Solitons in Higher Dimensions

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(Received February 29, 1988)

We first review the recent developments of studies on solitons in higher dimensions. Next we extract two characteristics about solitons in higher dimensions : (i) these solitons are written by the special functions such as the Bessel function, (ii) existence of the transformation which connects 1+1d soliton equation and its cylindrical or spherical equation. We check that to what extent these two characteristics hold in the recently found examples of the various higher dimensional solitons.

§ 1. Introduction

In this article, we will investigate the problems of solitons in higher dimensions. Solitons in one spatial plus one time dimension (abbreviated hereafter as 1+1d) have been extensively studied in the last twenty years or so, and many interesting features of solitons have been disclosed.[1)~3)] On the contrary, in the higher dimensional case, the studies of solitons are less developed and remain as one of the interesting and challenging, present and future research subjects.

As to the higher dimensional generalizations of the 1+1d soliton systems, we notice that there are two cases. In one case, people consider the form of one-soliton which is the same as that of 1+1d system, and consider the problem of superposition of such solitons in arbitrarily different propagation directions in 2+1d or 3+1d. In the other case, people consider the one-soliton of the higher dimensional system to be different from the 1+1d soliton, for example, soliton having cylindrical symmetry or spherical symmetry. Our interests here are about the latter case.

In this article, we like to make our arguments using the examples of the concrete physical equations as much as possible. For this purpose, we pick up widely-used 1+1d model soliton equations such as the KdV equation, the Toda equation, the Boussinesq equation, the nonlinear Schrödinger (NLS) equation, the sine-Gordon equation[1)] and the Heisenberg spin equation,[4)] all of which are known to be the so-called completely integrable systems. We will study how much the higher dimensional generalizations of these widely-used 1+1d soliton systems are possible.

In § 2, for each of the above-mentioned equations, we briefly review the researches up to now including the numerical studies. Next we like to extract the common characteristics among the higher dimensional solitons.

Concerning the properties of the higher dimensional solitons, in 1983 we have proposed a conjecture that the higher dimensional (2+1d) solitons (previously called as explode-decay solitons or ripplons) are described by the special functions such as the Bessel function.[5] At that stage, this property has been derived from the three examples of the 2+1d KdV equation, the 2+1d Toda equation and the 2+1d coupled NLS equation. We will see that the above property holds also in many other higher dimensional soliton equations. In § 3, we consider the problem of the existence of transformations which connect the 1+1d soliton equation and its higher dimensional equation. As the first example of such transformations, in 1979 Hirota found that the 1+1d KdV equation and the cylindrical KdV equation are connected by the simple variable transformations.[6] Later we have found two more examples of the similar nature: the transformation which connects the 1+1d Toda equation and the cylindrical Toda equation, and the transformation which connects the 1+1d higher-order water-wave equation and its cylindrical equation. From these three examples, in 1985 we have presented the conjecture such that there exists variable transformation which connects 1+1d soliton equation and its cylindrical equation.[7] We will see whether or not such a transformation exists for other soliton equations whose higher dimensional solitons have been found recently.

§ 2. Studies of higher dimensional solitons and their relation to the special functions

As stated in the introduction, we will consider the problems using the concrete examples of the physical equations starting from the KdV equation.

2.1. *The KdV equation*

The KdV equation including the 1+1d,[8] the cylindrical[9)~12] and the spherical version[13] is written as

$$u_t+6uu_x+u_{xxx}+(d'-1)u/(2t)=0\,. \tag{2·1}$$

Here and in the following, the subscripts x, t, $\cdots$ represent partial derivatives (except the subscript n of the Toda equation which will appear later). The parameter d' represents the space dimension. The choice of the value $d'=1$, 2 and 3 corresponds to the 1+1d, the cylindrical and the spherical KdV equations respectively. At the first sight of Eq. (2·1), it is not easy to immediately understand that d' corresponds to space dimension. It happened that the space dimension is taken into account in the ordinary sense in the original system of equations. Afterwards the special perturbational approximation is adopted which makes the appearance of the dimensionality somewhat difficult to read directly in the final result of Eq. (2·1). As is well-known, the 1+1d KdV equation is completely integrable. Similarly, the cylindrical KdV equation has been clarified to be completely integrable. Historically this was the first successful example of the cylindrical generalization of the 1+1d soliton equation. The cylindrical KdV equation was first proposed by Maxon and Viecelli in 1974 together with the numerical simulation,[9] its Lax pair was found by Dryuma,[10] and the derivation of soliton solutions with the inverse scattering method was performed by Calogero and Degasperis in 1978,[11] and with the Bäcklund transformation method by the present author in 1980.[12] The one-soliton solution of the cylindrical KdV equation is written as[11),12)]

$$u=(2\log f)_{xx}\,, \tag{2·2}$$

$$f=1+\epsilon f^{(1)}\,,\quad \epsilon=\text{arbitrary constant}\,, \tag{2·3}$$

$$f^{(1)}=(12t)^{-1/3}\int_s^\infty ds' Ai^2(s')\,, \tag{2·4}$$

$$s=(x+x_1)(12t)^{-1/3}\,,\quad x_1=\text{arbitrary constant}\,. \tag{2·5}$$

The symbol Ai represents the Airy function whose differential equation and the Bessel function relation are given by

$$(d^2/dx^2)Ai(x)=xAi(x)\,,$$

$$Ai(x)=(1/3)x^{1/2}\{I_{-1/3}(z)-I_{1/3}(z)\}\,,$$

$$Ai(-x)=(1/3)x^{1/2}\{J_{1/3}(z)+J_{-1/3}(z)\}\,,\quad z\equiv(2/3)x^{3/2}\,, \tag{2·6}$$

where J_n and I_n are respectively the Bessel and the modified Bessel functions of order n.[14),15)] We note that for the 1+1d KdV equation, the one-soliton solution has the same form up to Eq. (2·3) but Eq. (2·4) becomes simply $f^{(1)}=\exp(kx-k^3t)$ with k being an arbitrary constant. It often happens that the analytical expressions for the higher dimensional solitons have more complicated form than the 1+1d solitons.

