(4-D)}I_1^{\beta} \geqslant H_s\,.$$

These inequalities become equalities when evaluated on a radially symmetric soliton solution without zeros, which proves the stability of ion–acoustic solitons in a magnetized plasma in the case of many dimensions [14]. As a particular result, this also offers a stability proof for the classic KdV solitons. For one-dimensional KdV solitons, this approach was first used by Benjamin [13] in 1972. But at that time, the integrability of the KdV equation via the inverse scattering method had already been known owing to Gardner, Green, Cruscal, and Miura [48]. In particular, it had already been established in [48] that the KdV soliton is a structurally stable object.

4. Solitons in the three-wave system

We consider soliton solutions for system (25)–(27) in the form

$$\psi_1(x,t) = \psi_{1s}(x)\exp\left[\mathrm{i}(\lambda_1 + \lambda_2)t\right],$$
$$\psi_2(x,t) = \psi_{2s}(x)\exp(\mathrm{i}\lambda_1 t)\,,$$
$$\psi_3(x,t) = \psi_{3s}(x)\exp(\mathrm{i}\lambda_2 t)\,,$$

where $\psi_{1s}(x)$, $\psi_{2s}(x)$, and $\psi_{3s}(x)$ are assumed to be real valued, with no zeros (ground state) and decaying at infinity. These functions satisfy the equations

$$L_1\psi_{1s} = -\psi_{2s}\psi_{3s}\,, \qquad L_1 = -(\lambda_1 + \lambda_2 + \Omega) + \frac{1}{2}\,\omega_1''\partial_x^2\,, \tag{53}$$

$$L_2\psi_{2s} = -\psi_{1s}\psi_{3s}\,, \qquad L_2 = -\lambda_1 + \frac{1}{2}\,\omega_2''\partial_x^2\,, \tag{54}$$

$$L_3\psi_{3s} = \psi_{1s}\psi_{2s}\,, \qquad L_3 = -\lambda_2 + \frac{1}{2}\,\omega_3''\partial_x^2\,. \tag{55}$$

Solutions of system (53)–(55) correspond to stationary points of the Hamiltonian for two fixed Manley–Rowe integrals N_1 and N_2:

$$\delta(H + \lambda_1 N_1 + \lambda_2 N_2) = 0\,. \tag{56}$$

The soliton solutions exponentially decay at infinity if the following three conditions are satisfied:

$$\mu_1^2 \equiv d_1(\lambda_1 + \lambda_2 + \Omega) > 0\,, \quad \mu_2^2 \equiv d_2\lambda_1 > 0\,, \quad \mu_3^2 \equiv d_3\lambda_2 > 0\,, \tag{57}$$

where $d_l = 1/\omega_l''$.

This result can be obtained differently if we determine relations between the terms in the Hamiltonian and the Manley–Rowe integrals N_1 and N_2 in the soliton solution (for details, see review [4]). With this aim in view, Eqn (53) should be multiplied by ψ_{1s} and then integrated over x. As a result, we obtain

$$-(\lambda_1 + \lambda_2 + \Omega)n_1 + D_1 = -I\,. \tag{58}$$

An analogous procedure applied to Eqns (54) and (55) gives

$$-\lambda_1 n_2 + D_2 = -I\,, \tag{59}$$

$$-\lambda_2 n_3 + D_3 = -I\,, \tag{60}$$

where

$$n_l = \int |\psi_{ls}|^2\,dx\,, \quad D_l = \frac{1}{2}\int \omega_l'' \left|\frac{d\psi_{ls}}{dx}\right|^2 dx\,, \quad I = \int \psi_{1s}\psi_{2s}\psi_{3s}\,dx\,.$$

These integral relations have to be complemented by a condition that follows from the variational problem by applying to it scaling transformations that preserve N_1 and N_2:

$$\psi_{ls}(x) \to a^{-1/2}\psi_{ls}\left(\frac{x}{a}\right).$$

Under such transformations, the Hamiltonian becomes a function of the parameter a,

$$H(a) = \sum_l \int \left(\Omega_l|\psi_{ls}|^2 + \frac{1}{2a^2}\,\omega_l''|\psi_{lsx}|^2\right) dx - \frac{2}{a^{1/2}}\int \psi_{1s}\psi_{2s}\psi_{3s}\,dx\,.$$

Using the last relation together with Eqn (56), it is straightforward to obtain that

$$\left.\frac{dH}{da}\right|_{a=1} = 0\,, \text{ or } \sum_l \int \omega_l''|\psi_{lsx}|^2\,dx - \int \psi_{1s}\psi_{2s}\psi_{3s}\,dx = 0\,. \tag{61}$$

Taking relations (58)–(60) into account and bearing in mind the positivity of n_l (by definition), we arrive at conditions (57). Hence, for the existence of solitons, all operators L_l must have the same sign definiteness (positive or negative). We note that this requirement holds for all physical dimensions $D \leqslant 3$. Accordingly, all matrices $\omega_l^{\alpha\beta}$ must have the same sign definiteness, for example, be positive definite. In particular, if one of the matrices is not sign definite, then multidimensional solitons are absent. A similar situation occurs for envelope solitons in a medium with normal dispersion (see, e.g., Ref. [49]).

It should also be mentioned that the sign definiteness of the matrices $\omega_l^{\alpha\beta}$ translates into a simple physical requirement that Cherenkov resonances between the soliton and linear waves be absent (see Refs [50, 51]).

4.1 Nonlinear stability of solitons in the three-wave system

We demonstrate how the Lyapunov stability can be inferred for soliton solutions of the three-wave system. Because we here have three fields (the three amplitudes ψ_i), we need to consider two spaces, $L_{3,3}$ and W_2^1, with the norms

$$\|u\|_{L_{3,3}} = \left[\int \left(|\psi_1|^3 + |\psi_2|^3 + |\psi_3|^3\right) d^D x\right]^{1/3},$$

$$\|u\|_{W_2^1} = \left[\tilde\lambda_1 \int \left(|\psi_1|^2 + |\psi_2|^2\right) d^D x + \tilde\lambda_2 \int \left(|\psi_1|^2 + |\psi_3|^2\right) d^D x + \frac{1}{2}\sum_l \int \partial_\alpha\psi_l^*\omega_l^{\alpha\beta}\,\partial_\beta\psi_l\,d^D x\right]^{1/2},$$

where the constants $\tilde\lambda_{1,2} > 0$ and tensors $\omega_l^{\alpha\beta}$ are supposed to be positive definite. In this case, the Sobolev inequality becomes

$$\|u\|_{L_{3,3}} < M\|u\|_{W_2^1}\,. \tag{62}$$

We note that the norm $\|u\|_{L_{3,3}}$ and the interaction Hamiltonian are related by the simple inequality

$$\|u\|_{L_{3,3}}^3 \geqslant \frac{3}{2}\int (\psi_1^*\psi_2\psi_3 + \text{c.c.})\,d^D x\,. \tag{63}$$

The multiplicative variant of the Sobolev inequality is obtained similarly to that of the NLS equation. In particular, for $D = 1$, we have

$$J \leqslant M_1(\tilde\lambda_1 N_1 + \tilde\lambda_2 N_2)^{5/4} I^{1/4}\,,$$

where

$$J = \int (\psi_1^*\psi_2\psi_3 + \text{c.c.})\,d^D x\,, \qquad I = \frac{1}{2}\sum_l \int \partial_\alpha\psi_l^*\omega_l^{\alpha\beta}\,\partial_\beta\psi_l\,d^D x$$

and $\tilde\lambda_1$ and $\tilde\lambda_2$ are arbitrary positive parameters. Minimization with respect to $\tilde\lambda_1$ and $\tilde\lambda_2$ gives

$$J \leqslant C(N_1 N_2)^{5/8} I^{1/4}\,. \tag{64}$$

The next step consists in finding the best constant C as a minimum of the functional

$$C_{\text{best}} = \min_{[\psi]} F[\psi]\,, \qquad F = \frac{J}{(N_1 N_2)^{5/8} I^{1/4}}\,.$$

It follows that this minimum is provided by the ground-state soliton

$$C_{\text{best}} = F[\psi_s]\,. \tag{65}$$

Using refined inequality (64), we arrive at the required estimate for the Hamiltonian with zero detuning, $\Omega = 0$,

$$H \geqslant I - 2I_s^{3/4} I^{1/4} \geqslant H_s(\Omega = 0)\,,$$

which proves the stability of the (ground-state) soliton solution in the case of one dimension for zero detuning. As previously, the last inequalities transform into equalities for the soliton solution. Following this scheme, the soliton stability can be proved for any physical dimension $D \leqslant 3$.

For a nonzero detuning, the Hamiltonian in the one-dimensional case becomes

$$H = \Omega \int |\psi_1|^2 \, \mathrm{d}x + \tilde{H} ,$$

where $\tilde{H}$ coincides with the Hamiltonian at $\Omega = 0$:

$$\tilde{H} = \sum_l \int \frac{1}{2} \, \omega_l'' |\psi_{lx}|^2 \, \mathrm{d}x - \int (\psi_1^* \psi_2 \psi_3 + \text{c.c.}) \, \mathrm{d}x .$$

We now analyze how the frequency detuning Ω influences the stability of solitons. Obviously, the soliton existence domain is strongly asymmetric with respect to the sign of the detuning Ω. For $\Omega > 0$, the straightforward inequality

$$H \geqslant H_{\mathrm{s}}(\Omega = 0)$$

holds, from which it follows that the Hamiltonian is bounded from below by the value of H_{s} at the ground-state soliton solution with zero detuning.

For $\Omega < 0$, we have

$$H \geqslant H_{\mathrm{s}}(\Omega = 0) - |\Omega| \int |\psi_1|^2 \, \mathrm{d}x , \tag{66}$$

where the integral $\int |\psi_1|^2 \, \mathrm{d}x$ is always bounded from below by $\min(N_1, N_2)$. Accordingly, the estimate for H is written as

$$H \geqslant H_{\mathrm{s}}(\Omega = 0) - |\Omega| \min(N_1, N_2) .$$

We have thus proved the stability of ground-state solitons (those without nodes) describing a coupled state of three wave packets. Notably, in the absence of detuning, the soliton realizes a minimum of the Hamiltonian, which is rigorously proved with the help of majorizing Sobolev inequalities. That the Hamiltonian of the three-wave system is bounded was first demonstrated by Kanashev and Rubenchik (1981) [10] for isotropic media. Later, Turitsyn (1995) [52] showed that the Hamiltonian attains a minimum at the ground-state soliton (without zeros) for zero detuning for the coupling between the first and second harmonics, when the dispersion operators are Laplacians. In 1997, Berge, Bang, Rasmussen, and Mezentsev [53] demonstrated the boundedness of H for nonzero detuning for the coupling between the first and second harmonics. The general case of a bounded H for the three-wave system is discussed in Refs [54, 55].

5. Linear stability

We next address the problem of the linear stability of solitons considered in Sections 2–4.

5.1 Linear stability of one-dimensional solitons of the NLS equation

We begin with the one-dimensional NLS equation, assuming that

$$\psi(x,t) = \big(\psi_0(x) + u + \mathrm{i}v\big) \exp \frac{\mathrm{i}\lambda^2 t}{2} , \qquad \psi_0 \gg u, v .$$

The linearized equations for a perturbation are Hamiltonian equations

$$u_t = \frac{1}{2} \frac{\delta \tilde{H}}{\delta v} , \qquad v_t = -\frac{1}{2} \frac{\delta \tilde{H}}{\delta u} , \tag{67}$$

where u and v are canonically conjugate quantities and $\tilde{H}$ is the second variation of $F = H + \lambda^2 N/2$,

$$\tilde{H} = \langle v | L_0 | v \rangle + \langle u | L_1 | u \rangle , \tag{68}$$

with $L_0 = \lambda^2 - \partial_x^2 - 2\psi_0^2$ and $L_1 = \lambda^2 - \partial_x^2 - 6\psi_0^2$. The first term in $\tilde{H}$ is the mean value of L_0, playing the role of kinetic energy; the second term, $\langle u | L_1 | u \rangle$, plays the role of potential energy.

