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DOI

[10.2307/2586632](https://doi.org/10.2307/2586632)

Publication date

1999

Published in

Journal of Symbolic Logic

[Link to publication](#)

Citation for published version (APA):

Brendle, J., & Löwe, B. (1999). Solovay-type characterizations for forcing algebras. *Journal of Symbolic Logic*, 64, 1307-1323. <https://doi.org/10.2307/2586632>

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Solovay–type characterizations for forcing–algebras

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April 12, 2001

Abstract

We give characterizations for the (in ZFC unprovable) sentences “Every Σ_2^1 -set is measurable” and “Every Δ_2^1 -set is measurable” for various notions of measurability derived from well-known forcing partial orderings.

1 Introduction

In recent years, forcing notions which were originally devised to carry out some consistency proof have emerged more and more as independent mathematical objects which should be studied in their own right, from various angles. One such endeavor has been to investigate notions of measurability (that is, σ -algebras) associated with forcing orderings adding a generic real. This has a long tradition since the notions related to COHEN and random forcing are the BAIRE property and LEBESGUE measurability which have always been in the focus of set-theoretic research (*cf.* the results of [SOLOVAY 1970] and [SHELAH 1984]). Other algebras which have been around for quite a while include the MARCZEWSKI-measurable sets [MARCZEWSKI 1935] which correspond to SACKS forcing and the completely RAMSEY sets which are connected with MATHIAS forcing. In all of these cases, measurability of the analytic sets has been proved long ago, and it has been known that one can get non-measurable sets on the Δ_2^1 -level in the constructible universe \mathbf{L} . Furthermore, SOLOVAY (see 5.1) proved in the sixties that the statement “all Σ_2^1 -sets are LEBESGUE-measurable” is equivalent to “over each $\mathbf{L}[a]$, there is a measure-one set of

¹Part of this research was done while the first author was supported by DFG-grant Nr. Br 1420/1-1 and the second author by DAAD-grant Ref.316-D/96/20969 in the program HSP II/AUFE and a grant of the Studienstiftung des Deutschen Volkes.

AMS Subject Classification : 03E15 54A05 28A05 03E35

random reals” which is in turn equivalent to “for all a , the union of all null sets coded in $\mathbf{L}[a]$ is null”, thereby reducing a statement about measurability of projective sets to what might be termed a *transcendence principle* over \mathbf{L} . The value of such characterizations, apart from their intrinsic beauty, is obvious: they make it much easier to check whether Σ_2^1 -measurability holds in a given model of set theory. So it is a natural question whether statements like “all Δ_2^1 -sets are \mathbb{P} -measurable” and “all Σ_2^1 -sets are \mathbb{P} -measurable” can be characterized in SOLOVAY’s fashion as transcendence principles over \mathbf{L} , for other forcing notions \mathbb{P} adding a generic real.

In this work, we show this can be done in several cases. The most interesting results concern HECHLER forcing \mathbb{D} , the standard c.c.c. forcing notion adjoining a dominating real, and the related *dominating topology* \mathcal{D} on ω^ω (see the definition in 2, (ii)). The notion of measurability associated with \mathbb{D} is, of course, the property of BAIRE with respect to \mathcal{D} . We show that all Δ_2^1 -sets have the BAIRE property in \mathcal{D} iff all Σ_2^1 -sets have the BAIRE property in the standard topology on ω^ω (Theorem 5.8). Using a combinatorial result on the dominating topology due to [LABĘDZKI–REPICKÝ 1995] which builds, in turn, on the combinatorics of HECHLER forcing developed in [BRENDLE–JUDAH–SHELAH 1992], we then get, as a rather easy consequence of the characterization on the Δ_2^1 -level, that all Σ_2^1 -sets have the BAIRE property in \mathcal{D} iff $\aleph_1^{\mathbf{L}[a]} < \aleph_1$ for all reals a (Theorem 5.11). This confirms a conjecture put forward by JUDAH (private communication). It’s the only case we know of where the consistency strength of Σ_2^1 - \mathbb{P} -measurability is already an inaccessible. This should be compared to the result of [SHELAH 1984] showing that the consistency strength of Σ_3^1 -LEBESGUE-measurability is an inaccessible.

We also investigate various other notions of measurability, e.g. \mathbb{M} -measurability which is derived from MILLER’s rational perfect set forcing \mathbb{M} . We show that all Δ_2^1 -sets are \mathbb{M} -measurable iff all Σ_2^1 -sets are iff $\omega^\omega \cap \mathbf{L}[a]$ is not dominating in ω^ω for all reals a (Theorem 6.1). In all the cases we consider here, the proof of the projective statement assuming the transcendence principle follows either from known game-theoretic arguments or by rewriting the corresponding proof for the standard BAIRE property. Our main technical results (4.1, 6.1, but also 5.7 which follows from 4.1 and 3.4), then, deal with the other direction — taken care of by a FUBINI-argument in case of the BAIRE property and LEBESGUE measurability which does not apply in our case — and have all a similar flavour: each time, we construct a $\Delta_2^1(a)$ -partition of the reals along a carefully chosen scale of $\mathbf{L}[a]$. For example, to prove Theorem 6.1 mentioned above, we produce, under the assumption that $\omega^\omega \cap \mathbf{L}[a]$ is dominating, a $\Delta_2^1(a)$ -super-BERNSTEIN set, where $A \subseteq \omega^\omega$ is called super-BERNSTEIN iff both A and $\omega^\omega \setminus A$ meet every superperfect set.