Compared with the well-clarified case of the cylindrical KdV equation, the study of the spherical KdV equation has been very limited. In 1974, Maxon and Viecelli presented the spherical KdV equation together with the numerical simulation of the propagation of the soliton-like wave.[13)] Since then no exact analytic solution to the spherical KdV equation has been reported. It still remains as an open unsolved problem.

For the higher dimensional generalizations of the KdV equation, there are another approaches. Besides the x and t derivatives of the 1+1d KdV equation, we can add y and z derivative terms as follows:

$$(u_t+6uu_x+u_{xxx})_x+3au_{yy}+3bu_{zz}=0\,. \tag{2·7}$$

Here a and b are constants. The cases $a\neq 0$, $b=0$, and $a\neq 0$, $b\neq 0$ respectively correspond to the 2+1d KdV (also called as KP[16)]) equation and the 3+1d KdV equation. At first sight, Eqs. (2·1) and (2·7) look very different. However they are directly related by the variable transformations. For simplicity, we take $a=1$, $b=1$ when a, b are nonzero. Consider the variable transformation

$$X=x+y^2/(12t)\,, \tag{2·8}$$

$$u(x,y,t)=U(X,t)\,. \tag{2·9}$$

Equation (2·9) indicates that we assume the dependence of u upon x and y to be permitted only through the particular form of X given by Eq. (2·8). Then by the direct calculation we have

$$(u_t+6uu_x+u_{xxx})_x+3u_{yy}=\{U_t+6UU_X+U_{XXX}+U/(2t)\}_X\,. \tag{2·10}$$

This indicates that the solution of the cylindrical KdV equation provides us a solution of the 2+1d KdV equation by the way of Eqs. (2·8) and (2·9). This means that the 2+1d KdV (KP) equation has not only the 1+1d type soliton solution but also the Airy function type soliton solution. The solution can be written as Eqs. (2·2)~(2·4) with s given by

$$s=(x+x_1)(12t)^{-1/3}+(y+y_1)^2(12t)^{-4/3}a^{-1}, \tag{2·11}$$

where x_1 and y_1 are arbitrary constants. Since this soliton solution has not precise cylindrical symmetry, instead of the word "cylindrical" soliton we previously called it "explode-decay" soliton or "ripplon".[17]

Similarly we can consider the variable transformation

$$X=x+(y^2+z^2)(12t)^{-1}, \tag{2·12}$$

$$u(x, y, z, t)=U(X, t). \tag{2·13}$$

Then by the direct calculation we have

$$(u_t+6uu_x+u_{xxx})_x+3u_{yy}+3u_{zz}=(U_t+6UU_X+U_{XXX}+U/t)_X. \tag{2·14}$$

We have seen that indeed the 2+1d KdV equation and the 3+1d KdV equation are related respectively to the cylindrical and the spherical KdV equation by the variable transformation.

It is known that the 2+1d KdV (KP) equation is completely integrable. However very little is known about the method to solve the 3+1d KdV equation. The extension of the 2+1d KdV (KP) equation to the 3+1d KdV equation in the form of Eq. (2·7) is physically very natural, but seems to be difficult to solve mathematically.

There are other types of the generalizations of the 2+1d KdV (KP) equation. The cylindrical KP equation was considered by Johnson in 1980 for surface waves in a fluid which are characterized by small deviation from axial symmetry and is written as[18]

$$\{u_t+6uu_x+u_{xxx}+(d'-1)u/(2t)\}_x+3au_{yy}/(2t^2)=0, \tag{2·15}$$

where a is an arbitrary constant and the value of d' is taken to $d'=2$. This equation was shown to fit into the inverse scattering method and the one-soliton written by the Airy function has been obtained by Dryuma in 1983.[19] For Eq. (2·15), so far no one has considered the value of d' other than $d'=2$. Here we take the value $d'=3$ and call it the "spherical KP" equation. We have found that the spherical KP equation thus defined also has the one-soliton solution written by the Airy function. We leave the details of the derivations to Appendix A. As shown there, the one-soliton solution of the cylindrical KP equation is given by Eqs. (2·2)~(2·4) with s given by

$$s=x(12t)^{-1/3}+cyt^{-1/3}+(3/2)(12)^{1/3}ac^2t^{-4/3}, \tag{2·16}$$

where c is an arbitrary constant. The one-soliton solution of the spherical KP equation is given by Eq. (2·2) with f given by