The stability or instability of a soliton is therefore determined by the properties of the operators L_0 and L_1. The first property of L_0 follows directly from the stationary NLS equation,

$$L_0 \psi_0 = \lambda^2 \psi_0 - \partial_x^2 \psi_0 - 2\psi_0^3 \equiv 0 , \tag{69}$$

which means that ψ_0 is the ground state of L_0 (has no zeros). Accordingly, $\langle v | L_0 | v \rangle \geqslant 0$. The stability or instability of solitons is therefore determined by the sign of the potential energy.

The operator L_1 has an eigenfunction with zero eigenvalue,

$$L_1 \, \frac{\partial \psi_0}{\partial x} = 0 .$$

This is a neutrally stable mode that correspond to the shift of the soliton and has a single node ($x = 0$). Therefore, the Schrödinger operator L_1 has the only lower eigenvalue $E < 0$, which corresponds to the ground state ϕ_0. This apparently suggests that $\langle u | L_1 | u \rangle < 0$ and, correspondingly, that an instability develops. But this is not the case because u is constrained by $\langle u | \psi_0 \rangle \equiv \int u \psi_0 \, \mathrm{d}x = 0$, which is a consequence of the conservation of the number of particles. For this reason, the stability or instability of a soliton is determined from the solution of the spectral problem

$$L_1 |\phi\rangle = E |\phi\rangle + C |\psi_0\rangle \tag{70}$$

(where C is a Lagrange multiplier) under the additional constraint $\langle u | \psi_0 \rangle = 0$.

Expanding $|\phi\rangle$ in a series in the complete set of eigenfunctions $\{\phi_n\}$ of L_1 ($L_1 \phi_n = E_n \phi_n$),

$$\phi = \sum_n C_n \phi_n ,$$

we find

$$C_n = C \, \frac{\langle \phi_n | \psi_0 \rangle}{E_n - E} , \qquad C_1 \equiv 0 .$$

Inserting these expressions into the solvability condition $\langle \phi | \psi_0 \rangle = 0$, we obtain the dispersion relation

$$f(E) \equiv \sum_n{}' \, \frac{\langle \psi_0 | \phi_n \rangle \langle \phi_n | \psi_0 \rangle}{E_n - E} = 0 . \tag{71}$$

The prime at the sum indicates that the term with the energy $E_1 = 0$ is excluded (the translation mode $\phi_1 = \partial \psi_0 / \partial x$ corresponds to the energy E_1).

We next consider the energy range between the ground-state energy $E_0 < 0$ and the first positive level E_2. In this interval, $f(E)$ increases monotonically ($\partial f/\partial E > 0$) from $-\infty$ at $E = E_0$ to $+\infty$ at $E = E_2$. If $f(E)$ is negative at $E = 0$, then the dispersion equation does not have negative eigenvalues and the soliton is stable. If $f(0) > 0$, then there exists a single eigenvalue $E < 0$ and hence the soliton is unstable.

To find $f(0)$, note that

$$f(0) = \sum_n{}' \frac{\langle\psi_0|\phi_n\rangle\langle\phi_n|\psi_0\rangle}{E_n} \equiv \langle\psi_0|L_1^{-1}|\psi_0\rangle\,.$$

Differentiating the stationary NLS equation

$$-\lambda^2\psi_0 + \partial_x^2\psi_0 + 2|\psi_0|^2\psi_0 = 0$$

with respect to λ^2, we then obtain

$$L_1\left(\frac{\partial\psi_0}{\partial\lambda^2}\right) = -\psi_0\,,$$

or

$$\langle\psi_0|L_1^{-1}|\psi_0\rangle = -\left\langle\psi_0\left|\frac{\partial\psi_0}{\partial\lambda^2}\right.\right\rangle = -\frac{1}{2}\frac{\partial N_{\rm s}}{\partial\lambda^2}\,.$$

As a result, we arrive at the so-called Vakhitov–Kolokolov criterion [19]: *if*

$$\frac{\partial N_{\rm s}}{\partial\lambda^2} > 0\,, \tag{72}$$

then the soliton is stable; *for the other sign of the derivative, it is unstable*.

This criterion is valid for any dimension D and the dependence of the nonlinear term on $|\psi|^2$ (i.e., if $|\psi|^2$ in the NLS equation is replaced with an arbitrary function $f(|\psi|^2)$).

The criterion has a simple physical interpretation. The quantity $\varepsilon = -\lambda^2/2$ for the stationary NLS equation represents the energy of the bound state, the soliton. If, on adding a 'single' particle, the energy of the bound state decreases, the situation is stable. If, on adding a 'single' particle, the level $-\lambda^2$ shifts to the continuum spectrum, the soliton is unstable.

The dependence of $N_{\rm s}$ on λ can easily be established for a cubic NLS equation, $N_{\rm s} \propto \lambda^{2-D}$, which implies the stability of one-dimensional solitons of the cubic NLS equation and the instability of three-dimensional ones. The case of two dimensions is degenerate.

5.2 Stability of solitons for the coupling of the first and second harmonics

We consider system of Eqns (30) and (31) that describes the interaction of the first and second harmonics and show how the procedure for deriving the Vakhitov–Kolokolov criterion can be applied in this case. Soliton solutions for the coupling of the first and second harmonics $\psi_1(x,t) = \psi_{1\rm s}(x)\exp(\mathrm{i}\lambda^2 t)$ and $\psi_2(x,t) = \psi_{2\rm s}(x)\exp(2\mathrm{i}\lambda^2 t)$ are found from the equations for the amplitudes $\psi_{1\rm s}$ and $\psi_{2\rm s}$:

$$-\lambda^2\psi_1 + \frac{1}{2}\,\omega_1''\psi_1 = -2\psi_2\psi_1\,, \tag{73}$$

$$-2\lambda^2\psi_2 - \Omega\psi_2 + \frac{1}{2}\,\omega_2''\,\partial_x^2\psi_2 = -\psi_1^2\,. \tag{74}$$

Here, the solutions $\psi_{1\rm s}$ and $\psi_{2\rm s}$ are assumed to be real valued and have no nodes, i.e., correspond to the ground-state soliton solution (the index s at ψ_1 and ψ_2 is omitted). We consider small perturbations on the background of this soliton solution by writing

$$\psi_1(x,t) = (\psi_{1\rm s} + u_1 + \mathrm{i}v_1)\exp(\mathrm{i}\lambda^2 t)\,,$$
$$\psi_2(x,t) = (\psi_{2\rm s} + u_2 + \mathrm{i}v_2)\exp(2\mathrm{i}\lambda^2 t)\,.$$

The linearization of system (30), (31) leads to the linear (Hamiltonian) equations

$$u_t = \frac{1}{2}\frac{\delta\tilde H}{\delta\mathbf{v}}\,,\qquad v_t = -\frac{1}{2}\frac{\delta\tilde H}{\delta\mathbf{u}}\,, \tag{75}$$

where $\tilde H$, just as for the NLS equation, is the second variation of $F = H + \lambda^2 N$,

$$\tilde H = \langle\mathbf{v}|L_0|\mathbf{v}\rangle + \langle\mathbf{u}|L_1|\mathbf{u}\rangle\,. \tag{76}$$

Here, $\mathbf{u}$ and $\mathbf{v}$ are vectors with components u_1, u_2 and v_1, v_2, and N is defined in Eqn (32). The second-order differential operators L_0 and L_1 are in this case 2×2 matrix operators

$$L_{0,1} = \begin{pmatrix} \lambda^2 - \dfrac{1}{2}\,\omega_1''\partial_x^2 \mp 2\psi_2 & -2\psi_1 \\ -2\psi_1 & 2\lambda^2 - \dfrac{1}{2}\,\omega_2''\partial_x^2 - \Omega \end{pmatrix}.$$

Both operators, just as in the NLS case, are self-adjoint. From the standpoint of quantum mechanics, these operators correspond to the Schrödinger operator for a nonrelativistic particle of spin $S = 1/2$ in an inhomogeneous magnetic field. It is common knowledge that the oscillation theorem is not applicable to such operators. The ground-state eigenfunction does not have zeros, but the correspondence between the number of zeros and the level number, inherent in the scalar Schrödinger operator, is absent here. This lack of correspondence, as is to become clear in what follows, does not allow drawing the same conclusions about the soliton stability as for the NLS equation. The Vakhitov–Kolokolov criterion for such matrix systems can only be used as a sufficient condition of instability; the same statement is also valid for the three-wave system.

As regards the properties of the operators L_0 and L_1, they are similar to the properties of analogous operators for the NLS equation. The nonnegativity of L_0 follows from the fact that if, instead of v_1 and v_2, we introduce new functions χ_1 and χ_2 defined by the relations

$$v_1 = \psi_1\chi_1\,,\qquad v_2 = \psi_2\chi_2\,,$$

then $\langle v|L_0|v\rangle$ can be represented as

$$\langle v|L_0|v\rangle = \frac{1}{2}\int(\omega_1''\psi_1^2\chi_1'^2 + \omega_2''\psi_2^2\chi_2'^2)\,\mathrm{d}x + \int\psi_1^2\psi_2(2\chi_1 - \chi_2)^2\,\mathrm{d}x\,.$$

The nonnegativity of L_0 becomes obvious from this representation, and for the eigenvector of the ground state, it follows that

$$\chi_1 = c_1\,,\qquad \chi_2 = c_2\,,\qquad 2c_1 = c_2\,,$$

or

$$v_0 = \begin{bmatrix}\psi_1\\ 2\psi_2\end{bmatrix}.$$

This eigenvector, as in the NLS case, explicitly enters the expression $\delta N = 2\int(\psi_1 u_1 + 2\psi_2 u_2)\,dx \equiv 2\langle v_0|u\rangle$, which is equal to zero by virtue of the conservation of N. Just as in the NLS case, $\delta N = 0$ is the solvability condition for linear system (75).

The subsequent analysis resembles that carried out for the NLS equation. To begin, we consider the eigenvalue problem for the operator L_1,

$$L_1|\phi\rangle = E|\phi\rangle + C|v_0\rangle\,, \qquad (77)$$

and then expand $|\phi\rangle$ in a complete set of eigenfunctions $\{|\phi_n\rangle\}$ of L_1. As a result, the solution of Eqn (77) is written in the form

$$|\phi\rangle = C\sum_n \frac{|\phi_n\rangle\langle\phi_n|v_0\rangle}{E_n - E}\,.$$

Analogously, the solvability condition leads to the dispersion relation

$$f(E) \equiv {\sum_n}' \frac{\langle v_0|\phi_n\rangle\langle\phi_n|v_0\rangle}{E_n - E} = 0\,. \qquad (78)$$

The prime at the summation symbol indicates, as previously, the absence of the term with $E = 0$, because $\langle\Psi_x|v_0\rangle = 0$ and $L_1|\Psi_x\rangle = 0$, where

$$\langle\Psi_x| = (\psi_{1x}, \psi_{2x})\,.$$

Up to this point, everything replicates the NLS case. The difference becomes apparent in the analysis of the function $f(E)$. Now the oscillation theorem is not valid. Accordingly, there could be several levels below the level $E = 0$. For this reason, the dispersion relation can have negative roots $E < 0$ independent of the sign of the derivative $\partial N_s/\partial\lambda^2$. As a result, only a sufficient instability criterion can be formulated; it has the same form as for NLS solitons:

$$\frac{\partial N_s}{\partial\lambda^2} < 0\,. \qquad (79)$$

But we cannot decide on the stability in the general case. The stability criterion

$$\frac{\partial N_s}{\partial\lambda^2} > 0$$

would hold only if, for energies less than $E_0 = 0$, the operator L_1 has only one (ground-state) level, but it is not the general case. Thus, the Vakhitov–Kolokolov stability criterion applied to vector models ensures only a sufficient condition for soliton instability.