This paper is organized as follows. In section 2, we introduce the notions of forcing we are interested in, define what we mean by the corresponding notion of measurability and fix our notation. Section 3 contains general results on

the connection between the various measurability notions we study. The next three sections contain the main results: in section 5 we study HECHLER forcing; sections 4 and 6 deal with LAVER and MILLER forcing, respectively. We conclude with a brief remark about SACKS forcing in section 7, and an overview on our results as well as an open problem in section 8. All sections depend on sections 2 and 3; the higher-numbered sections can be read independently of each other; however, 5.7 uses 4.1.

2 Main definitions and notation

- (i). $\mathbb{C} := \langle \omega^{<\omega}, \supseteq \rangle$ is called COHEN forcing. For each condition s we define $[s] := \{f \in \omega^\omega : s \subseteq f\}$. The sets $([s])_{s \in \omega^{<\omega}}$ are a topology base of the so called BAIRE space whose topology we denote by \mathcal{B} . Sometimes it may be necessary to regard COHEN forcing on the CANTOR space 2^ω . In this case we will denote the topology by \mathcal{C} .
- (ii). We call $\mathbb{D} := \omega \times \omega^\omega$ HECHLER forcing, when we have the following partial ordering on it:

$$\langle N, f \rangle \leq \langle M, g \rangle \iff N \geq M, f|_M = g|_M, f \geq g$$

We put $[N, f] := \{x \in \omega^\omega : f|_N \subseteq x \text{ and } x(n) \geq f(n) \text{ for all } n\}$. Again, the sets $([N, f])_{\langle N, f \rangle \in \mathbb{D}}$ are a topology base of the *dominating topology* \mathcal{D} . Obviously the dominating topology is finer than the BAIRE topology, because if we define the following real number

$$x_s(n) := \begin{cases} s(n) & \text{if } n < |s| \\ 0 & \text{else} \end{cases}$$

then $[|s|, x_s] = [s]$. As we know from [ŁABĘDZKI–REPICKÝ 1995], \mathcal{D} is a c.c.c. BAIRE space.

In contrast to these two forcings whose conditions form a topology base on ω^ω (and which we call therefore *topological forcings*) we consider the following three *non-topological forcings*:

- (iii). A tree $L \subseteq \omega^{<\omega}$ is called LAVER tree, if all nodes above the stem are ω -splitting nodes². We call the set of all LAVER trees ordered by inclusion LAVER forcing \mathbb{L} .
- (iv). A tree $M \subseteq \omega^{<\omega}$ is called *superperfect*, if every splitting node is an ω -splitting node and every node has a (not necessarily immediate) successor which is a splitting node (and therefore an ω -splitting node). MILLER forcing \mathbb{M} is the set of all superperfect trees ordered by inclusion.

²A node is called *splitting* if it has more than one immediate successor, and it is called *ω -splitting* if it has infinitely many immediate successors

- (v). In analogy to the definition of \mathbb{M} we call a tree $P \subseteq 2^{<\omega}$ *perfect*, if below every node there is a splitting node and define SACKS forcing \mathbb{S} to be the set of all perfect trees ordered by inclusion.

Given a tree $T \subseteq \omega^{<\omega}$, let $[T] := \{f \in \omega^\omega : f|n \in T \text{ for all } n \in \omega\}$ denote the set of its branches. For $s \in T$, let $\text{Succ}(s)$ be the set of immediate successors of s in T . $\text{Split}(T)$ stands for the set of splitting nodes of T .

We can associate each of these forcing in a natural way with a notion of measurability. In the definition of the topological forcings \mathbb{C} and \mathbb{D} , we remarked that the forcings form topology bases for \mathcal{B} and \mathcal{D} respectively. The forcings are therefore quite naturally connected to the σ -algebra of sets with the BAIRE property in these topologies. The \mathcal{B} - and \mathcal{D} -meager sets are also called \mathbb{C} - and \mathbb{D} -null sets.

In the case of non-topological forcings $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}, \mathbb{L}\}$ we define a set of real numbers A ($A \subseteq \omega^\omega$ or $A \subseteq 2^\omega$ according to the definition of \mathbb{P}) to be \mathbb{P} -*measurable* if

$$\forall p \in \mathbb{P} \exists p' \leq p ([p'] \cap A = \emptyset \text{ or } [p'] \cap \omega^\omega \setminus A = \emptyset)$$

and to be \mathbb{P} -*null* if

$$\forall p \in \mathbb{P} \exists p' \leq p ([p'] \cap A = \emptyset)$$

The ideal of \mathbb{P} -null sets we denote by (p^0) and the set of complements of \mathbb{P} -null sets we denote by (p^1) .

For pointclasses Γ , we abbreviate the sentence “every set in Γ is \mathbb{P} -measurable” by $\Gamma(\mathbb{P})$. In addition to that we define a set A to be *weakly* \mathbb{P} -*measurable* if either A or its complement contains the branches through some element of \mathbb{P} . As above, we abbreviate the sentence “every set in Γ is weakly \mathbb{P} -measurable” by $w\Gamma(\mathbb{P})$. We will call a pointclass Γ *topologically reasonable* if it is closed under continuous preimages and has the following property:

$$\text{For } A \in \Gamma \text{ and } Q \text{ closed, we have } A \cap Q \in \Gamma$$

2.1 Lemma

Let \mathbb{P} be any of the forcings considered in this work, and let Γ be a topologically reasonable pointclass. Then the following are equivalent:

- (i). $w\Gamma(\mathbb{P})$
- (ii). $\Gamma(\mathbb{P})$

Proof :