$$f=\int_s^\infty ds' Ai^2(s')\,,$$

$$s=x(12t)^{-1/3}-y^2t(6a)^{-1}(12t)^{-1/3}\,. \tag{2·17}$$

Before concluding this subsection, we mention about the classical Boussinesq equation or Kaup's higher-order water-wave (=HOWW) equation.[20] This equation corresponds to the generalization of the KdV equation in the following sense. The equation contains the smallness parameters in the coefficients. In the small limit of the smallness parameters together with time stretching, the equation reduces to the KdV equation. As the cylindrical generalization of this equation, we can consider the cylindrical HOWW equation which reduces to the cylindrical KdV equation in the same limiting procedure. The cylindrical HOWW equation is written as[21]

$$p_t-(pq_x-q_{xxx})_x+p/t=0\,, \tag{2·18}$$

$$p-q_t+(q_x)^2/2=0\,, \tag{2·19}$$

where the smallness parameters are taken to be unity. The HOWW equation corresponds to the above coupled equations with the last term of Eq. (2·18), p/t, dropped. We have found that the cylindrical HOWW equation has the one-soliton solution given by[21]

$$p=-2(\log ff^*)_{xx}\,,\quad q=-2i\log(f/f^*)\,, \tag{2·20}$$

$$f=B^2(s)-i(\mathrm{sgn}\,t)[s^2B^2(s)+\{dB(s)/ds\}^2]\,, \tag{2·21}$$

$$s=x/\sqrt{4|t|}\,, \tag{2·22}$$

$$d^2B(s)/ds^2+s^2B(s)=0\,, \tag{2·23}$$

$$B(s)=b_+B_+(s)+b_-B_-(s)\,,\quad B_\pm(s)=s^{1/2}J_{\pm 1/4}(s^2/2)\,,$$

$$b_+\,,\ b_-=\text{arbitrary constants}\,. \tag{2·24}$$

In Eq. (2·20), the star indicates complex conjugate.

2.2. *The Toda equation*

The Toda lattice equations are written in the 1+1d, 2+1d and 3+1d as

$$\Delta_i u_n-\exp(-u_n+u_{n-1})+\exp(-u_{n+1}+u_n)=0\,,\quad (i=1,2,3) \tag{2·25}$$

where Δ_1, Δ_2 and Δ_3 are 1d, 2d and 3d Laplacian operators respectively. In Eq. (2·25), the subscript n represents not the derivative but the integer number such as the lattice site number in the original Toda lattice equation.[22] We can introduce V_n by $V_n=\exp(-u_n+u_{n-1})-1$ and rewrite Eq. (2·25) as

$$\Delta_i\log(1+V_n)-V_{n+1}+2V_n-V_{n-1}=0\,.\quad (i=1,2,3) \tag{2·26}$$

As well-known, the Laplacian operators can be written either in the rectangular coordinates or the cylindrical and spherical coordinates. We introduce the usual cylindrical coordinates r, θ with the relations $r=(x^2+y^2)^{1/2}$, $\theta=\arctan(y/x)$ and the spherical coordinates r, θ and ϕ with the relations $r=(x^2+y^2+z^2)^{1/2}$, $\theta=\arccos(z/r)$ and $\phi=\arctan(y/x)$. As usual, we have the expressions of the Laplacian operators as

$$\Delta_1=\partial^2/\partial x^2\,, \tag{2·27}$$

$$\Delta_2=\partial^2/\partial x^2+\partial^2/\partial y^2=\partial^2/\partial r^2+r^{-1}\partial/\partial r+r^{-2}\partial^2/\partial\theta^2\,, \tag{2·28}$$

$$\begin{aligned}\Delta_3=&\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2=\partial^2/\partial r^2+2r^{-1}\partial/\partial r\\&+r^{-2}(\partial^2/\partial\theta^2+\cot\theta\partial/\partial\theta+\sin^{-2}\theta\partial^2/\partial\phi^2)\,.\end{aligned} \tag{2·29}$$

The 1+1d Toda equation is known to be completely integrable.[23] Similarly, the 2+1d Toda equation has been clarified to be completely integrable.[24)~26] However, up to now, the exact solutions of the 3+1d Toda equation have been obtained only in the special cases.[27),28] Much remains unknown about the 3+1d Toda equation.

The one-soliton solution of the cylindrical Toda equation was found by the present author and is given by

$$V_n=(f_{n+1}f_{n-1}/f_n^2)-1\,, \tag{2·30}$$

$$f_n=1+\epsilon f_n^{(1)}\,,\quad \epsilon=\text{arbitrary constant}\,, \tag{2·31}$$

$$f_n^{(1)}=\sum_{k=0}^{\infty}J_{n+k}^2(r)=J_n^2(r)+J_{n+1}^2(r)+\cdots\,, \tag{2·32}$$

where J_n is the Bessel function of order n.[14),15] This solution has been derived both by Hirota's bilinear method and the inverse scattering method.[26] The cylindrical one-soliton given by Eqs. (2·30)~(2·32) represents the cylindrical localized wave which is decreasing in any directions of n and r (or x, y). The Bäcklund transformation for the cylindrical Toda equation has been obtained by Saitoh et al.[29]