Nevertheless, the combination of the Vakhitov–Kolokolov criterion and the Lyapunov method may grant the full answer about stability. To conclude this section, we discuss one example in which, based on the combined method, it is possible to draw a more or less definitive conclusion about the soliton stability by numerically integrating system (73), (74). The dependences of H and N (for one-dimensional soliton solutions) on λ was numerically found in [56] for a nonzero frequency detuning $\Omega \neq 0$. Numerical integration has shown that for $\Omega < 0$, both dependences are monotonic: N increases with λ and H decreases. As a result, it was shown that only a single branch of soliton solutions with an unambiguous dependence $H(N)$ exists. For $\Omega > 0$, the dependence $N(\lambda)$ contains two branches. The first occupies the domain $0 < \lambda < \lambda_{\min}$. The function $N(\lambda)$ increases monotonically as λ approaches zero. At $\lambda = \lambda_{\min}$, the function $N(\lambda)$ attains a minimum. In agreement with criterion (79), all this branch of soliton solutions is unstable. For $\lambda > \lambda_{\min}$, N increases monotonically, but the linear stability criterion cannot be applied to these solutions. However, the dependence $H(\lambda)$ allows making a certain conclusion about stability. The function $H(\lambda)$ has a maximum at the point $\lambda = \lambda_{\min}$, and hence H as a function of N has a cusp singularity at this point, which separates two branches of soliton solutions. The upper branch corresponds to larger values of H than the lower one. If we assume that no other soliton solutions exist in this range of N (which is by no means easy to establish numerically), then the lower branch belongs to the stable family of soliton solutions.

6. Wave collapses

Wave collapse is the other variant of evolution for nonlinear wave systems, in a certain sense an alternative to the soliton scenario. As we have seen in Sections 2–4, Lyapunov stable solitons are encountered when, for a fixed number of particles N or other integrals of motion, the Hamiltonian is a functional bounded from below, its lower bound corresponding to a Lyapunov-stable soliton. A question naturally arises as to what happens if the Hamiltonian is not bounded from below. How does the system behave in this case? One possible mode of behavior is collapse, i.e., singularity formation in finite time. The type of the emerging singularity depends on the concrete physical problem. In this review, we limit ourselves mainly to the analysis of the wave collapse described by NLS equation (1) for $D \geqslant 2$.

It is well known (see, e.g., Refs [17, 29, 39]) that as the space dimension D increases, the role of nonlinear effects also increases; similarly to the phase transition theory, the role of cooperative effects becomes more prominent with the increasing number of neighbors, for example, in the Ising model. For the NLS equation, this is seen, in particular, from how H varies under scaling transformations $\psi \to a^{-D/2}\psi(\mathbf{r}/a)$ that preserve the number of particles,

$$H(a) = \frac{1}{2}\left(\frac{I_1}{a^2} - \frac{I_2}{a^D}\right)\,. \qquad (80)$$

In one dimension ($D = 1$), $H(a)$ has a minimum that corresponds to a (stable) soliton. In two dimensions, $I_{1s} = I_{2s}$; therefore, $H(a) \equiv 0$ for the entire soliton family. For $D = 3$, $H(a)$ has a maximum instead of a minimum (in reality, a saddle point), which points to the instability of the soliton. Additionally, $H(a)$ turns out to be an unbounded function as $a \to 0$, which is one of the collapse criteria [39] (see also Ref. [57]). In this case, the collapse can be regarded as a nonlinear stage of soliton instability.

To illustrate the foregoing, we resort to the variational approach. As a test function for Eqn (1), we choose ψ in the form

$$\psi(\mathbf{r}, t) = a^{-3/2}\psi_s\left(\frac{\mathbf{r}}{a}\right)\exp\left(\mathrm{i}\lambda^2 t + \mathrm{i}\mu r^2\right),$$

where $a = a(t)$ and $\mu = \mu(t)$ are unknown functions of time. Substituting this ansatz into the action

$$S = \frac{\mathrm{i}}{2}\int(\psi_t\psi^* - \text{c.c.})\,\mathrm{d}t\,\mathrm{d}\mathbf{r} - \int H\,\mathrm{d}t$$

and integrating over spatial variables, we obtain a reduced action for the two functions $a = a(t)$ and $\mu = \mu(t)$. It is easy to verify that the two Lagrange equations, for $a(t)$ and $\mu(t)$, reduce to a single second-order equation for a, which has the form of Newton's equation

$$C\ddot{a} = -\frac{\partial H}{\partial a}\,, \tag{81}$$

where $C = \int \xi^2|\psi_0(\xi)|^2\,\mathrm{d}\xi$ plays the role of the mass of a particle and function (80) has the meaning of potential energy. The behavior of $a(t)$ depends on the total energy

$$E = C\,\frac{\dot{a}^2}{2} + H(a)\,,$$

on the initial position of the 'particle,' and on the dimension D. For $D = 1$, the soliton realizes a minimum of the potential energy $H(a)$, which is the cause of its stability. For $D = 3$, if the particle is initially at a maximum point, the system, depending on its motion direction (toward the center $a = 0$ or away from it), either collapses ($\psi \to \infty$) or expands ($\psi \to 0$) because of dispersion. For the collapsing mode, which corresponds to the fall of the particle on the center, in the vicinity of the singularity, $a(t)$ obeys the power law

$$a(t) \sim (t_0 - t)^{2/5}\,, \tag{82}$$

where the collapse time t_0 is finite and equal to the particle fall time in the potential $H(a)$. As shown in Ref. [39], such an asymptotic form of $a(t)$ in the vicinity of the singular point coincides, up to a constant, with the exact collapsing semiclassical solution, which tends to a compact distribution as $t \to t_0$,

$$|\psi| \to \lambda\sqrt{1-\xi^2} \;\; \text{for} \;\; \xi = \frac{r}{a(t)} \leqslant 1\,,$$

with $\lambda \sim (t_0 - t)^{-3/5}$.

The consideration above already invites several conclusions. First, the role of nonlinearity increases with the space dimension. Stable solitons are encountered in systems of low dimensions, whereas in higher dimensions, we should expect explosive events instead of solitons, which evolve into singularities. Second, one of the collapse criteria is the unboundedness of the Hamiltonian at small scales, in which case the collapse can be interpreted as a fall of a particle on the center in a self-consistent potential which is unbounded at small scales [39].

6.1 The role of radiation

In reality, however, the picture of singularity formation is more complex than that sketched above. From the very beginning, we are dealing with a spatially distributed system, i.e., one with an infinitely many degrees of freedom. It is therefore obvious that a reduction of the NLS equation with the help of the variational approach to the system of ordinary differential equations as in Eqns (81) does not account for wave radiation. This process, as explained above, is of principal importance for soliton interactions. As shown in Ref. [58], it contributes crucially to the relaxation of a pulse to a soliton in the one-dimensional NLS equation integrated by the inverse scattering method [40]. In the last case, the variational approach [59, 60], albeit rather popular presently, provides an incorrect answer by predicting soliton oscillations. A rigorous analysis shows that the initial pulse relaxes to a soliton in an oscillatory manner owing to wave radiation. Moreover, the oscillation frequency is far from that given by the variational approach.

In the case where H is unbounded from below, when we anticipate collapse, the radiation of small-amplitude waves is one of the mechanisms governing the singularity formation. To demonstrate this, we can resort to the same arguments about the role of radiation in soliton formation as in Section 3.

Let the Hamiltonian be negative in a certain domain G, $H_{\mathrm{G}} < 0$. Then the inequality [cf. Eqn (49)]

$$\max_{x\in\mathrm{G}} |\psi|^2 \geqslant \frac{|H_{\mathrm{G}}|}{N_{\mathrm{G}}}$$

holds within the domain G. We assume that outside the domain G, the magnitude of $|\psi|$ is small. We consider the role of wave radiation from this domain. The radiation, obviously, carries away positive energy because nonlinear effects for the radiated waves are small at large distances. As a consequence, because of radiation, H_{G} becomes progressively larger in absolute value, while N_{G} decreases. This is the reason why $\max|\psi|^2$ increases unlimitedly, for the Hamiltonian is unbounded from below. We can therefore state that the radiation promotes collapse. In a nonlinear wave system, the radiation therefore plays a role of friction [17, 27, 29]. Moreover, the radiation accelerates the compression of the collapsing domain, which, in particular, results in compression obeying the self-similarity law

$$r \sim (t_0 - t)^{1/2}\,, \tag{83}$$

which differs from the semiclassical compression in (82). The arguments above are, however, insufficient to draw a conclusion about the finiteness of the collapse time. The answer is provided by the virial theorem.

6.2 Virial theorem

The exact criterion of the singularity formation in the NLS framework can be inferred from the virial theorem. In classical mechanics, this theorem can easily be derived by computing the second time derivative of the inertia moment and then averaging the result to find the relation between the mean values of the kinetic and potential energies of particles, whenever the interaction between the particles obeys a power law.

In 1971, Vlasov, Petrishchev, and Talanov [18] found that the virial theorem can be applied to the two-dimensional NLS equation. The analog of the moment of inertia for the two-dimensional NLS equation is the mean square size of the distribution $\langle r^2\rangle = N^{-1}\int r^2|\psi|^2\,\mathrm{d}\mathbf{r}$. By direct calculations, it can be verified that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int r^2|\psi|^2\,\mathrm{d}\mathbf{r} = 2H\,. \tag{84}$$

Because H is a conserved quantity, Eqn (84) can be integrated twice,

$$\int r^2|\psi|^2\,d\mathbf{r} = Ht^2 + C_1 t + C_2\,, \qquad (85)$$

where C_1 and C_2 are the additional integrals of motion, whose existence is related to two Noether-type symmetries: the lens transformation (established by Talanov [22] in 1970) and scaling transformations [61, 62].

Now it can be easily seen that for any distribution with a negative Hamiltonian

$$H < 0\,, \qquad (86)$$

$\langle r^2 \rangle$ vanishes in a finite time independent of C_1 and C_2, which, because N is conserved, implies the formation of a singularity in the field ψ [18]. Condition (86) represents the celebrated Vlasov–Petrishchev–Talanov criterion. This criterion, formulated in 1971, is a fundamental result in the theory of wave collapses. It was the first rigorous result for nonlinear wave systems with dispersion, and showed that the formation of a singularity in a finite time is possible despite the linear wave dispersion, which hinders (for example, in linear optics) the appearance of point singularities, foci.

We emphasize that criterion (86) is only a *sufficient*, but not a necessary condition for the onset of collapse. For example, if in a certain part of the system, well isolated from the rest, the Hamiltonian H is negative, collapse would unfold there independent of whether the total Hamiltonian of the full system is positive or negative.

6.3 Strong collapse

We note that for $D = 2$, a soliton is associated with the equality $H = 0$ with the particle number $N = N_s$. Moreover, for $N < N_s$, the inequality

$$H \geqslant \frac{1}{2} I_1 \left(1 - \frac{N}{N_s}\right) > 0$$

holds and, as a consequence, collapse is forbidden [47]. In this case, the amplitude of waves tends to zero as $t \to \infty$ because of dispersion or diffraction. It can therefore be argued that the soliton for the two-dimensional NLS equation represents a separatrix between the manifolds of collapsing and noncollapsing distributions.

It follows from relation (85) that for states with $H < 0$, the characteristic size a of the collapsing domain behaves as

$$a \sim (t_0 - t)^{1/2}\,,$$

in full agreement with self-similarity law (83).