As the backward direction is obvious, we prove the forward direction: Suppose $\Gamma(\mathbb{P})$ is false. Then there is an $A \in \Gamma$ which is not \mathbb{P} -measurable, *i.e.* there is a $P \in \mathbb{P}$ such that for all $Q \leq P$:

$$[Q] \cap A \neq \emptyset$$

and

$$[Q] \cap \omega^\omega \setminus A \neq \emptyset$$

Let σ be an homeomorphism between $[P]$ and ω^ω (or 2^ω in the case that \mathbb{P} is defined on the CANTOR space). Then because of the properties postulated for Γ , $A \cap [P]$ and $A' := \sigma(A \cap [P])$ are in Γ . Because of $w\Gamma(\mathbb{P})$ we have $Q' \in \mathbb{P}$ with either $[Q'] \subseteq A'$ or $[Q'] \subseteq \omega^\omega \setminus A'$. Applying σ^{-1} yields:

$$\sigma^{-1}[Q'] \subseteq A \cap [P] \text{ or } \sigma^{-1}[Q'] \subseteq [P] \cap (\omega^\omega \setminus A)$$

But this is a contradiction.

q.e.d.

For this lemma we do not need closure under continuous preimages, closure under homeomorphism would suffice.

Let \mathfrak{M} be a model of ZFC and \mathbf{BC} be a fixed coding of the BOREL sets. For a code c we denote the decoded set with A_c . If \mathbb{P} is any of the defined forcing notions then $\mathbf{BC}_{\mathbb{P}}(\mathfrak{M})$ denotes the set of all real numbers in \mathfrak{M} which code a Borel \mathbb{P} -null set (we have enough absoluteness properties for this to make sense). For abbreviation we define:

$$\mathbf{N}_{\mathbb{P}}(\mathfrak{M}) := \bigcup \{A_c : c \in \mathbf{BC}_{\mathbb{P}}(\mathfrak{M})\}$$

For the c.c.c. forcings considered here, one can prove that the \mathbb{P} -generic reals over \mathfrak{M} are exactly those not in $\mathbf{N}_{\mathbb{P}}(\mathfrak{M})$. This result allows the following definition:

2.2 Definition

- (i). $\mathbf{Coh}(\mathfrak{M}) := \omega^\omega \setminus \mathbf{N}_{\mathbb{C}}(\mathfrak{M})$
- (ii). $\mathbf{Hech}(\mathfrak{M}) := \omega^\omega \setminus \mathbf{N}_{\mathbb{D}}(\mathfrak{M})$

2.3 Definition

Let $A \subseteq \omega^\omega$. A real $x \in \omega^\omega$ is called

- *unbounded* over A , if:

$$\forall a \in A \exists^\infty n : a(n) \leq x(n)$$

- *dominating* over A , if:

$$\forall a \in A \forall^\infty n : a(n) \leq x(n)$$

Shortly we write $x \leq^* y : \iff \forall^\infty n : x(n) \leq y(n)$ for “ y dominates x ”.

Let $B \subseteq \omega^\omega$. B is called

- σ -bounded in A , if there is an $a \in A$ dominating over B
- unbounded in A , if it is not σ -bounded in A
- dominating in A , if no $a \in A$ is unbounded over B

Apart from this, we use standard notions and notation of Descriptive Set Theory and Forcing Theory (see e.g. [JECH 1978] or [BARTOSZYŃSKI–JUDAH 1995]).

3 General Results

We provide a few results on the connection between some notions of measurability which hold for arbitrary topologically reasonable pointclasses Γ .

3.1 Theorem

For any topologically reasonable pointclass Γ , $\Gamma(\mathbb{D})$ implies $\Gamma(\mathbb{C})$

Proof :

We define a mapping $\varphi : \omega^\omega \rightarrow 2^\omega$ via

$$\varphi(f)(n) := f(n) \bmod 2$$

for $f \in \omega^\omega$ and $n \in \omega$. Note that φ is onto, continuous and open, regardless of whether we topologize ω^ω with \mathcal{D} or \mathcal{B} . (Of course, 2^ω always carries the topology \mathcal{C}).

Now let $A \subseteq 2^\omega$ be a \mathcal{C} -nonmeager set in Γ . It suffices to show that there is $s \in 2^{<\omega}$ such that $[s] \cap A$ is comeager in $[s]$. Since φ is continuous when going from \mathcal{B} to \mathcal{C} and Γ is topologically reasonable, $B = \varphi^{-1}(A)$ is in Γ as well. As φ is onto, continuous and open as a map from $(\omega^\omega, \mathcal{D})$ to $(2^\omega, \mathcal{C})$, B is \mathcal{D} -nonmeager. By assumption we find $[N, f]$ such that $[N, f] \cap B$ is comeager in $[N, f]$. Hence there is a G_δ -set $C \subseteq [N, f] \cap B$ dense in $[N, f]$. Assume $C = \bigcap_i C_i$ where the C_i form a decreasing sequence of open sets, and $C_i = \bigcup_{|\sigma|=i} [N_\sigma, f_\sigma]$ (where $N_{\langle \rangle} = N$ and $f_{\langle \rangle} = f$) are such that

- (i). $\bigcup_j [N_{\sigma \cdot \langle j \rangle}, f_{\sigma \cdot \langle j \rangle}] \subseteq [N_\sigma, f_\sigma]$ is dense for all σ , and
- (ii). $N_\sigma \geq i$ for all i and σ with $|\sigma| = i$.

It is clear that all the C_i can be written in this form.