Very recently we have obtained a generalization of the solution (2·30)~(2·32) which represents the exact quasi-cylindrical soliton solution satisfying the periodic boundary condition $V_n=V_{n+N}$ with N being an arbitrary integer.[30] This solution is deformed from pure cylindrical symmetry (has dependence also upon ϕ) when N is finite, and reduces to the cylindrical one-soliton, Eqs. (2·30)~(2·32), when N is infinity. The expression is written as Eqs. (2·30) and (2·31) with $f_n^{(1)}$ given by[30]

$$f_n^{(1)}=f_n^{(1,1)}+f_n^{(1,2)}+f_n^{(1,3)}\,, \tag{2·33}$$

$$\begin{aligned}&f_n^{(1,1)}=J_n^2(r)+J_{n-1}^2(r)+\cdots+J_1^2(r)+J_0^2(r)/2\,,\\&f_n^{(1,2)}=\sum_{k=1}^{\infty}\{J_{Nk+n}^2(r)+J_{Nk+n-1}^2(r)+\cdots+J_{Nk-n}^2(r)\}\,,\\&f_n^{(1,3)}=\sum_{k=1}^{\infty}\sum_{k'=-\infty}^{\infty}\sum_{\tilde{k}=0}^{\infty}J_{n+Nk'-k}(r)J_{n+Nk'-k+N\tilde{k}}(r)2\cos(N\tilde{k}\theta)\,.\end{aligned} \tag{2·34}$$

It was shown that in the limit of N infinity, both $f_n^{(1,2)}$ and $f_n^{(1,3)}$ vanish while $f_n^{(1,1)}$ remains non-zero and that the solution in this limit is equivalent to the solution (2·30) ~(2·32).

We note that from the known cylindrical one-soliton of the 2+1d Toda equation, we can obtain exact one-soliton solution of the 3+1d Toda equation of the following form:

$$\tilde{\Delta}_3 \log(1+V_n) - V_{n+1} + 2V_n - V_{n-1} = 0\,, \tag{2·35}$$

$$\tilde{\Delta}_3 = \partial^2/\partial x^2 + \partial^2/\partial y^2 - \partial^2/\partial t^2, \tag{2·36}$$

by the Lorentz transformation. Namely, in the solution (2·30)~(2·32), we replace $r=(x^2+y^2)^{1/2}$ by the Lorentz transformed form, say, with the constant velocity v in the x-direction $[\{(x-vt)(1-v^2)^{-1/2}\}^2+y^2]^{1/2}$, and the resultant form satisfies Eqs. (2·35) and (2·36). It is worthy to investigate collisions of two such running cylindrical solitons for the 3+1d Toda equation, (2·35) and (2·36). However so far no studies have been done including the numerical simulations.

Next we consider the 3+1d Toda equation given by Eq. (2·25) or (2·26) with $i=3$. We consider in the spherical coordinates. We assume the solution to have the form

$$1+V_n = r^{-2}\{1+\tilde{V}_n(\theta, \phi)\}\,. \tag{2·37}$$

Under this simplifying assumption, the 3+1d Toda equation (2·26) reduces to the form

$$\begin{aligned}(\partial^2/\partial\theta^2 + \cot\theta\,\partial/\partial\theta + \sin^{-2}\theta\,\partial^2/\partial\phi^2)\log\{1+\tilde{V}_n(\theta,\phi)\}\\ - \tilde{V}_{n+1}(\theta,\phi) + 2\tilde{V}_n(\theta,\phi) - \tilde{V}_{n-1}(\theta,\phi) - 2 = 0\,.\end{aligned} \tag{2·38}$$

The one-soliton solution of Eq. (2·38) depending only upon the variable θ has been recently found.[27] The solution is written as

$$1+V_n = r^{-2}\{1+\tilde{V}_n(\theta)\} = (N+n)(N+1-n)r^{-2} f_{n+1} f_{n-1}/f_n^2\,, \tag{2·39}$$

$$f_n = f_n^{(0)} + \epsilon f_n^{(1)}\,, \quad \epsilon = \text{arbitrary constant}\,, \tag{2·40}$$

$$f_n^{(0)} = a_{n-1}a_{n-2}\cdots a_{-N}\,, \quad a_k \equiv (N-k)(N+k+1)\,,$$

$$f_n^{(1)} = \{P_N^n(\cos\theta)\}^2 + a_{n-1}\{P_N^{n-1}(\cos\theta)\}^2 + a_{n-1}a_{n-2}\{P_N^{n-2}(\cos\theta)\}^2 + \cdots\,. \tag{2·41}$$

Here N is an arbitrary integer constant and P_N^n represents the associated Legendre function.[14),15)] It was shown that by taking the limit of large N and small angle θ properly with r being fixed to the constant, the solution (2·39)~(2·41) reduces to the cylindrical one-soliton of the 2+1d Toda equation given by Eqs. (2·30)~(2·32).[27] There is one point to be noticed about the boundary condition. The solution (2·39)~(2·41) satisfies the so-called "Toda molecule" boundary condition or $1+V_n=0$ when n=finite integer N and $-N$. It remains unsolved to find the 3+1d solution under the "infinite lattice" boundary condition or V_n=constant when $n=\pm\infty$. We also note that for the 3+1d Toda equation, it is interesting to obtain soliton-solution which has pure spherical symmetry namely the solution depending only on the variable r, $V_n = V_n(r)$. The solution of this type is not known so far. The situation is somewhat similar to the spherical KdV case.