The rigorous analysis in [63], however, shows that

$$a^2(t) \sim \frac{t_0 - t}{\log|\log(t_0 - t)|}\,,$$

while the spatial profile ψ asymptotically approaches the soliton one (see also Refs [64–68]). The collapse in such cases is accompanied by exponentially weak radiation. The energy confined to the singularity coincides with N up to a constant factor and is *finite* and equal to the energy of the two-dimensional soliton. This is why such a collapse is called the *strong collapse* [39].

We note that two-dimensional NLS equation (1) is regarded as a critical NLS equation. For the critical NLS equation, as we have seen, scaling transformations represent an extra symmetry. As a consequence, in particular, the dispersion ($\sim I_1$) and nonlinear ($\sim I_2$) terms in H behave similarly under such transformations. For the NLS equation with the power-law nonlinearity in (46), the critical behavior is associated with $\sigma = 1 + 2/D$. For smaller σ, the Hamiltonian is bounded from below for a fixed number of particles, and the soliton solution, which corresponds to the minimum of H, turns out to be stable.

6.4 Collapse in a boundary layer

The critical behavior does not occur solely for the NLS equation. For example, the two-dimensional model

$$u_t = \frac{\partial}{\partial x}\hat{k}u - 6uu_x = \frac{\partial}{\partial x}\frac{\delta H}{\delta u} \qquad (87)$$

with the Hamiltonian

$$H = \int \left(\frac{1}{2}u\hat{k}u - u^3\right) d\mathbf{r} \equiv \frac{1}{2} I_1 - I_2$$

also belongs to the class of critical models. Here, $\hat{k}$ is the integral operator whose Fourier transform is the modulus $|\mathbf{k}| = (k_x^2 + k_y^2)^{1/2}$. Equation (87) describes low-frequency oscillations in a boundary layer for large Reynolds numbers, $\mathrm{Re} \gg l$. The mean profile of velocity (parallel to the x axis) $v = U(z)$ $(0 \leqslant z < \infty)$ is assumed to be a monotonically increasing function that tends to a constant as $z \to \infty$. The dimensionless amplitude u is related to velocity fluctuations along the flow as

$$\delta v_x \approx -6huU'(z)\,, \qquad (88)$$

where $h = U(0)/U'(0)$ is the boundary layer thickness. Equation (81), first derived by Shrira [69], is a two-dimensional generalization of the Benjamin–Ono equation describing long waves in stratified fluids. In one dimension, Eqn (87) was derived in Refs [70, 71] with account for (small) viscosity.

Scaling transformations preserving the x-component of the momentum $P_x = 1/2\int u^2\,d\mathbf{r}$, similarly to those in (52),

$$u(\mathbf{r}) \to \frac{1}{a^{D/2}}\,u\left(\frac{\mathbf{r}}{a}\right),$$

where $D = 1, 2$, lead to the following dependence of the Hamiltonian H on the scale parameter a:

$$H(a) = \frac{1}{2}\frac{I_1}{a} - \frac{I_2}{a^{D/2}}\,.$$

It then follows that this model becomes critical for $D = 2$. For $D = 1$, the Hamiltonian is bounded from below, and its minimum corresponds to the Benjamin–Ono soliton. This solution, having the form $u = u_s(x - Vt, y)$, can be found exactly:

$$u_s = \frac{2V}{3\left[(x - Vt)^2 V^2 + 1\right]}\,, \qquad V > 0\,. \qquad (89)$$

However, this soliton turns out to be unstable with respect to two-dimensional perturbations [72, 73]. In the long-wave

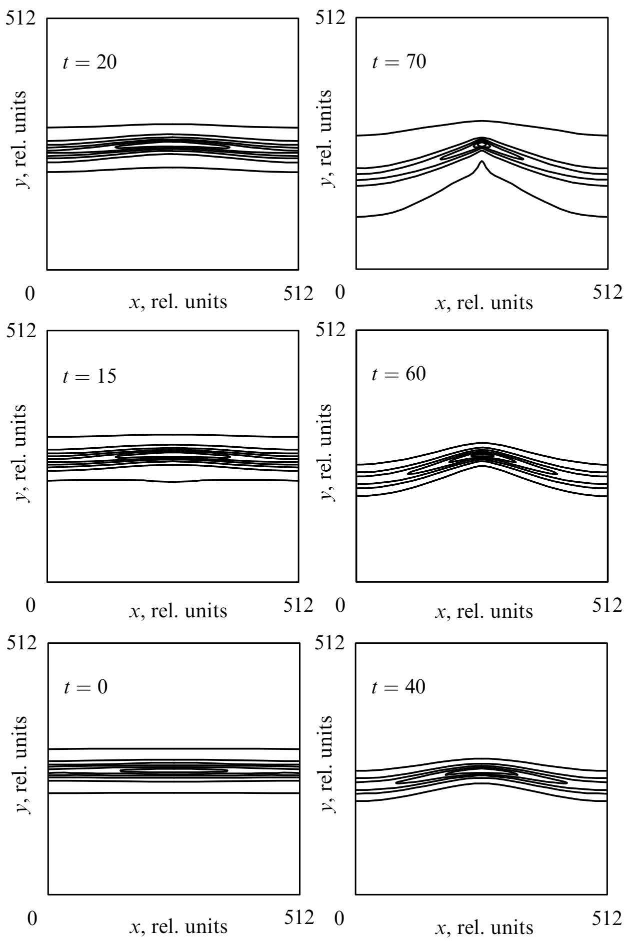

Figure 2. The evolution of $u(x, y, t)$ (isolines) for initial condition (90), demonstrating the solution instability and cluster formation.

limit, the increment of this instability is [72]

$$\Gamma = \frac{k_y V}{\sqrt{2}} .$$

Figure 2 shows the evolution of $u(x, y, t)$ illustrating this instability. The initial condition is the perturbed periodic wave

$$u_{\text{ini}}(x, y) = u_{\text{p}}(x)\left(1 + 0.1 \cos \frac{2\pi y}{L_y}\right), \tag{90}$$

where $u_{\text{p}}(x)$ is the exact one-dimensional solution of the Benjamin–Ono equation in the form of a periodic wave,

$$u_{\text{p}}(x) = \frac{k}{3} \frac{\sinh(k/V)}{\cosh(k/V) - \cos(kx)} - 1 . \tag{91}$$

As $k \to 0$, solution (91) transforms into soliton solution (89). It can be shown that the solution in the form of a periodic wave can be rewritten as a periodic lattice of solitons with the period $2\pi/k$; wave (91) accommodates one soliton per period.

For the chosen variant of initial conditions (90), $k = 0.0625$, $V = 0.4$, and the size of the domain L_x coincides with the wave period in (90).

The instability described above is analogous to the Kadomtsev–Petviashvili (KP) instability [74, 75] for acoustic solitons in media with positive dispersion (see also Refs [76,

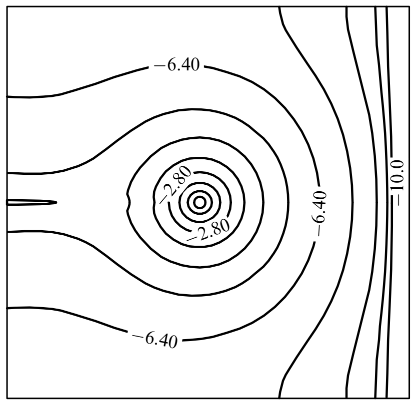

Figure 3. Isolines $u(x, y)$ for $t = 45$ in the collapse mode. The contours correspond to values of $u(x, y)$ in the range from -10.0 to 0.8 with the interval 0.90000.

77]). (See Section 6.5 on the cause of this instability.) In two dimensions, this model has a solution in the form of a cylindrically symmetric soliton without zeros. Such a solution, found numerically in Ref. [78], plays the same role as the Townes mode for the two-dimensional NLS equation [23]: for $P_x < P_{x,\text{cr}}$, where $P_{x,\text{cr}}$ is the value of the x-component of momentum in the ground-state (without zeros) soliton solution, the Hamiltonian turns out to be positive with a minimum that tends to zero as $u \to 0$. In this case, collapse is impossible, just as in the analogous case of the two-dimensional NLS equation. The Hamiltonian is unbounded for states with negative values of H. It was numerically confirmed in [72, 73] that the collapse in system (88) occurs for $H < 0$, and the collapsing solution is cylindrically symmetric and approaches the soliton distribution as $t \to t_0$. The initial condition was chosen as

$$u(\mathbf{r}) = \frac{2/3|\mathbf{V}|}{1 + (V_x x)^2 + (V_y y)^2} .$$

The parameters V_x and V_y were varied so as to satisfy the conditions $P_x > P_{x,\text{cr}}$ and $H < 0$. On approaching the singularity, the peak anisotropy disappeared and the distribution became symmetric. A typical example of collapse is shown in Fig. 3.

To conclude this section, we point to a series of interesting experiments [79, 80], performed over several years, on the excitation of coherent structures in a transient boundary layer over a plate. The structures were excited with the help of a vibrating mechanical system placed at the front edge of the plate. According to experimental data, one-dimensional solitons developed at the initial stage, and later, far downstream from the leading edge, the solitons lost stability, giving way to so-called thorns — three-dimensional coherent structures. Further, self-focusing of these structures was observed. At even later stages of thorn development, vortices were formed and separated. Theoretical arguments and numerical modeling performed in Refs [72, 73] qualitatively explain the entire sequence of experimental observations, up to the formation of vortices, where Eqn (87) loses applicability. It is worth noting that a more elaborate analysis [70, 71], based on one-dimensional model (87), i.e., performed in the framework of the one-dimensional Benjamin–Ono equation, showed a rather good agreement between the theory and experiment.

A question that naturally arises is why instability and self-focusing in such prominent phenomena as one-dimensional solitons have not been observed in other experiments. In our opinion, the answer to this question is related to the character of collapse in boundary layers — the collapse is strong, i.e., its appearance requires a *finite* pulse energy. If the pulse amplitude is insufficient, the phenomenon does not occur.

6.5 Weak collapse

In three dimensions ($D = 3$), virial relation (84) for the NLS equation takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int r^2|\psi|^2\,\mathrm{d}\mathbf{r} = 2H - \frac{1}{2}\,I_2\,. \tag{92}$$

Accordingly, because I_2 is positive, equality (92) can be replaced by the inequality

$$N\langle r^2\rangle < Ht^2 + C_1 t + C_2\,, \tag{93}$$

where C_1 and C_2 are integration constants determined from the initial conditions. Relation (93) provides the same collapse criterion as for $D = 2$: $H < 0$ [17].

However, the criterion $H < 0$ for the NLS equation in three dimensions is rather rough. According to Refs [81, 82], this criterion can be improved. A more precise criterion is given by two conditions:

$$H < H_{\mathrm{s}}\,, \qquad I_1 > I_{1\mathrm{s}}\,. \tag{94}$$

Criterion (94) indicates once again that the soliton corresponds to a saddle point. When the system passes through this saddle point, collapse becomes possible.

A more precise inequality, which follows from virial relation (92) for $D = 3$, has the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int r^2|\psi|^2\,\mathrm{d}\mathbf{r} \leqslant 3(H - H_{\mathrm{s}})\,. \tag{95}$$

Integration of (95) gives

$$N\langle r^2\rangle \leqslant \frac{3}{2}\,(H - H_{\mathrm{s}})t^2 + C_1 t + C_2\,,$$

where H_{s} is the value of H in the ground-state (without zeros) soliton solution.