Let $\bar{\varphi} : \omega^{<\omega} \rightarrow 2^{<\omega}$ be defined by

$$\bar{\varphi}(s)(n) := s(n) \bmod 2$$

for $s \in \omega^{<\omega}$ and $n < |s|$, and put $s_\sigma = \bar{\varphi}(f_\sigma|N_\sigma)$ and $s = s_{\langle \rangle}$. Next find $H_\sigma \subseteq \omega$ such that

- (iii). the $[s_{\sigma \cdot \langle j \rangle}]$ for $j \in H_\sigma$ are pairwise disjoint, and

(iv). $\bigcup_{j \in H_\sigma} [s_{\sigma \hat{\langle j \rangle}}]$ is dense in $[s_\sigma]$.

Again this is easily done by (i) above. Now define recursively $E_0 = \{\langle \rangle\}$, $E_{i+1} = \bigcup_{\sigma \in E_i} \{\sigma \hat{\langle j \rangle} : j \in H_\sigma\}$, put $D_i = \bigcup_{\sigma \in E_i} [s_\sigma]$, and let $D = \bigcap_i D_i$. By (iv), D is dense in $[s]$. We claim that $D \subseteq A$, thus completing the proof.

Given $x \in D$, there is a unique $y \in \omega^\omega$ such that $x \in [s_{y|n}]$ for all $n \in \omega$, by clause (iii). Furthermore, (ii) entails that $\bigcap_n [s_{y|n}] = \{x\}$. Now, $\bigcap_n [N_{y|n}, f_{y|n}]$ also contains a unique element $g \in \omega^\omega$, by (i) and (ii). Clearly, $\varphi(g) = x$. Since $g \in C \subseteq B = \varphi^{-1}(A)$, we get $x \in A$, as required.

q.e.d.

Note that this result is nothing but a topological version of the well-known fact that if $f \in \omega^\omega$ is HECHLER over a model \mathfrak{M} of set theory, then $\varphi(f)$ is COHEN over \mathfrak{M} . For the next result (Theorem 3.4 below), we need the following notion from [BRENDLE–HJORTH–SPINAS 1995] (p. 294):

3.2 Definition

Let $\bar{W} = \langle w_\sigma, s_\sigma : \sigma \in \omega^{<\omega} \rangle$ be such that

- $\text{dom}(s_{\langle \rangle})$ and w_σ are finite subsets of ω
- $s_\sigma : w_{\sigma| |\sigma|-1} \rightarrow \omega$ for $\sigma \neq \langle \rangle$, $s_{\langle \rangle} : \text{dom}(s_{\langle \rangle}) \rightarrow \omega$ are functions
- $\omega = \text{dom}(s_{\langle \rangle}) \cup \bigcup_{n \in \omega} w_{f|n}$ for all $f \in \omega^\omega$, the union being pairwise disjoint
- $s_\sigma(i) > \sigma(|\sigma| - 1)$ for all $i \in w_{\sigma| |\sigma|-1}$ and all σ

Then we can define the set $C = C(\bar{W}) \subseteq \omega^\omega$ such that $g \in C$ iff $g = \bigcup_n s_{f|n}$ for some $f \in \omega^\omega$. C is called a *nice set*; it is necessarily closed and dominating.

3.3 Theorem ([BRENDLE–HJORTH–SPINAS 1995], Theorem 1.1)

Every dominating analytic set contains a nice set.

3.4 Theorem

For any topologically reasonable pointclass Γ , $\Gamma(\mathbb{D})$ implies $\Gamma(\mathbb{L})$.

Proof :

Let $A \in \Gamma$ and let T be a LAVER tree. By 2.1, we can assume $T = \omega^{<\omega}$. We have to find a LAVER tree $S \leq T$ such that either $[S] \subseteq A$ or $[S] \cap A = \emptyset$. We define a function $\varphi : \omega^\omega \rightarrow \omega^\omega$ recursively by

$$\begin{aligned} \varphi(x)(0) &= x(0) \\ \varphi(x)(n+1) &= x(\varphi(x)(n)) \end{aligned}$$

Clearly φ is continuous. Put $B := \varphi^{-1}(A)$. By assumption $B \in \Gamma$. Hence B has the property of BAIRE in the topology \mathcal{D} . Thus we can find an open set $[N, f]$ in \mathcal{D} such that either B or $\omega^\omega \setminus B$ is \mathcal{D} -comeager in $[N, f]$. Without loss

the former holds. Hence there is a G_δ -set $C \subseteq [N, f] \cap B$ dense in $[N, f]$. Note that C must be dominating in ω^ω , for otherwise we could find $g \in \omega^\omega$ above f with $[N, g] \cap C = \emptyset$, contradicting C 's density. By Theorem 3.3, C contains a nice set $D = C(\bar{W})$ where $\bar{W} = \langle w_\sigma, s_\sigma : \sigma \in \omega^{<\omega} \rangle$. Since $D \subseteq B$, we get $\varphi[D] \subseteq A$. We are left with showing that $\varphi[D]$ contains the set of branches through a LAVER tree S .

To this end, define recursively a function $\bar{\varphi}$ with range $\omega^{<\omega}$ and domain all functions from a finite subset of ω to ω , as follows. If $0 \notin \text{dom}(s)$, let $\bar{\varphi}(s) = \langle \rangle$. Otherwise put

$$\bar{\varphi}(s)(0) := s(0)$$

Assume $\bar{\varphi}(s)(i)$ has been defined. If $\bar{\varphi}(s)(i) \notin \text{dom}(s)$, we're done and have $|\bar{\varphi}(s)| = i + 1$. Otherwise, put

$$\bar{\varphi}(s)(i + 1) := s(\bar{\varphi}(s)(i))$$

Now construct recursively a LAVER tree S such that for any $t \in S$ there is $\sigma \in \omega^{<\omega}$ such that $\bar{\varphi}(\bigcup_{j \leq |\sigma|} s_{\sigma|j}) = t$ (\star). Clearly (\star) implies $[S] \subseteq \varphi[D]$.