2.3. *The Boussinesq equation*

The 1+1d Boussinesq equation is written as

$$u_{tt} - 3(u^2)_{xx} - u_{xxxx} = 0\,. \tag{2·42}$$

In Eq. (2·42), if necessary the term $-u_{xx}$ can be added by replacing u by $u+1/6$. We can extend the 1+1d Boussinesq equation to higher dimension as

$$u_{tt}-3(u^2)_{xx}-u_{xxxx}+u_{yy}+u_{zz}=0\,. \tag{2·43}$$

Note that Eq. (2·43) is formally symmetric with t, y and z. We introduce the variable r by $r=(t^2+y^2+z^2)^{1/2}$ and assume u to be $u=u(r, x)$. Then Eq. (2·43) reduces to

$$u_{rr}+(2/r)u_r-3(u^2)_{xx}-u_{xxxx}=0\,, \tag{2·44}$$

which we called the spherical Boussinesq equation. We have found that Eq. (2·44) has soliton solutions.[31] Hirota found that each order of solutions can be expressed by each order of Wronskians consisting of the Hermite function.[32] The simplest one-soliton solution of Eq. (2·44) is given by Eq. (2·2) with f given by[31),32]

$$f=4s^4+3\propto\begin{vmatrix} H_3(S) & H_2(S) \\ d/dS\ H_3(S) & d/dS\ H_2(S)\end{vmatrix}, \tag{2·45}$$

$$S=\sqrt{2}s, \qquad s=(1/2)x(t^2+y^2+z^2)^{-1/4}\,. \tag{2·46}$$

Here H_n is the Hermite function (polynomial) of order n.[14),15] About the 3+1d Boussinesq equation, so far the above studies[31),32] are the only one we know.

We can consider the 2+1d Boussinesq equation given by Eq. (2·43) with the last term u_{zz} dropped. So far nothing has been reported about the cylindrical Boussinesq solitons corresponding to such 2+1d Boussinesq equation. It is also an unsolved open problem.

2.4. *The nonlinear Schrödinger (=NLS) equation*

The simplest form of the NLS equations can be written in the respective form of the 1+1d, 2+1d and 3+1d as

$$iu_t+\Delta_j u+u^*uu=0\,, \qquad (j=1, 2, 3) \tag{2·47}$$

where Δ_1, Δ_2, and Δ_3 are the same as before given by Eqs. (2·27) ~ (2·29). As is well known, the 1+1d NLS equation is completely integrable.[1] However the solutions to the higher dimensional NLS equation in the form of Eq. (2·47) are mostly unknown. In the cylindrical symmetric case of the 2+1d NLS equation of the type (2·47), the numerical simulation was performed by Lomdahl et al. Their results indicate that (i) the outward ring waves either expand infinitely or reach a maximum size and then shrink depending on the energy, (ii) inward waves collapse in a finite time.[33] As to the exact analytic solutions, it is known that the 2+1d NLS equation (2·47) has one-dimensionally alligned explode-decay type solution.[34] We consider a set of the variable transformations as

$$u(x, y, t)=U(X, Y, T)t^{-1}\exp\{i(x^2+y^2)/(4t)\}\,, \tag{2·48}$$

$$X=x/t, \qquad Y=y/t, \qquad T=-1/t\,. \tag{2·49}$$

Then from Eqs. (2·48) and (2·49) we have the relation

$$iu_t+u_{xx}+u_{yy}+u^*uu=(iU_T+U_{XX}+U_{YY}+U^*UU)t^{-3}\exp\{i(x^2+y^2)/(4t)\}\,. \tag{2·50}$$

Equation (2·50) indicates that if U is the solution of the 2+1d NLS equation then u is also the solution and the converse is true. If U is taken to be the ordinary hyperbolic secant (sech) type 1+1d soliton, then via Eqs. (2·48) and (2·49), we obtain the solution u which is one-dimensionally alligned explode-decay type (due to the factors x/t and t^{-1}) soliton solution. Except this particular case, so far very little is known about the analytic solutions of the 2+1d and 3+1d NLS equations of the type (2·47).

As a more tractable model, we have the 2+1d coupled NLS equation

$$\begin{aligned} &iu_t-\beta u_{xx}+\gamma u_{yy}+\delta u^*uu-2wu=0\,,\\ &\beta w_{xx}+\gamma w_{yy}-\beta\delta(u^*u)_{xx}=0\,, \end{aligned} \tag{2·51}$$

where β, γ and δ are real constants. This equation is known to be completely integrable and have various explode-decay type solitons expressed by the Hermite function, the Airy function and the Bessel function of order $\pm 1/4$.[5]

2.5. *The sine-Gordon equation*

The sine-Gordon equations are written in 1+1d, 2+1d and 3+1d as

$$\Delta_i u-u_{tt}-\sin u=0\,,\qquad (i=1,2,3) \tag{2·52}$$

where Δ_i's are the same as before. The 1+1d sine-Gordon equation is known to be completely integrable.[1] For the studies of the cylindrical or spherical solitons of Eq. (2·52), only a few numerical simulations are known which reported the behaviors more or less similar to the case of the cylindrical NLS equation mentioned after Eq. (2·47) of the previous subsection.[35] So far we have no exact analytic studies about the 2+1d and 3+1d of Eq. (2·52).