We next consider a self-similar substitution for Eqn (1),

$$\psi = \frac{1}{(t_0 - t)^{1/2+\mathrm{i}\alpha}}\,\chi(\xi)\,. \tag{96}$$

Here, the self-similarity variable $\xi = r(t_0 - t)^{-1/2}$ corresponds to Eqn (83) and the function $\chi(\xi)$ is assumed to be spherically symmetric. Inserting ψ into the NLS equation results in the equation for the function χ [39]

$$\mathrm{i}\left[\left(\frac{1}{2} + \mathrm{i}\alpha\right)\chi + \frac{1}{2}\,\xi\chi_{\xi}\right] + \frac{1}{2}\,\chi_{\xi\xi} + \frac{1}{\xi}\,\chi_{\xi} + |\chi|^2\chi = 0\,, \tag{97}$$

where α plays the role of a spectral parameter. We are interested only in regular solutions of Eqn (97) that decay at infinity. Obviously, as $\xi \to \infty$, χ satisfies the linear equation

$$\left(\frac{1}{2} + \mathrm{i}\alpha\right)\chi + \frac{1}{2}\,\xi\chi_{\xi} = 0\,. \tag{98}$$

It defines the asymptotic form of χ:

$$\chi \to \frac{C}{\xi^{\,1+2\mathrm{i}\alpha}}\,, \tag{99}$$

where C is some constant, which can be taken real and, moreover, positive without loss of generality. The requirement that the solution be regular eliminates the ambiguity in the choice of α and C. In fact, we are dealing with a nonlinear spectral problem for Eqn (97), whose numerical solution gives

$$\alpha = 0.545\,, \qquad C = 1.01\,. \tag{100}$$

We now discuss the characteristics of the self-similar solution. First, for any fixed point of physical space with a coordinate r, the corresponding self-similar coordinate ξ tends to infinity as $t \to t_0$. The self-similar solution in this case passes to its asymptotic form (99), acquiring a singularity

$$\psi \to \frac{C}{r^{\,1+2\mathrm{i}\alpha}} \tag{101}$$

in the physical variable r, which is independent of time t.

Second, the self-similar solution cannot exist simultaneously in the entire space: it can be realized only in some domain with coordinates $r < r_0$, where the size r_0 must be constrained by the quantity $N/(4\pi C^2)$.

Singularity (101), which 'sprouts' in the center of this domain as $r \to 0$, is integrable. At a first glance, self-similar ansatz (96) results in the nonconservation of the integral

$$N = \int|\psi|^2\,\mathrm{d}\mathbf{r} = (t_0 - t)^{1/2}\int|\chi_{\xi}|^2\,\mathrm{d}\xi \tag{102}$$

because of the appearance of the factor $(t_0 - t)^{1/2}$. On the other hand, the conservation of N implies that integral (102) must be infinite. This is the case when the self-similar solution is considered in the entire space. But in any finite domain $r < r_0$, the integral N remains finite. Indeed, upon substituting (96) in Eqn (102), for $r < r_0$, we obtain

$$N = (t_0 - t)^{1/2}4\pi\int_0^{\xi_*}\xi^2\big|\chi(\xi)\big|^2\,\mathrm{d}\xi\,, \qquad \xi_* = r_0(t_0 - t)^{-1/2}\,. \tag{103}$$

With the asymptotic form in (99), the integral in Eqn (103) increases linearly as the upper limit increases as $t \to t_0$. We suppose that the size r_0 is sufficiently large. Then the value of the integral N in the domain $r < r_0$ must approach its value at the instant of collapse $t = t_0$. In other words, the relation

$$\int_0^{\infty}\left(|\chi|^2 - \frac{C^2}{\xi^2}\right)\xi^2\,\mathrm{d}\xi = 0 \tag{104}$$

must hold. It was verified numerically that relation (104) holds with high accuracy for $\chi(\xi)$ and C found by integrating Eqn (97).

The solution constructed here corresponds to a *weak collapse*. In this state, speaking formally, zero energy is trapped in the singularity at $r = 0$ [39]. In fact, this implies that if ψ_0 is a characteristic amplitude at which the absorption of energy occurs [Eqn (1) is then not valid], then the amount of energy lost to dissipation in a single act is of the order of

$$\Delta N \sim \psi_0^2 r_0^3 \sim \frac{1}{\psi_0}\,,$$

where $r_0 \sim 1/\psi_0$ is the characteristic scale of the dissipation domain.

The self-similar solution related to the weak collapse mode describes the most rapid collapse in which radiation plays the primary role; in contrast to the semiclassical collapse mode (82), it essentially accelerates the process of singularity formation; as a result, only a small fraction of the energy reaches the singularity.

We note that the weak collapse mode was first simulated in numerical experiments in [83] and then in [84].

6.6 Black-hole regime

As we have seen in Section 6.5, the time-independent singularity in (101) forms in the regime of weak collapse, which corresponds to the attracting potential

$$U = -\frac{C^2}{r^2}\,, \tag{105}$$

with a constant $C^2 > 1$. As is known from quantum mechanics (see, e.g., Ref. [85]), a particle can fall on the center in this case. Moreover, as the center $r = 0$ is approached, such a fall in quantum mechanics becomes closer and closer to the semiclassical one.

For all waves surrounding the singularity, potential (105) serves as an attractor, i.e., the singularity plays the role of a funnel for these waves. Because the energy dissipating in the singularity for weak collapse is infinitely small in physical terms, the process of pulling particles into the singularity from its periphery — the postcollapse — can be considered quasistationary. The postcollapse is characterized by a finite flux of the number of particles into the singularity. As was first shown in Refs [86, 87] (see also Refs [88–90]), the density $|\psi|^2$ behaves in the vicinity of the singularity in this mode as

$$|\psi|^2 = \frac{1}{2r^2|\log r|}\,.$$

The incessant flow into the funnel, mediated by waves trapped in the singularity from the periphery, ensures the existence of long-lived 'burning' points — locations where all the energy supplied from outside is 'burnt'. Such a regime is also called the *black-hole regime*.

7. The role of dispersion in collapse

Dispersion affects collapse quite significantly. In stationary self-focusing light, wave dispersion is unimportant: the appearance of a singularity — the focus — stems from the concurring nonlinearity from the Kerr effect and diffraction. In this case, the two-dimensional NLS equation is applicable, with the role of time played by the coordinate z along the beam propagation. In three dimensions, however, everything depends on the sign of dispersion, positive or negative; in optics, in particular, it depends on whether the dispersion is anomalous or normal. Three-dimensional NLS equation (1) is applicable for anomalous dispersion. In the case of normal dispersion, the Laplace operator in Eqn (1) must be replaced with the hyperbolic operator $\Delta_\perp - \partial^2/\partial z^2$, where $\Delta_\perp = \partial_{xx} + \partial_{yy}$. This is why the NLS equation of the form

$$\mathrm{i}\,\frac{\partial\psi}{\partial t} + \Delta_\perp\psi - \psi_{zz} + |\psi|^2\psi = 0 \tag{106}$$

is often called the hyperbolic nonlinear Schrödinger equation. The different sign of the second-order operator stems from the sign of the frequency derivative of the group velocity, $\partial V_{\mathrm{gr}}/\partial\omega$. The derivative $\partial V_{\mathrm{gr}}/\partial\omega$ is negative for normal dispersion.

The hyperbolic operator in Eqn (106) essentially modifies the character of nonlinear interaction. For the anomalous dispersion, all 'directions' are equivalent, but for the normal dispersion, we have repulsion instead of attraction in Eqn (1) along the z direction, whereas attraction is preserved in the lateral direction. Correspondingly, quasiparticles are attracted to each other in the lateral direction, which leads to beam narrowing in the plane perpendicular to the z axis. Put differently, in NLS equation (106), the mass of particles in the longitudinal direction is negative, and therefore the nonlinearity promotes an increase in the beam longitudinal scale. In this respect, the main question is whether the lateral beam compression is capable of producing a singularity in spite of the longitudinal beam expansion.

We demonstrate that in the framework of hyperbolic Schrödinger equation (106) in three dimensions, the collapse of a wave packet as a whole is impossible at the stage most favorable for collapse, when a pulse is compressed in all three directions. The idea of the proof is to use integral estimates for the virial equations.

Similarly to Eqn (1), Eqn (106) is Hamiltonian with

$$H = \int |\nabla_\perp\psi|^2\,\mathrm{d}\mathbf{r} - \int |\psi_z|^2\,\mathrm{d}\mathbf{r} - \int \frac{1}{2}\,|\psi|^4\,\mathrm{d}\mathbf{r} \equiv I_\perp - I_z - I_2\,. \tag{107}$$

The different signs of the first and second integrals correspond to the difference in the behavior in the lateral and longitudinal directions.

We consider the behavior of mean square sizes $\langle z^2\rangle$ and $\langle r_\perp^2\rangle$, along and transverse to the z axis. Calculations similar to those in Eqn (84) give

$$N\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle r_\perp^2\rangle = 4\left(2\int|\nabla_\perp\psi|^2\,\mathrm{d}\mathbf{r} - \int|\psi|^4\,\mathrm{d}\mathbf{r}\right), \tag{108}$$

$$N\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle z^2\rangle = 8\int|\psi_z|^2\,\mathrm{d}\mathbf{r} + 2\int|\psi|^4\,\mathrm{d}\mathbf{r}\,. \tag{109}$$

From Eqn (108) for $\langle r_\perp^2\rangle$ and $I_\perp$, just as from Eqn (109) for $\langle z^2\rangle$ and I_z, we can obtain the uncertainty relations

$$I_\perp\langle r_\perp^2\rangle \geqslant N\,, \qquad I_z\langle z^2\rangle \geqslant \frac{N}{4}\,. \tag{110}$$

Using relations (110) and the definition of H in (107), we can estimate the right-hand sides of Eqns (108) and (109):

$$N\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle r_\perp^2\rangle = 8H + 8I_z \geqslant -4H + 2\,\frac{N}{\langle z^2\rangle}\,, \tag{111}$$

$$N\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle z^2\rangle = -4H + 4I_z + 4I_\perp > -4H + 4\,\frac{N}{\langle r_\perp^2\rangle}\,. \tag{112}$$

We now consider the contraction mode in all directions, the most preferential from the standpoint of collapse, when

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle r_\perp^2\rangle < 0\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle z^2\rangle < 0\,,$$

and demonstrate that the collapse, understood as the contraction to zero of the mean square transverse and longitudinal sizes ($\langle r_\perp^2\rangle \to 0$, $\langle z^2\rangle \to 0$), is impossible in this case.

First, we prove that the mean square longitudinal scale z^2 of the wave packet cannot vanish if $\mathrm{d}\langle z^2\rangle/\mathrm{d}t < 0$. From Eqn (109), also using Eqn (110), we derive a closed-form inequality for $\langle z^2\rangle$:

$$N\,\frac{\mathrm{d}^2\langle z^2\rangle}{\mathrm{d}t^2} \geqslant 8I_z \geqslant 2\,\frac{N}{\langle z^2\rangle}\,. \tag{113}$$

Formula (113) is a second-order differential inequality. Its first integration, analogous to finding the energy integral in mechanics, with $\mathrm{d}\langle z^2\rangle/\mathrm{d}t < 0$, gives the relation

$$\mathcal{E}(t) = \frac{1}{2}\left(\frac{\mathrm{d}\langle z^2\rangle}{\mathrm{d}t}\right)^2 - 2\log\langle z^2\rangle \leqslant \mathcal{E}(0)\,, \tag{114}$$

where $\mathcal{E}(0)$ is the initial value of $\mathcal{E}(t)$. As $\langle z^2\rangle \to 0$, the left-hand side of (114) tends to infinity because of the logarithmic term, which would violate the inequality. Hence, shrinking $\langle z^2\rangle \to 0$ is impossible.