First put $t := \bar{\varphi}(s_{\langle \rangle})$ into S . Then assume $t \in S$ has property (\star) with witness σ . We have to define the successors of t in S . Put $s := \bigcup_{j \leq |\sigma|} s_{\sigma|j}$. Then by definition of $\bar{\varphi}$, $t(|t| - 1) \notin \text{dom}(s)$. Hence there is $\tau \supseteq \sigma$ minimal such that $t(|t| - 1) \in w_\tau$. Now, if n is large enough, we will have $\bar{\varphi}(s_n)(|t|) = s_n(t(|t| - 1)) \notin \text{dom}(s_n)$ where $s_n = \bigcup_{j \leq |\tau| + 1} s_{\tau \hat{\ } \langle n \rangle | j}$. Therefore $t_n = \bar{\varphi}(s_n)$ for such n has length $|t| + 1$ and also has property (\star) with witness $\tau \hat{\ } \langle n \rangle$. Thus we can put such t_n into S . This completes the recursive construction of the LAVER tree S , and the proof of the Theorem.

q.e.d.

Results like 3.1 and 3.4 can be subsumed in the following diagram.

3.5 Corollary

Let Γ be a topologically reasonable pointclass. Then one has the following implications:

$$\begin{array}{ccccc} \Gamma(\mathbb{D}) & \implies & \Gamma(\mathbb{C}) & & \\ \downarrow & & \downarrow & & \\ \Gamma(\mathbb{L}) & \implies & \Gamma(\mathbb{M}) & \implies & \Gamma(\mathbb{S}) \end{array}$$

Proof :

$\Gamma(\mathbb{D}) \implies \Gamma(\mathbb{C})$ and $\Gamma(\mathbb{D}) \implies \Gamma(\mathbb{L})$ were proved in Theorems 3.1 and 3.4, respectively.

The directions $\Gamma(\mathbb{L}) \implies \Gamma(\mathbb{M}) \implies \Gamma(\mathbb{S})$ are easy consequences of 2.1. To see e.g. the second implication, let $A \subseteq 2^\omega$ be a set in Γ , and let $S \subseteq 2^{<\omega}$ be a SACKS tree. By 2.1, we can assume $S = 2^{<\omega}$. Let $\varphi : \omega^\omega \rightarrow 2^\omega$ be the canonical map which identifies the BAIRE space with the irrationals in 2^ω (i.e., the $x \in 2^\omega$ such that $\{i : x(i) = 1\}$ is infinite). Since φ is continuous, $\varphi^{-1}(A)$ belongs

to Γ . Hence we can find $M \in \mathbb{M}$ with $[M] \subseteq \varphi^{-1}(A)$ or $[M] \subseteq \omega^\omega \setminus \varphi^{-1}(A)$. Assume without loss the former. Since $\varphi[M]$ is an uncountable G_δ -set, we can find $T \in \mathbb{S}$ with $[T] \subseteq \varphi[M] \subseteq A$, as required.

To see that $\Gamma(\mathbb{C})$ implies $\Gamma(\mathbb{M})$, simply note that every non-meager set with the property of BAIRE contains the set of branches through a superperfect tree.

q.e.d.

We sketch another connection between two regularity properties which we shall need in section 4 when dealing with LAVER forcing. To this end, we introduce the following three notions the first of which is Definition 2.1 in [GOLDSTERN *et al.* 1995] while the last is on p. 296 in [BRENDLE–HJORTH–SPINAS 1995]:

3.6 Definition

- (i). A set $A \subseteq \omega^\omega$ is called *strongly dominating* iff

$$\forall f \in \omega^\omega \exists x \in A \forall^\infty k : f(x(k-1)) < x(k)$$

- (ii). A set $A \subseteq \omega^\omega$ is called ℓ -regular if either A contains the set of branches through a Laver tree or A is not strongly dominating.
- (iii). A set $A \subseteq \omega^\omega$ is called *strongly u -regular* if either A contains a nice set or A is not dominating.

Note that the second and third notions are very similar, and analogous facts can be proved about both. It was shown in Lemma 2.3 of [GOLDSTERN *et al.* 1995] that every BOREL set is ℓ -regular. Standard modifications of the game-theoretic argument used in the proof (SOLOVAY's unfolding trick) show the same conclusion is true for analytic sets — this is, of course, analogous to Theorem 3.3 above, but it's also a consequence of 3.3 and the following proposition:

3.7 Proposition

For a topologically reasonable pointclass Γ , strong u -regularity for Γ implies ℓ -regularity for Γ .

Proof :

Let $A \in \Gamma$ be strongly dominating. Let $\varphi : \omega^\omega \rightarrow \omega^\omega$ be the function constructed in the proof of Theorem 3.4. By the proof of 3.4, it suffices to show that $B := \varphi^{-1}(A)$ is dominating — for then we can use strong u -regularity to get a nice set $C \subseteq B$ and the argument of 3.4 shows that $\varphi[C] \subseteq A$ contains a LAVER tree.

To see that B is dominating, let $g \in \omega^\omega$ be an arbitrary increasing function such that $\varphi(g)(j-1) \geq j$ for all j . Find $x \in A$ such that $x(n+1) > \varphi(g)(x(n))$ for all $n \in \omega$. Define $y \in \omega^\omega$ such that

$$\begin{aligned} y(0) &= x(0) \\ y(i) &= x(1) && \text{for } 1 \leq i \leq x(0) \\ y(i) &= x(n+1) && \text{for } n \geq 1 \text{ and } x(n-1) < i \leq x(n) \end{aligned}$$

Then $\varphi(y) = x$ and hence $y \in B$. Furthermore,

$$y(i) = x(n+1) > \varphi(g)(x(n)) = g(\varphi(g)(x(n) - 1)) \geq g(x(n)) \geq g(i)$$

for $x(n-1) < i \leq x(n)$, because $\varphi(g)(x(n) - 1) \geq x(n)$. Thus we have $y \geq^* g$, as required.

q.e.d.