2.6. *The Heisenberg spin equation*

The Heisenberg spin equations are written in 1+1d, 2+1d and 3+1d as

$$\begin{aligned} &\boldsymbol{s}_t-\boldsymbol{s}\times\Delta_i\boldsymbol{s}=0\,,\qquad (i=1,2,3)\\ &\boldsymbol{s}=s_1(\boldsymbol{r},t)\boldsymbol{e}_1+s_2(\boldsymbol{r},t)\boldsymbol{e}_2+s_3(\boldsymbol{r},t)\boldsymbol{e}_3\,,\\ &\boldsymbol{s}\cdot\boldsymbol{s}=s_1^2+s_2^2+s_3^2=s^2=\text{constant}\,, \end{aligned} \tag{2·53}$$

where Δ_i's are the same as before, s_1, s_2 and s_3 are scalers, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are three-dimensional orthogonal unit vectors satisfying the relations $\boldsymbol{e}_1\times\boldsymbol{e}_2=-\boldsymbol{e}_2\times\boldsymbol{e}_1=\boldsymbol{e}_3$, $\boldsymbol{e}_1\times\boldsymbol{e}_1=0$ and the similar relations with cyclic rotations of indexes 1, 2 and 3. Lakshmanan showed in 1977 that the 1+1d case of Eq. (2·53) is equivalent to the 1+1d NLS equation, thus is completely integrable.[4] Mikhailov and Yaremchuk have found the cylindrical solitons to this system in 1982.[36] We do not know whether or not their cylindrical soliton is related to some special function. At present we have no studies about the exact soliton solutions of the 3+1d Heisenberg spin equation.

2.7. List of relations between higher dimensional solitons and the special functions

We have seen that most of the cylindrical or spherical solitons are indeed written by the special functions. For the known cases described in this section, we make a list. In this list, we write the name of the equations and the related special functions which appear in the one-soliton solution of the corresponding equation.

(1) cylindrical KdV eq.	——	$Ai(J_{1/3})$,
(2) KP eq.	——	Ai,
(3) cylindrical KP eq.	——	Ai,
(4) spherical KP eq.	——	Ai,
(5) cylindrical higher-order water-wave eq.	——	$J_{1/4}$,
(6) cylindrical Toda eq.	——	J_n,
(7) 2+1d periodic Toda eq.	——	J_n,
(8) 3+1d Toda eq.	——	P_N^n,
(9) spherical Boussinesq eq.	——	H_n,
(10) 2+1d coupled NLS eq.	——	H_n, $Ai(J_{1/3})$, $J_{1/4}$.

From this list, we see that the conjecture about the higher dimensional solitons and the special function mentioned in the introduction seems to hold fairly well.

§ 3. Transformations which connect soliton equations of different dimensions

In this section, we study the existence of the transformations between soliton equations of different dimensions.

3.1. The cases of the KdV related equations

We consider the following variable transformation:

$$u(x, t)=x/(6t)+a^2t^{-2}U(X, T), \tag{3·1}$$

$$X=axt^{-1}, \qquad T=-a^3(2t^2)^{-1}, \qquad a=\text{arbitrary constant}. \tag{3·2}$$

Algebraic manipulations of Eqs. (3·1) and (3·2) lead to the relation

$$u_t+6uu_x+u_{xxx}=(a/t)^5\{U_T+6UU_X+U_{XXX}+U/(2T)\}. \tag{3·3}$$

This shows that if u satisfies the 1+1d KdV equation then U satisfies the cylindrical KdV equation and the converse is true. This was found by Hirota in 1979 (the notation was slightly different).[6] Similarly we consider the variable transformation

$$u(x, y, t)=x/(6t)+a^2t^{-2}U(X, Y, T), \tag{3·4}$$

$$X=axt^{-1}, \qquad Y=by, \qquad T=-a^3(2t^2)^{-1}, \qquad a, b=\text{arbitrary constants}. \tag{3·5}$$

Algebraic manipulations of Eqs. (3·4) and (3·5) lead to the relation

$$(u_t+6uu_x+u_{xxx})_x+3\alpha u_{yy}$$
$$=a^6t^{-6}[\{U_T+6UU_X+U_{XXX}+U/(2T)\}_X+(3/4)\alpha a^2b^2U_{YY}/T^2]. \quad (3\cdot6)$$

Here α is a constant. Equation (3·6) shows that the KP equation and the cylindrical KP equation given by Eq. (2·15) are connected by the variable transformation which is essentially the same as the one connecting the 1+1d KdV equation and its cylindrical version.

Next we consider the case of the HOWW equation. We consider the variable transformation

$$q(x, t)=-x^2/(2t)+Q(X, T), \qquad p(x, t)=a^2t^{-2}P(X, T), \quad (3\cdot7)$$

$$X=ax/t, \qquad T=-a^2/t, \qquad a=\text{arbitrary constant}. \quad (3\cdot8)$$

Equations (3·7) and (3·8) lead to the relation

$$p_t-(pq_x-q_{xxx})_x=(a/t)^4\{P_T-(PQ_X-Q_{XXX})_X+P/T\}, \quad (3\cdot9)$$

$$p-q_t+q_x{}^2/2=(a/t)^2\{P-Q_T+Q_X{}^2/2\}. \quad (3\cdot10)$$

Equations (3·9) and (3·10) show that the HOWW equation and its cylindrical version Eqs. (2·18) and (2·19), are connected by the variable transformation.[21]

3.2. *The case of the Toda equation*

We consider the variable transformation

$$u_n(x)=-2nx+U_n(R), \qquad R=e^x. \quad (3\cdot11)$$

Equation (3·11) leads to the relation

$$u_{n,\,xx}-\exp(-u_n+u_{n-1})+\exp(-u_{n+1}+u_n)$$
$$=R^2\{(\partial^2/\partial R^2+R^{-1}\partial/\partial R)U_n-\exp(-U_n+U_{n-1})+\exp(-U_{n+1}+U_n)\}. \quad (3\cdot12)$$