We now show that the collapse in the transverse direction (when $\langle r_\perp^2\rangle \to 0$) is also impossible. For this, we multiply inequality (111) by $\mathrm{d}\langle z^2\rangle/\mathrm{d}t < 0$, and inequality (112) by $\mathrm{d}\langle r_\perp^2\rangle/\mathrm{d}t < 0$, add the results, and integrate the resulting equation over time from zero to t. This yields

$$E(t) = N\,\frac{\mathrm{d}\langle r_\perp^2\rangle}{\mathrm{d}t}\,\frac{\mathrm{d}\langle z^2\rangle}{\mathrm{d}t} - 8H\langle z^2\rangle + 4H\langle r_\perp^2\rangle - 2N\log\langle z^2\rangle - 4N\log\langle r_\perp^2\rangle \leqslant E(0)\,, \tag{115}$$

where $E(0)$ is the value of $E(t)$ at the initial instant. From inequality (115), it immediately follows that collapse is impossible at the stage of uniform contraction because the first term in the right-hand side of (115) is positive by definition, the terms proportional to H are finite, and the logarithmic term becomes infinitely large as $\langle r_\perp^2\rangle \to 0$. This does not agree with the fact that the function $E(t)$ is bounded from above by the initial value $E(0)$. Hence, the collapse of a three-dimensional wave package as a whole is impossible at the most 'dangerous' stage of shrinking in all directions [91] (see also Ref. [92]).

Does this imply that the collapse in such a system is completely impossible? Strictly speaking, no, because, first, the Vlasov–Petrishchev–Talanov criterion is only a sufficient one and, second, the collapse, if it is possible, should be sought among states that correspond to the longitudinal dispersion of the gas of quasiparticles, which leads to an increase in the longitudinal scale. On the other hand, longitudinal stretching results in a decrease in the particle line density. For the two-dimensional Schrödinger equation, as we have seen, collapse is prohibited for $N < N_{\mathrm{cr}}$. It is worth noting that in spite of keen interest in this problem (see, e.g., Refs [93–95]), it is still awaiting its full solution.

8. Collapse in Kadomtsev–Petviashvili equations

Finally, we discuss the question of collapse in Kadomtsev–Petviashvili (KP) systems. We begin with the classic KP equation with positive dispersion

$$\frac{\partial}{\partial x}(u_t + u_{xxx} + 6uu_x) = 3\Delta_\perp u\,, \tag{116}$$

which describes the propagation of a beam of weakly nonlinear acoustic waves in a medium with weak dispersion. The second term in the left-hand side of Eqn (116) is responsible for dispersion effects, the third term describes the effects of sound wave steepening, and the term in the right-hand side describes the diffraction ($\Delta_\perp = \partial_y^2 + \partial_z^2$). Equation (116) is written in the coordinate frame moving at the speed of sound c_{s} along the x axis. All the terms responsible for nonlinearity, dispersion, and diffraction are small compared to the 'fast' propagation at the speed of sound, and therefore the KP equation has an additional integral, the adiabatic invariant $P = (1/2)\int u^2\,\mathrm{d}\mathbf{r}$, which coincides with the x-projection of the momentum. It is also important that P is the total energy up to a constant factor.

It is pertinent to say a few words about sound waves with positive dispersion. For example, they include fast magnetoacoustic waves in plasmas with a small parameter β (the ratio of the thermal plasma pressure to the magnetic field pressure), $\beta \ll 1$, for propagation directions that are not close to the longitudinal and lateral ones (with respect to the magnetic field). One more example of positive dispersion is offered, under certain conditions, by sound waves in liquid helium [96]. Finally, we mention surface waves in shallow water when the (positive) dispersion due to capillarity exceeds that due to a finite depth (see, e.g., Ref. [97]).

Equation (116), which is Hamiltonian, can be written in the form

$$u_t = \frac{\partial}{\partial x}\,\frac{\delta H}{\delta u} \tag{117}$$

with the Hamiltonian

$$H = \frac{1}{2}\int u_x^2\,\mathrm{d}\mathbf{r} + \frac{3}{2}\int(\nabla_\perp w)^2\,\mathrm{d}\mathbf{r} - \int u^3\,\mathrm{d}\mathbf{r} \equiv \frac{I_1}{2} + \frac{3I_2}{2} - I_3\,. \tag{118}$$

Here, w plays the role of the hydrodynamic potential, $u = w_x$.

Solutions in the form of a multi-dimensional soliton propagating along the x axis, $u = u_{\mathrm{s}}(x - Vt, r_\perp)$, are found by integrating the equation

$$\frac{\partial^2}{\partial x^2}(-Vu + u_{xx} + 3u^2) = 3\Delta_\perp u\,. \tag{119}$$

The linear operator in Eqn (119),

$$\frac{\partial^2}{\partial x^2}\left(-V + \frac{\partial^2}{\partial x^2}\right) - 3\Delta_\perp\,,$$

is positive definite if the soliton velocity is positive, $V > 0$. As was first noted by Petviashvili [98], this requirement is in fact the existence condition for soliton solutions of the KP equation. Physically, this condition implies the absence of a Cherenkov resonance between the moving soliton and small-amplitude waves [50, 99].

In one dimension, the solution of Eqn (119) is the KdV soliton

$$u_{\mathrm{s}} = \frac{2\kappa^2}{\cosh^2\kappa(x - 4\kappa^2 t)}\,,$$

and in two dimensions, it is the lump solution [100]

$$u_{\mathrm{s}} = 4V'\,\frac{1 + 4V'^2y^2 - V'(x - Vt)^2}{1 + 4V'^2y^2 + V'(x - Vt)^2}\,.$$

In three dimensions, solutions of Eqn (119) can only be found numerically.

The soliton solution of Eqn (119) corresponds to a stationary point of Hamiltonian (118) with the momentum fixed:

$$\delta(H + VP) = 0\,.$$

Using the explicit expression for the Hamiltonian, Eqn (118), together with integral estimates [77] (see also Refs [3, 51]), it is possible to show that in one and two dimensions, the Hamiltonian for the KP equation is a functional bounded from below if the momentum P is fixed. These soliton solutions, however, turn out to be unstable under three-dimensional perturbations, which was discovered for the first time for one-dimensional solitons in the long-wave limit by Kadomtsev and Petviashvili [74]. Later, with the help of the inverse scattering method, Zakharov found the exact expression for the growth rate of this instability [76]. The cause of the instability is that the speed of a soliton in physical variables decreases as its amplitude increases. As a consequence, if such a soliton is modulated in the lateral direction, the regions with larger amplitude take over those with a smaller amplitude. This leads to a focusing-type instability [75].

For $D = 3$, Hamiltonian (118) is unbounded from below, which, notably, follows by applying the scaling transformations [57, 101]

$$u \to \alpha^{-1/2}\beta^{-1}u\left(\frac{x}{\alpha}\,,\,\frac{\mathbf{r}_\perp}{\beta}\right),$$

which preserve P, to Hamiltonian (118),

$$H(\alpha, \beta) = \frac{I_1}{2\alpha^2} + \frac{3I_2\alpha^2}{2\beta^2} - \frac{I_3}{\alpha^{1/2}\beta}\,, \tag{120}$$

which, as a function of α and β, has a fixed point at $\alpha = \beta = 1$, corresponding to a three-dimensional soliton. It is clear that at the fixed point, the conditions

$$\frac{\partial H}{\partial \alpha} = \frac{\partial H}{\partial \beta} = 0\,,$$

are satisfied; they are equivalent to the following relations between the integrals:

$$I_1 = \frac{3I_3}{2}\,, \qquad I_2 = \frac{I_3}{3}\,.$$

Equipped with these relations, it is easy to see that this fixed point is a saddle. This fact indicates a possible instability of the three-dimensional soliton under small perturbations. As regards the instability under finite perturbations, it follows from the unboundedness of the Hamiltonian as $\alpha, \beta \to 0$. Indeed, if we consider parabolas $\beta \propto \alpha^2$, the first two terms in H behave as α^{-2}, while the cubic term is proportional to $\alpha^{-5/2}$. This indicates that H is unbounded (from below) at small scales owing to the nonlinear term. The unboundedness of H, as we have seen in other examples, is a criterion for the collapse to occur. Under scaling transformation along the parabolas $\beta \propto \alpha^2$, the exponents in the dispersion term (equal to -2) and in the cubic term $(-5/2)$ do not coincide, and hence the possible collapse is not critical and should be weak.

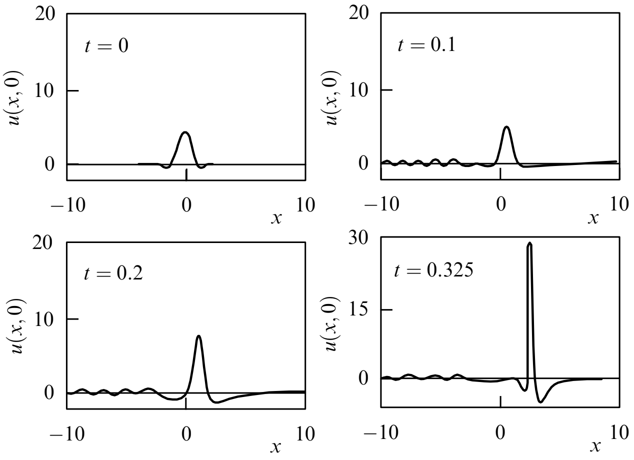

Figure 4. The dependence of u on the axis $r_\perp = 0$ for four moments of time.

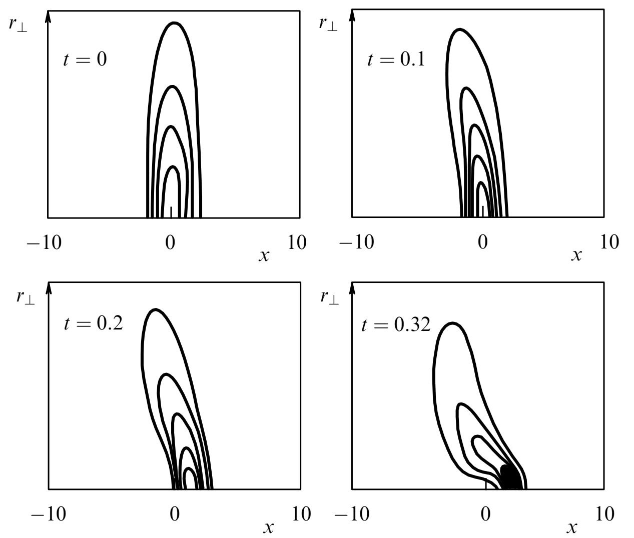

Figure 5. Isolines of $u(x, r_\perp)$ for the same time instants as in Fig. 4.

The self-similar substitution

$$u = \tau^{-2/3}u\left(\frac{x}{\tau^{1/3}}\,,\,\frac{\mathbf{r}_\perp}{\tau^{2/3}}\right)$$

corresponds to this collapse regime, with $\tau = t_0 - t$ and t_0 the collapse time.

Numerical integration of three-dimensional KP equation (116) for cylindrically symmetric distributions [57, 101] has shown that a singularity forms for initial conditions with a negative value of the Hamiltonian.