4 LAVER Measurability

In contrast to the topological forcings (see section 5), for the three non-topological forcings the notions of Δ_2^1 - and Σ_2^1 -measurability are equivalent. For LAVER forcing we will prove:

4.1 Theorem

The following are equivalent:

- (i). $\forall a \in \omega^\omega : \omega^\omega \cap \mathbf{L}[a]$ is σ -bounded in ω^ω
- (ii). $\Delta_2^1(\mathbb{L})$
- (iii). $\Sigma_2^1(\mathbb{L})$

For our proof, we need the following characterization part of which is a consequence of 3.7.

4.2 Proposition

The following are equivalent:

- (i). Every Σ_2^1 -set is strongly u -regular
- (ii). Every Σ_2^1 -set is ℓ -regular
- (iii). $\forall a \in \omega^\omega : \omega^\omega \cap \mathbf{L}[a]$ is σ -bounded in ω^ω

Proof :

(i) \Rightarrow (ii): By Proposition 3.7.

(ii) \Rightarrow (iii): This will follow from (ii) \Rightarrow (i) in 4.1, because Σ_2^1 - ℓ -regularity clearly implies $w\Sigma_2^1(\mathbb{L})$, and hence $\Sigma_2^1(\mathbb{L})$ by Lemma 2.1.

(iii) \Rightarrow (i): This was remarked on p. 296 in [BRENDLE-HJORTH-SPINAS 1995]. The proof is identical to the proof of Theorem 4.2 of [SPINAS 1994].

q.e.d.

Proof of 4.1 :

(i) \Rightarrow (iii): This is immediate from the direction (iii) \Rightarrow (ii) in 4.2.

(ii) \Rightarrow (i): Suppose we had an a so that $\mathbf{L}[a] \cap \omega^\omega$ is not σ -bounded, then:

$$\forall x \in \omega^\omega \exists y \in \mathbf{L}[a] \cap \omega^\omega \exists^\infty n \in \omega : y(n) > x(n)$$

Let $\langle g_\alpha : \alpha < \omega_1 \rangle$ be the $\Sigma_2^1(a)$ -good well-ordering of $\mathbf{L}[a]$. From this we can define a $\Sigma_2^1(a)$ -scale $\langle f_\alpha : \alpha < \omega_1 \rangle$ in $\mathbf{L}[a]^3$ which is unbounded in ω^ω and additionally has the property

$$\forall \alpha < \omega_1 \forall n < \omega : f_{\alpha+1}(n) \geq f_\alpha(n+1),$$

by standard tricks. With this scale of reals we define the following sets:

4.3 Definition

$$x \in A_\alpha : \iff (\forall \beta < \alpha : x^* \geq f_\beta) \wedge \exists^\infty n (x(n) < f_\alpha(n))$$

$$A := \bigcup_{\alpha \text{ is even}} A_\alpha$$

$$B := \bigcup_{\alpha \text{ is odd}} A_\alpha$$

As usual, limit ordinals are counted as even.

As is easily checked, the family $\langle A_\alpha : \alpha < \omega_1 \rangle$ is pairwise disjoint and covers all of ω^ω . Therefore A and B are complementary. Because the scale was $\Sigma_2^1(a)$, both A and B are $\Delta_2^1(a)$ -sets.

Next take a LAVER tree L . Without loss of generality we may assume that for all nodes $s \in L$ we have the following property:

$$\forall t \in \text{Succ}(s) : t(|s|) > s(|s| - 1)$$

Now we define recursively for any $s \in L$:

$$g_s(n) := s(n) \text{ for } n < |s|$$

$$g_s(n) := \min\{t(n) : t \in \text{Succ}(g_s|n)\} \text{ for } n \geq |s|$$

Then $g_s \in [L]$ and $g_s|_m \in L$ for all $m < \omega$. Because of our assumption on L the g_s are strictly increasing after the stem of L .

Now find α so that f_α lies infinitely often above each g_s for $s \in L$. This is possible by the unboundedness of the sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$. To prove the theorem we have to show that for the arbitrarily chosen LAVER tree L there is a branch through L in A as well as in B . To this end we will prove the following stronger claim:

³*I.e.* a dominating subset of $\omega^\omega \cap \mathbf{L}[a]$ well-ordered by \leq^* .

Let $\gamma \geq \alpha$. Then there is $x \in [L] \cap A_{\gamma+1}$.

For this, we make the following recursive construction. Define s_0 to be the stem of L . If s_i is already defined, choose $t \in \text{Succ}(s_i)$ so that $t(|s_i|) \geq f_\gamma(|s_i|)$. We know that $f_{\gamma+1}$ has infinitely many points where it is above g_t . Take $n \geq |s_i|$ minimal with this property. Then for all m with $|s_i| < m \leq n$:

$$f_\gamma(m) \leq f_{\gamma+1}(m-1) \leq g_t(m-1) < g_t(m)$$

Since we also have

$$f_\gamma(|s_i|) \leq t(|s_i|) = g_t(|s_i|)$$

we know that f_γ lies below g_t between $|s_i|$ and n and $f_{\gamma+1}(n) > g_t(n)$. Hence define $s_{i+1} := g_t|n + 1$.