Equation (3·12) shows that the 1+1d Toda equation and its cylindrical version are connected by the variable transformation. This was found by the present author in 1985.[7] The above transformation can be equivalently rewritten as

$$1+v_n(x)=R^2\{1+V_n(R)\}, \qquad R=e^x, \quad (3\cdot13)$$

$$\partial^2/\partial x^2\log(1+v_n)-v_{n+1}+2v_n-v_{n-1}$$
$$=R^2\{(\partial^2/\partial R^2+R^{-1}\partial/\partial R)\log(1+V_n)-V_{n+1}+2V_n-V_{n-1}\}. \quad (3\cdot14)$$

Very recently, Hirota has found the transformation between the 2+1d Toda equation and the quasi-spherical Toda equation (2·38). He considers the transformation[28]

$$x=\log\tan(\theta/2)=(1/2)\log\{(1-\cos\theta)/(1+\cos\theta)\}, \qquad y=\phi, \quad (3\cdot15)$$

$$u_n(x, y)=-2n\log\sin\theta+U_n(\theta, \phi). \quad (3\cdot16)$$

We note that Eq. (3·15) gives the relation

$$\partial^2/\partial x^2+\partial^2/\partial y^2=(\sin^2\theta)(\partial^2/\partial\theta^2+\cot\theta\partial/\partial\theta+\sin^{-2}\theta\partial^2/\partial\phi^2)\,. \tag{3·17}$$

Equations (3·16) and (3.17) lead to the relation

$$\begin{aligned}&(\partial^2/\partial x^2+\partial^2/\partial y^2)u_n-\exp(-u_n+u_{n-1})+\exp(-u_{n+1}+u_n)\\&\quad=(\sin^2\theta)\{(\partial^2/\partial\theta^2+\cot\theta\partial/\partial\theta+\sin^{-2}\theta\partial^2/\partial\phi^2)U_n(\theta,\phi)\\&\qquad-\exp(-U_n+U_{n-1})+\exp(-U_{n+1}+U_n)+2n\}\,.\end{aligned} \tag{3·18}$$

This shows that the 2+1d Toda equation (2·25) and the quasi-spherical Toda equation are connected by the variable transformation. Instead of Eq. (3·16), we can equivalently consider the transformation

$$1+v_n(x,y)=(\sin^2\theta)\{1+V_n(\theta,\phi)\} \tag{3·19}$$

with Eq. (3·15) unchanged. Then Eqs. (3·15) and (3·19) lead to the relation

$$\begin{aligned}&(\partial^2/\partial x^2+\partial^2/\partial y^2)\log(1+v_n)-v_{n+1}+2v_n-v_{n-1}\\&\quad=(\sin^2\theta)\{(\partial^2/\partial\theta^2+\cot\theta\partial/\partial\theta+\sin^{-2}\theta\partial^2/\partial\phi^2)\log(1+V_n)\\&\qquad-V_{n+1}+2V_n-V_{n-1}-2\}\,.\end{aligned} \tag{3·20}$$

This shows the connection between the 2+1d Toda equation, Eq. (2·26), and the quasi-spherical Toda equation, Eq. (2·38).

3.3. *The case of the Boussinesq equation*

At present the transformation which connects the 1+1d Boussinesq equation (2·42) and the spherical Boussinesq equation given by Eq. (2·44) is unknown. Therefore we will be satisfied with less general arguments. We consider the similarity reductions of the two equations. We assume that u appearing in Eqs. (2·42) and (2·44) are of the form $u=t^{-1}v(s)$, $s=xt^{-1/2}$ and $u=r^{-1}v(s)$, $s=xr^{-1/2}$ respectively. Then we have the similarity-reduced 1+1d Boussinesq equation and the similarity-reduced spherical Boussinesq equation respectively as

$$s^2v_{ss}+7sv_s+8v-12(v^2)_{ss}-4v_{ssss}=0\,, \tag{3·21}$$

$$s^2v_{ss}+3sv_s-12(v^2)_{ss}-4v_{ssss}=0\,. \tag{3·22}$$

We consider the transformation

$$v(s)=s^2/6+a^2V(S)\,,\qquad S=as.\qquad(a^4=-3\text{ or }a=3^{1/4}i^{1/2}) \tag{3·23}$$

Algebraic manipulations of Eq. (3·23) lead to

$$s^2v_{ss}+7sv_s+8v-12(v^2)_{ss}-4v_{ssss}=-3\sqrt{3}i[S^2V_{SS}+3SV_S-12(V^2)_{SS}-4V_{SSSS}]\,. \tag{3·24}$$

Equation (3·24) shows that the two equations (3·21) and (3·22) are connected by the variable transformation.

3.4. *The case of the NLS equation*

For the 2+1d NLS equation (2·47), we put $u(x,y,t)=v(x,t)\exp\{iy^2/(4t)\}$.