Figure 4 shows the profile of u on the axis $r_\perp = 0$ at four time instants. The spatial distribution of $u(x, r_\perp)$ for these same instants is given in Fig. 5. The evolving distribution has a horseshoe shape, which corresponds to the linear stage of the KP instability. Numerical experiments have corroborated that this collapse is a weak one. Zero boundary conditions were chosen for $u(x, r_\perp)$ in numerical experiments; because of the KP equation nonlocality, they did not ensure that the Hamiltonian is preserved, since radiation from the integration domain is possible. Figure 6 plots the Hamiltonian H, the momentum P_x computed over the integration domain, and the maximum amplitude as functions of time. As can be seen,

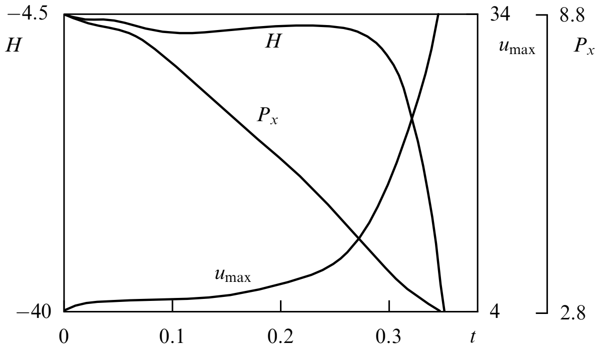

Figure 6. The dependences of the Hamiltonian H, the momentum P_x, and the maximum amplitude $u_{\max}$ on time.

a decrease of P_x and $|H|$ with time is accompanied by a simultaneous increase in $\max_{\mathbf{r}} u$, which points precisely to the weak nature of the collapse.

The question of the sufficient criterion of collapse and, accordingly, the finiteness of the singularity formation time for three-dimensional KP equation (116) is presently open, although the possibility of a collapse is indicated by the fact that the Hamiltonian H is unbounded from below, if the momentum $P = (1/2)\int u^2\,\mathrm{d}\mathbf{r}$ is fixed, and also by the above results of numerical simulations of the collapse.

8.1 Virial inequalities

We demonstrate how the Vlasov–Petrishchev–Talanov criterion can be used in studies of collapses for the generalized three-dimensional KP equation

$$\frac{\partial}{\partial x}\left(u_t + u_{xxx} + n(n-1)u^{n-2}u_x\right) = 3\Delta_\perp u\,. \tag{121}$$

The case $n = 3$ in Eqn (121) corresponds to the classic KP equation (116). The KP equation with $n = 4$ emerges in the description of acoustic waves in antiferromagnets for certain propagation angles [102].

We begin from Eqn (116) assuming $n = 4$. We consider the quantity

$$\langle r_\perp^2\rangle = \frac{1}{2P}\int r_\perp^2 u^2\,\mathrm{d}\mathbf{r}\,,$$

which, by virtue of the conservation of $P = \int u^2\,\mathrm{d}\mathbf{r}$, can be interpreted as the mean beam cross section. As was first shown in Ref. [102], an analog of virial theorem (92) can be formulated for $\langle r_\perp^2\rangle$ as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int r_\perp^2 u^2\,\mathrm{d}\mathbf{r} = 48H - 8\int u_x^2\,\mathrm{d}\mathbf{r}\,, \tag{122}$$

where H is a conserved quantity, the Hamiltonian, written for $n = 4$ as

$$H = \int\left(\frac{u_x^2}{2} + 3\,\frac{\nabla_\perp w^2}{2} - u^4\right)\mathrm{d}\mathbf{r}\,.$$

This immediately leads to the estimate

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int r_\perp^2 u^2\,\mathrm{d}\mathbf{r} < 48H\,. \tag{123}$$

Hence, for $H < 0$, the beam collapses [102], formally shrinking to $\langle r_\perp^2\rangle = 0$. It can be shown that the collapse criterion $H < 0$ pertains to all integer values $n > 4$, but, unfortunately, the case $n = 3$, which corresponds to the classic KP equation with quadratic nonlinearity, escapes this proof.

We dwell on one more example involving the generalized KP equation. This is the model proposed in Ref. [103] for the description of self-focusing of ultrashort pulses, which does not rely on averaging over the frequency of a quasimonochromatic wave. The pulse spectrum is assumed to be broad and to lie in the nonresonance domain. However, the nearest resonance (the transition frequency ω_0) and the plasma resonance (at the plasma frequency ω_{p}) are incorporated in this model. An interesting feature is that the dispersion dependence contains intervals of both anomalous and normal dispersion. Other limitations involve the Kerr nonlinearity and small-angle approximation, which is characteristic of the KP equation,

$$\frac{2n_0}{c}\,\frac{\partial^2 E}{\partial z\,\partial\tau} + \frac{4\pi\delta\chi^3}{(c\omega_0)^2}\,\frac{\partial^2 E^3}{\partial\tau^2} - \frac{4\pi\chi}{(c\omega_0)^2}\,\frac{\partial^4 E}{\partial\tau^4} + \frac{\omega_{\mathrm{p}}^2}{c^2}\,E = \Delta_\perp E\,. \tag{124}$$

Equation (124) is written for the electric field amplitude E in the case of a linearly polarized wave. Here, $\varepsilon_0 = n_0^2 = 1 + 4\pi\chi$ is the static dielectric permittivity, c is the speed of light, $\tau = t - zn_0/c$ is the time in the comoving frame of reference, and $4\pi\delta\chi^3\omega_0^{-2}$ is the Kerr constant, which is assumed positive. Compared with Eqn (5) in Ref. [103], Eqn (124) neglects linear wave damping. Using dimensionless variables, Eqn (124) can be rewritten in the form

$$\frac{\partial}{\partial\tau}\left(\frac{\partial u}{\partial z} + 3u^3\,\frac{\partial u}{\partial\tau} - b\,\frac{\partial^3 u}{\partial\tau^3}\right) + au = \Delta_\perp u\,, \tag{125}$$

with positive constants a and b. Equation (125), which is the generalized KP equation, is Hamiltonian, like the original KP equation (116),

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial\tau}\,\frac{\delta H}{\delta u}\,,$$

with the Hamiltonian

$$H = \frac{1}{2}\int\left(-bu_\tau^2 + aw^2 + (\nabla_\perp w)^2 - \frac{1}{2}\,u^4\right)\mathrm{d}\tau\,\mathrm{d}\mathbf{r}_\perp\,,$$

where $u = w_\tau$, i.e., w plays the role of the hydrodynamic potential. The expression for H indicates that its quadratic part is not necessarily positive definite in the general case and becomes so only in the low-frequency domain, where the first term bu_τ^2 can be neglected as being small compared with the second one. It is also obvious that the quadratic form is positive definite for $b = 0$. Just in these cases a sufficient criterion for the occurrence of collapse can be proposed, which is analogous to (123):

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\int r_\perp^2 u^2\,\mathrm{d}\tau\,\mathrm{d}\mathbf{r}_\perp = 4H - 8\int aw^2\,\mathrm{d}\tau\,\mathrm{d}\mathbf{r}_\perp \leqslant 4H\,.$$

Integration of the last inequality results in a sufficient condition for the occurrence of collapse: $H < 0$ [103]. Numerical integration of Eqn (125) demonstrated the tendency to beam self-focusing in the transverse direction. A rather interesting effect — the buildup of sharp gradients on the beam axis prior to self-focusing — was also discovered in [103].

9. Conclusions

We have presented two methods for exploring the stability of solitons, one linked to the Vakhitov–Kolokolov criterion and its generalizations to vector systems of the NLS type, and the other based on the analysis of stability in the Lyapunov sense. The combination of these two methods furnishes an effective approach to exploring soliton stability.

Another key element of the present review is the application of embedding theorems, which play an essential role in proving the soliton stability. In particular, the use of such an approach helped establish soliton stability for the three-wave system describing the coupling of electromagnetic waves in χ^2-media and for three-dimensional solitons of magnetized ion-acoustic waves described in terms of the anisotropic KdV equation [14].

We note that by the χ^2-media, we first and foremost mean crystals without a center of symmetry; only in that case does the three-wave matrix element differ from zero. Electromagnetic waves propagating in such crystals have anisotropic dispersion laws, which, notably, implies that the tensors $\omega_{\alpha\beta}$ cannot be simultaneously reduced to a diagonal form for every wave package in general. However, the method presented in this review requires no diagonalization of the dispersion tensors $\omega_{\alpha\beta}$. The only essential condition is that they have the same sign definiteness. Only in this case do solitons exist. The sign definiteness of the dispersion tensors allows introducing the appropriate Sobolev spaces and obtaining necessary integral estimates for the Hamiltonians. It is also important that solitons realizing the minimum of the Hamiltonian are stable with respect to not only small but also finite perturbations. In this sense, the Lyapunov stability criterion is equivalent to the energy principle.

When the Hamiltonian is an unbounded functional, wave collapse leading to a singularity stands out as the most probable scenario for nonlinear system behavior. Collapse in this case is similar to the process of particle fall in a self-consistent potential (see, e.g., Ref. [39]).

As the space dimension increases, the role of nonlinear effects also increases. Stable solitons are therefore observed at low dimensions, whereas collapses are typical for higher-dimensional systems. In collapsing systems, solitons represent a separatrix separating collapsing and noncollapsing distributions. Such (unstable) solitons define the threshold of wave collapse.

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References

1. Arnold V I *Teoriya Katastrof* (Catastrophe Theory) (Novoe v Zhizni, Nauke, Tekhnike. Ser. Matematika, Kibernetika, No. 9) (Moscow: Znanie, 1981) [Translated into English (Berlin: Springer, 1986)]
2. Arnold V I *Matematicheskie Metody Klassicheckoi Mekhaniki* (Mathematical Methods of Classical Mechanics) 3rd ed. (Moscow: Nauka, 1989) [Translated into English (New York: Springer, 1997)]
3. Kuznetsov E A, Rubenchik A M, Zakharov V E *Phys. Rep.* **142** 103 (1986)
4. Kuznetsov E A, Zakharov V E *Lecture Notes Phys.* **542** 3 (2000)
5. Hasegawa A, Tappet F *Appl. Phys. Lett.* **23** 142 (1973)
6. Agrawal G P *Nonlinear Fiber Optics* (New York: Academic Press, 1989) [Translated into Russian (Moscow: Mir, 1996)]
7. Bloembergen N *Nonlinear Optics* (Reading, Mass.: W.A. Benjamin, 1977)
8. Zakharov V E, Manakov S V *Zh. Eksp. Teor. Fiz.* **69** 1654 (1975) [*Sov. Phys. JETP* **42** 842 (1975)]
9. Karamzin Yu N, Sukhorukov A P *Zh. Eksp. Teor. Fiz.* **68** 834 (1975) [*Sov. Phys. JETP* **41** 414 (1975)]
10. Kanashev A A, Rubenchik A M *Physica D* **4** 122 (1981)
11. Zakharov V E, Kuznetsov E A *Usp. Fiz. Nauk* **167** 1137 (1997) [*Phys. Usp.* **40** 1087 (1997)]
12. Derrick G *J. Math. Phys.* **5** 1252 (1966)
13. Benjamin T B *Proc. R. Soc. London A* **328** 153 (1972)
14. Zakharov V E, Kuznetsov E A *Zh. Eksp. Teor. Fiz.* **66** 594 (1974) [*Sov. Phys. JETP* **39** 285 (1974)]
15. Krylov S F, Yan'kov V V *Zh. Eksp. Teor. Fiz.* **79** 82 (1980) [*Sov. Phys. JETP* **52** 41 (1980)]
16. D'yachenko A I et al. *Zh. Eksp. Teor. Fiz.* **96** 2026 (1989) [*Sov. Phys. JETP* **69** 1144 (1989)]
17. Zakharov V E *Zh. Eksp. Teor. Fiz.* **62** 1745 (1972) [*Sov. Phys. JETP* **35** 908 (1972)]
18. Vlasov S N, Petrishchev V A, Talanov V I *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **14** 1353 (1971) [*Radiophys. Quantum Electron.* **14** 1062 (1971)]
19. Vakhitov N G, Kolokolov A A *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **16** 1020 (1973) [*Radiophys. Quantum Electron.* **16** 783 (1973)]
20. Askar'yan G A *Zh. Eksp. Teor. Fiz.* **42** 1567 (1962) [*Sov. Phys. JETP* **15** 1088 (1962)]
21. Talanov V I *Pis'ma Zh. Eksp. Teor. Fiz.* **2** 218 (1965) [*JETP Lett.* **2** 138 (1965)]
22. Talanov V I *Pis'ma Zh. Eksp. Teor. Fiz.* **11** 303 (1970) [*JETP Lett.* **11** 199 (1970)]
23. Chiao R Y, Garmire F, Townes C H *Phys. Rev. Lett.* **13** 479 (1964)
24. Dawes E L, Marburger J H *Phys. Rev.* **179** 862 (1969)
25. Litvak A G *Zh. Eksp. Teor. Fiz.* **57** 629 (1969) [*Sov. Phys. JETP* **30** 344 (1969)]
26. Vlasov S N, Talanov V I *Samofokusirovka Voln* (Wave Self-Focusing) (N. Novgorod: IPF RAN, 1997)
27. Zakharov V E, in *Basic Plasma Physics* (Handbook of Plasma Physics, Vol. 2, Eds A A Galeev, R N Sudan) (Amsterdam: North-Holland, 1984) p. 81
28. Rasmussen J J, Rypdal K *Phys. Scripta* **33** 481 (1986)
29. Kuznetsov E A *Chaos* **6** 381 (1996)
30. Sulem C, Sulem P-L *The Nonlinear Schrödinger Equation* (New York: Springer, 1999)
31. Gross E P *Nuovo Cimento* **20** 454 (1961)
32. Pitaevskii L P *Zh. Eksp. Teor. Fiz.* **40** 646 (1961) [*Sov. Phys. JETP* **13** 451 (1961)]
33. Kagan Yu M *Usp. Fiz. Nauk* **178** 633 (2008) [*Phys. Usp.* **51** 603 (2008)]
34. Kagan Yu, Svistunov B V *Phys. Rev. Lett.* **79** 3331 (1997)
35. Kagan Yu M, Svistunov B V *Pis'ma Zh. Eksp. Teor. Fiz.* **67** 495 (1998) [*JETP Lett.* **67** 521 (1998)]
36. Kagan Yu M, Svistunov B V *Zh. Eksp. Teor. Fiz.* **105** 353 (1994) [*JETP* **78** 187 (1994)]
37. Flambaum V, Kuznetsov E, in *Singularities in Fluids, Optics, and Plasmas* (Eds R E Caflisch, G C Papanicolaou) (Dordrecht: Kluwer Acad., 1993) p. 197
38. Berge L, Rasmussen J J *Phys. Lett. A* **304** 136 (2002)