Now we put $x := \bigcup_{i \in \omega} s_i$. According to the construction, x dominates f_γ and $x(|s_i| - 1) < f_{\gamma+1}(|s_i| - 1)$ for all $i < \omega$. Thus $x \in A_{\gamma+1}$. Because all s_i were in L , we have $x \in [L]$.

Since $\gamma \geq \alpha$ was arbitrary we have elements of $[L]$ both in A and B , whence A and B cannot be \mathbb{L} -measurable.

q.e.d.

5 HECHLER-Forcing

This section is devoted to proving the characterizations of $\Sigma_2^1(\mathbb{D})$ and $\Delta_2^1(\mathbb{D})$ mentioned in the Introduction. For this we will need the well-known characterizations for $\Sigma_2^1(\mathbb{C})$ and $\Delta_2^1(\mathbb{C})$:

5.1 Theorem (SOLOVAY)

The following are equivalent:

- (i). $\Sigma_2^1(\mathbb{C})$
- (ii). $\forall a \in \omega^\omega : \mathcal{N}_{\mathbb{C}}(\mathbf{L}[a])$ is meager
- (iii). $\forall a \in \omega^\omega : \text{Coh}(\mathbf{L}[a])$ is comeager

5.2 Theorem ([JUDAH-SHELAH 1989], Theorem 3.1)

The following are equivalent:

- (i). $\Delta_2^1(\mathbb{C})$
- (ii). $\forall a \in \omega^\omega : \text{Coh}(\mathbf{L}[a]) \neq \emptyset$

For proofs *cf.* [BARTOSZYŃSKI-JUDAH 1995], p. 457 and p. 452*sqq.* To get from these equivalences results about $\Delta_2^1(\mathbb{D})$ and $\Sigma_2^1(\mathbb{D})$ we need a connection between \mathbb{C} and \mathbb{D} . This connection is provided by the theorems of MILLER and TRUSS:

5.3 Theorem ([TRUSS 1977], Lemma 6.2)

If \mathfrak{M} is a ZFC-model, d a dominating real over $\omega^\omega \cap \mathfrak{M}$ and $c \in \omega^\omega$ COHEN over $\mathfrak{M}[d]$, then $c + d$ is a HECHLER real over \mathfrak{M} .

5.4 Theorem ([TRUSS 1977], Theorem 6.5)

Let c be a COHEN real over \mathfrak{M} and d dominating over $\mathfrak{M}[c]$. Then the set of all COHEN reals over \mathfrak{M} is comeager.

5.5 Theorem ([MILLER 1981], Theorem 1.2)

Consider the partial orderings $\langle \omega^\omega, \leq^* \rangle$ and $\langle (c^0), \subseteq \rangle$. For $f \in \omega^\omega$ define

$$\tilde{f}(i) := \max\{f(j) + 1 : j \leq i\}$$

Then the function $T : \omega^\omega \rightarrow (c^0) : f \mapsto \{x \in \omega^\omega : x \leq^* \tilde{f}\}$ satisfies:
For every set X bounded in (c^0) , $T^{-1}(X)$ is bounded in ω^ω .

For a proof *cf.* [BARTOSZYŃSKI–JUDAH 1995], p. 39*sq.* As an easy corollary we get:

5.6 Corollary

Suppose that for every $a \in \omega^\omega$ there is a COHEN real over $\mathbf{L}[a]$. Then the following are equivalent:

- (i). $\Sigma_2^1(\mathbb{C})$
- (ii). $\forall a \in \omega^\omega : \mathbf{L}[a] \cap \omega^\omega$ is σ -bounded

Proof :

(i) \Rightarrow (ii): According to 5.1 $\mathbf{N}_{\mathbb{C}}(\mathbf{L}[a])$ is meager, therefore $\mathcal{M} := \langle A_c : c \in \mathbf{BC}_{\mathbb{C}}(\mathbf{L}[a]) \rangle$ is a bounded family in (c^0) . Hence according to 5.5 $T^{-1}(\mathcal{M})$ a bounded family in ω^ω . Because we constructed T in $\mathbf{L}[a]$, the set $T(x)$ is a meager set coded in $\mathbf{L}[a]$ for any $x \in \omega^\omega \cap \mathbf{L}[a]$. So we have $\omega^\omega \cap \mathbf{L}[a] \subseteq T^{-1}(\mathcal{M})$, and therefore the real numbers of $\mathbf{L}[a]$ are bounded.

(ii) \Rightarrow (i): Following the assumption we have over each $\mathbf{L}[a]$ a COHEN real c_a and a dominating real over $\mathbf{L}[a][c_a]$, hence we have with 5.4: The set of all COHEN reals over $\mathbf{L}[a]$ is comeager. Because a was arbitrary, the claim follows from 5.1.

q.e.d.

The following result which is a consequence of earlier theorems is the cornerstone of the proof of Theorem 5.8 below.

5.7 Corollary

$\Delta_2^1(\mathbb{D}) \Rightarrow \forall a \in \omega^\omega : \omega^\omega \cap \mathbf{L}[a]$ is σ -bounded

Proof :

Follows from Theorems 3.4 and 4.1.

q.e.d.

Of course, this result can also be proved directly, without any reference to LAVER forcing.

5.8 Theorem

The following are equivalent:

- (i). $\Delta_2^1(\mathbb{D})$
- (ii). $\forall a \in \omega^\omega : \text{Hech}(\mathbf{L}[a]) \neq \emptyset$
- (iii). $\Sigma_2^1(\mathbb{C})$

Proof :

(i) \Rightarrow (iii): Because of 3.1 we have $\Delta_2^1(\mathbb{C})$, especially there is a COHEN real over each $\mathbf{L}[a]$ according to 5.2. Because of 5.7 we get a dominating real over $\mathbf{L}[a]$, and from the COHEN and the dominating real we conclude $\Sigma_2^1(\mathbb{C})$ via 5.6.