Then the 2+1d NLS equation reduces to the equation[34)]

$$i\{v_t+v/(2t)\}+v_{xx}+v^*vv=0\,. \tag{3·25}$$

This equation is not the cylindrical equation in the sense of 2d Laplacian Eq. (2·28). However the appearance of the term $v_t+v/(2t)$ is the same as in the cylindrical KdV equation or Eq. (2·1) with $d'=2$. Thus we may call Eq. (3·25) as "quasi-cylindrical NLS equation". We consider the variable transformation

$$u(x,t)=U(X,T)t^{-1}\exp\{ix^2/(4t)\}\,,$$
$$X=x/t, \qquad T=-1/t\,. \tag{3·26}$$

Then we have

$$iu_t+u_{xx}+u^*uu=\{iU_T+U_{XX}+U^*UU+iU/(2T)\}t^{-3}\exp\{ix^2/(4t)\}\,. \tag{3·27}$$

Equation (3·27) shows that the 1+1d NLS equation and the "quasi-cylindrical" NLS equation are connected by the variable transformation. This was found by Leclert et al, in 1979.

3.5. *Summary*

We have seen the known examples of the variable transformations between soliton equations of different dimensions. In the following, we write the names of the two related equations together with the line. The line indicates the existence of the connecting transformation.

(1) 1+1d KdV eq.	——	cylindrical KdV eq.
(2) 1+1d KP eq.	——	cylindrical KP eq.
(3) 1+1d HOWW eq.	——	cylindrical HOWW eq.
(4) 1+1d Toda eq.	——	cylindrical Toda eq.
(5) 2+1d Toda eq.	——	quasi-spherical Toda eq.
(6) similarity-reduced 1+1d Boussinesq eq.	——	similarity-reduced spherical Boussinesq eq.
(7) 1+1d NLS eq.	——	quasi-cylindrical NLS eq.

Acknowledgements

I thank Mr. Koji Funaki for the help of typing the manuscript.

Appendix A
—— *Derivation of the One-Soliton Solutions of the Cylindrical and the Spherical KP Equation by the Bilinear Method* ——

For Eq. (2·15), we consider the dependent variable transformation given by Eq. (2·2). Then the equation reduces to the following bilinear equation:

$$\{D_xD_t+D_x^4+3a(2t^2)^{-1}D_y^2+(d'-1)(2t)^{-1}(\partial/\partial x)\}f\cdot f=0\,. \tag{A·1}$$

Here the operator D_x is defined for arbitrary functions $a(x)$ and $b(x)$ as

$$D_x^n a(x)\cdot b(x)\equiv(\partial/\partial x-\partial/\partial x')^n a(x)b(x')|_{x'=x}\,, \tag{A·2}$$

and similarly for D_t and D_y. We introduce the similarity variable s defined by Eq. (2·16) for $d'=2$ and by Eq. (2·17) for $d'=3$ respectively.

In Eq. (A·1), we assume that f depends only on the similarity variable s. Then both of two different expressions of Eq. (A·1) with the different choices of $d'=2$ and $d'=3$ reduce to the same one equation

$$\{D_s^4-4sD_s^2+2(\partial/\partial s)\}f\cdot f=0\,. \tag{A·3}$$

We see that both the cylindrical KP and the spherical KP equation reduce to the same ordinary differential equation. It is worthy to note that the cylindrical KdV equation which corresponds to the choice of parameters $d'=2$ and $a=0$ in Eqs. (2·15) and (A·1), also reduces to the same equation (A·3) if we consider the similarity variable given by Eq. (2·16) with $a=c=0$ or $s\equiv x(12t)^{-1/3}$. Equation (A·3) has the solution $f=f_0, f_1, f_2\cdots$ given by

$$f_0=1,\qquad f_1=Ai(s),$$

$$f_2=\int_s^\infty ds' Ai^2(s')=-\{sAi^2(s)-Ai'^2(s)\}=-\begin{vmatrix} Ai(s) & Ai'(s) \\ Ai'(s) & Ai''(s)\end{vmatrix}. \tag{A·4}$$

The solutions f_1 and f_2 can be checked by substituting the relations $f_1''=sAi$, $f_1^{(3)}=Ai+sAi'$, $f_1^{(4)}=s^2Ai+2Ai'$ and $f_2'=-Ai^2$, $f_2''=-2AiAi'$, $f_2^{(3)}=-2(Ai'^2+sAi^2)$, $f_2^{(4)}=-2(Ai^2+4sAiAi')$ into Eq. (A·3). The solution f_2 gives the solution of the spherical KP equation written in Eq. (2·17). We note that the solution f_2 in Eq. (A·4) has no constant term. In the case of the cylindrical KP equation ($d'=2$), we can obtain more general type of solution as follows. We assume the solution form Eq. (2·3) with $f^{(1)}$ given by $f^{(1)}=t^{-1/3}\tilde{f}^{(1)}(s)$. We substitute it into Eq. (A·1) and separate each order of ϵ. $O(\epsilon^0)$ is trivial and $O(\epsilon^2)$ becomes same as Eq. (A·3) with f replaced by $\tilde{f}^{(1)}(s)$, which has the solution $\tilde{f}^{(1)}(s)=f_2=\int_s^\infty ds' Ai^2(s')$. $O(\epsilon^1)$ reduces to the linear equation, $\{(\partial/\partial s)^4-4s(\partial/\partial s)^2-2(\partial/\partial s)\}\tilde{f}^{(1)}(s)=0$, which has the same solution $\tilde{f}^{(1)}(s)=f_2$. Thus the solution (2·2)~(2·4) with s given by Eq. (2·16) is actually the exact solution of the cylindrical KP equation.

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