39. Zakharov V E, Kuznetsov E A *Zh. Eksp. Teor. Fiz.* **91** 1310 (1986) [*Sov. Phys. JETP* **64** 773 (1986)]
40. Zakharov V E, Shabat A B *Zh. Eksp. Teor. Fiz.* **61** 118 (1971) [*Sov. Phys. JETP* **34** 62 (1972)]
41. Kivshar Yu S, Agrawal G P *Optical Solitons: From Fibers to Photonic Crystals* (Amsterdam: Academic Press, 2003) [Translated into Russian (Moscow: Fizmatlit, 2005)]
42. Segev M, Stegeman G *Phys. Today* **51** (8) 42 (1998)
43. Stegeman G, Segev M *Science* **286** 1518 (1998)
44. Kivshar Yu S *Opt. Quantum Electron.* **30** 571 (1998)
45. Ladyzhenskaya O A *Matematicheskie Voprosy Dinamiki Neszhimaemoi Vyazkoi Zhidkosti* (The Mathematical Theory of Viscous Incompressible Flow) (Moscow: Nauka, 1970) [Translated into English (New York: Gordon and Breach, 1969)]
46. Nirenberg L *Ann. Scuola Norm. Sup. Pisa* **20** 733 (1966)
47. Weinstein M I *Commun. Math. Phys.* **87** 567 (1983)
48. Gardner C S et al. *Phys. Rev. Lett.* **19** 1095 (1967)
49. Berge L, Kuznetsov E A, Rasmussen J J *Phys. Rev. E* **53** R1340 (1996)
50. Zakharov V E, Kuznetsov E A *Zh. Eksp. Teor. Fiz.* **113** 1892 (1998) [*JETP* **86** 1035 (1998)]
51. Kuznetsov E A, Dias F *Phys. Rep.* **507** 43 (2011)
52. Turitsyn S K *Pis'ma Zh. Eksp. Teor. Fiz.* **61** 458 (1995) [*JETP Lett.* **61** 469 (1995)]
53. Berge L et al. *Phys. Rev. E* **55** 3555 (1997)
54. Kuznetsov E A, in *Optical Solitons: Theoretical Challenges and Industrial Perspectives* (Eds V E Zakharov, S Wabnitz) (Berlin: Springer-Verlag, 1999) p. 31
55. Kuznetsov E A, in *Trudy Nauchnoi Shkoly "Nelineinye Volny — 2010"* (Proceedings of Scientific School "Nonlinear Waves — 2010") (Ed. A V Gaponov-Grekhov, V I Nekorkin) (N. Novgorod: IPF RAN, 2011) p. 205
56. Grimshaw R H J, Kuznetsov E A, Shapiro E G *Physica D* **152 – 153** 325 (2001)
57. Kuznetsov E A, Musher S L *Zh. Eksp. Teor. Fiz.* **91** 1605 (1986) [*Sov. Phys. JETP* **64** 947 (1986)]
58. Kuznetsov E A, Mikhailov A V, Shimokhin I A *Physica D* **87** 201 (1995)
59. Anderson D, Bonnedal M, Lisak M *Phys. Fluids* **22** 1838 (1979)
60. Anderson D, Lisak M, Berntson A *PRAMANA* **57** 917 (2001)
61. Vlasov S N, Talanov V I, Private Communication (1984)
62. Kuznetsov E A, Turitsyn S K *Phys. Lett. A* **112** 273 (1985)
63. Fraiman G M *Zh. Eksp. Teor. Fiz.* **88** 390 (1985) [*Sov. Phys. JETP* **61** 228 (1985)]
64. Zakharov V E, Shvets V F *Pis'ma Zh. Eksp. Teor. Fiz.* **47** 227 (1988) [*JETP Lett.* **47** 275 (1988)]
65. Landman M J et al. *Phys. Rev. A* **38** 3837 (1988)
66. Smirnov A M, Fraiman G M *Physica D* **52** 2 (1992)
67. Berge L, Pesme D *Phys. Lett. A* **166** 116 (1992)
68. Berge L *Phys. Rep.* **303** 259 (1998)
69. Shrira V I *Dokl. Akad. Nauk SSSR* **308** 732 (1989)
70. Zhuk V I, Ryzhov O S *Dokl. Akad. Nauk SSSR* **263** 56 (1982) [*Sov. Phys. Dokl.* **27** 177 (1982)]
71. Ryzhov O S *Zh. Vychisl. Mat. Mat. Fiz.* **30** 1804 (1990) [*USSR Comput. Math. Math. Phys.* **30** (6) 146 (1990)]
72. D'yachenko A I, Kuznetsov E A *Pis'ma Zh. Eksp. Teor. Fiz.* **59** 103 (1994) [*JETP Lett.* **39** 108 (1994)]
73. Dyachenko A I, Kuznetsov E A *Physica D* **87** 301 (1995)
74. Kadomtsev B B, Petviashvili V I *Dokl. Akad. Nauk SSSR* **192** 753 (1970) [*Sov. Phys. Dokl.* **15** 539 (1970)]
75. Kadomtsev B B *Kollektivnye Yavleniya v Plazme* (Cooperative Effects in Plasmas) (Moscow: Nauka, 1976) [Translated into English, in *Reviews of Plasma Physics* Vol. 22 (Ed. V D Shafranov) (New York: Kluwer Acad./Plenum Publ., 2001) p. 1]
76. Zakharov V E *Pis'ma Zh. Eksp. Teor. Fiz.* **22** 364 (1975) [*JETP Lett.* **22** 172 (1975)]
77. Kuznetsov E A, Turitsyn S K *Zh. Eksp. Teor. Fiz.* **82** 1457 (1982) [*Sov. Phys. JETP* **55** 844 (1982)]
78. Abramyan L A, Stepanyants Yu A, Shrira V I *Dokl. Akad. Nauk SSSR* **327** 460 (1992) [*Sov. Phys. Dokl.* **37** 575 (1992)]
79. Kachanov Yu S, Ryzhov O S *Sib. Fiz.-Tekh. Zh.* (1) 34 (1992)
80. Kachanov Y S, Ryzhov O S, Smith F T *J. Fluid Mech.* **251** 273 (1993)
81. Turitsyn S K *Phys. Rev. E* **47** R13 (1993)
82. Kuznetsov E A et al. *Physica D* **87** 273 (1995)
83. Budneva O B, Zakharov V E, Synakh V S *Fiz. Plazmy* **1** 606 (1975) [*Sov. J. Plasma Phys.* **1** 335 (1975)]
84. Zakharov V E, Kuznetsov E A, Musher S L *Pis'ma Zh. Eksp. Teor. Fiz.* **41** 125 (1985) [*JETP Lett.* **41** 154 (1985)]
85. Landau L D, Lifshitz E M *Kvantovaya Mekhanika. Nerelyativistskaya Teoriya* (Quantum Mechanics. Non-Relativistic Theory) (Moscow: Nauka, 1989) [Translated into English (Oxford: Pergamon Press, 1977)]
86. Vlasov S N, Piskunova L V, Talanov V I, in *IV Intern. Workshop on Nonlinear and Turbulent Processes in Physics, Kiev, October 9 – 22, 1989*
87. Zakharov V E, Kosmatov N E, Shvets V F *Pis'ma Zh. Eksp. Teor. Fiz.* **49** 431 (1989) [*JETP Lett.* **49** 492 (1989)]
88. Malkin V M, Shapiro E G *Zh. Eksp. Teor. Fiz.* **97** 183 (1990) [*Sov. Phys. JETP* **70** 102 (1990)]
89. Kosmatov N E, Shvets V F, Zakharov V E *Physica D* **52** 16 (1991)
90. Fibich G, Ren W, Wang X-P *Phys. Rev. E* **67** 056603 (2003)
91. Berge L, Kuznetsov E A, Rasmussen J J *Phys. Rev. E* **53** R1340 (1996)
92. Kuznetsov E A *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **46** 342 (2003) [*Radiophys. Quantum Electron.* **46** 307 (2003)]
93. Berge L et al. *Phys. Rev. Lett.* **89** 153902 (2002)
94. Zharova N A, Litvak A G, Mironov V A *Pis'ma Zh. Eksp. Teor. Fiz.* **75** 655 (2002) [*JETP Lett.* **75** 539 (2002)]
95. Zharova N A, Litvak A G, Mironov V A *Zh. Eksp. Teor. Fiz.* **123** 726 (2003) [*JETP* **96** 643 (2003)]
96. Gurevich V L *Kinetika Fononnykh Sistem* (Kinetics of Phonon Systems) (Moscow: Nauka, 1980)
97. Dias F, Kuznetsov E A *Phys. Lett. A* **263** 98 (1999)
98. Petviashvili V I *Fiz. Plazmy* **2** 450 (1976) [*Sov. J. Plasma Phys.* **2** 247 (1976)]
99. Nycander J *Chaos* **4** 253 (1994)
100. Bordag L A et al. *Phys. Lett. A* **63** 205 (1979)
101. Kuznetsov E A, Musher S L, Shafarenko A V *Pis'ma Zh. Eksp. Teor. Fiz.* **37** 204 (1983) [*JETP Lett.* **37** 241 (1983)]
102. Turitsyn S K, Fal'kovich G E *Zh. Eksp. Teor. Fiz.* **89** 258 (1985) [*Sov. Phys. JETP* **62** 146 (1985)]
103. Balakin A A et al. *Zh. Eksp. Teor. Fiz.* **131** 408 (2007) [*JETP* **104** 363 (2007)]