(iii) \Rightarrow (ii): According to 5.6, $\Sigma_2^1(\mathbb{C})$ implies the existence of a dominating real over each $\mathbf{L}[a]$. Together with the COHEN real we get from 5.1, we get with 5.3 a HECHLER real over each $\mathbf{L}[a]$.

(ii) \Rightarrow (i): This is exactly the same proof as in the corresponding direction of 5.2 (cf. [JUDAH–SHELAH 1989], Theorem 3.1, or [BARTOSZYŃSKI–JUDAH 1995], p. 452 *sqq.*).

q.e.d.

Notice that this result can be looked at as a “projective” version of the combinatorial result that the covering number of the ideal (d^0) is equal to the additivity of (c^0) (cf. [LABĘDZKI–REPICKÝ 1995], Theorem 3.6). We are now heading towards a characterization of $\Sigma_2^1(\mathbb{D})$. Apart from what has been proved so far, the following combinatorial tool is essential. Let \mathcal{A} be an almost disjoint system of subsets of ω . We define for $A \in \mathcal{A}$:

$$X_A := \{x \in \omega^\omega : \text{ran}(x) \cap A = \emptyset\}$$

As one can easily see X_A is a closed nowhere dense set in \mathcal{D} .

5.9 Theorem ([LABĘDZKI–REPICKÝ 1995], Theorem 6.2)

If X is a \mathbb{D} -null set, then there are at most countably many $A \in \mathcal{A}$, so that $X_A \subseteq X$.

5.10 Lemma

Suppose that the set $\text{Hech}(\mathbf{L}[a])$ has the BAIRE property, then it is either meager or comeager in \mathcal{D} .

Proof :

Suppose $\text{Hech}(\mathbf{L}[a])$ has the BAIRE property in \mathcal{D} . If $\text{Hech}(\mathbf{L}[a])$ is not meager then there is an open set $[N, f]$ in which $\text{Hech}(\mathbf{L}[a])$ is comeager. It suffices to show that below each open set $[M, g]$ there is another open set in which $\text{Hech}(\mathbf{L}[a])$ is comeager. Take an open set $[M, g]$ and define $\tilde{f}(i) := g(i)$ for $i < M$ and $\tilde{f}(i + M) := f(i + N)$. Then — since changing finite initial segments does not change the property of being HECHLER — $\text{Hech}(\mathbf{L}[a])$ is still comeager in $[M, \tilde{f}]$ and therefore in every subset. Obviously one can find in $[M, \tilde{f}]$ a real h lying completely above g . Then $[M, h]$ is a subset of both $[M, g]$ and $[M, \tilde{f}]$. Hence $\text{Hech}(\mathbf{L}[a])$ is comeager in $[M, h]$, as required.

q.e.d.

5.11 Theorem

The following are equivalent:

- (i). $\Sigma_2^1(\mathbb{D})$
- (ii). $\forall a \in \omega^\omega : \aleph_1^{\mathbf{L}[a]} < \aleph_1$

We divide the proof into two parts, one using 5.8 and the other using 5.9.

5.12 Proposition

$$\Sigma_2^1(\mathbb{D}) \iff \forall a \in \omega^\omega : \text{Hech}(\mathbf{L}[a]) \in (d^1)$$

Proof :

“ \Leftarrow ”: Let A be a $\Sigma_2^1(a)$ -set. By a well-known result of [SOLOVAY 1970] (cf. also [JECH 1978], p. 545), there is a BOREL set B such that $A \cap \text{Hech}(\mathbf{L}[a]) = B \cap \text{Hech}(\mathbf{L}[a])$. Since $\text{Hech}(\mathbf{L}[a])$ is \mathcal{D} -comeager, it follows that A has the BAIRE property in \mathcal{D} .

“ \Rightarrow ”: The set $N_{\mathbb{D}}(\mathbf{L}[a])$ is a $\Sigma_2^1(a)$ -set, because

$$x \in N_{\mathbb{D}}(\mathbf{L}[a]) \iff \exists c \in \text{BC}_{\mathbb{D}}(\mathbf{L}[a]) : x \in A_c$$

Hence it is \mathbb{D} -measurable. According to 5.10 it is either comeager or meager. We have to exclude the case that it is comeager. Suppose $\text{Hech}(\mathbf{L}[a]) = \omega^\omega \setminus N_{\mathbb{D}}(\mathbf{L}[a])$ is a meager set. Then it is included in some meager set coded in some $\mathbf{L}[b]$, hence there are no HECHLER reals over $\mathbf{L}[a, b]$ anymore. But $\Sigma_2^1(\mathbb{D})$ implies by 5.8 the existence of a HECHLER real over $\mathbf{L}[a, b]$, a contradiction.

q.e.d.

5.13 Proposition

$$\forall a \in \omega^\omega : \text{Hech}(\mathbf{L}[a]) \in (d^1) \iff \forall a \in \omega^\omega : \aleph_1^{\mathbf{L}[a]} < \aleph_1$$

Proof :

“ \Leftarrow ”: If $(2^{\aleph_0})^{\mathbf{L}[a]} = (\aleph_1)^{\mathbf{L}[a]}$ is countable, then there are at most countably many codes for \mathbb{D} -null sets in $\mathbf{L}[a]$, hence $\mathbf{N}_{\mathbb{D}}(\mathbf{L}[a])$ is a \mathbb{D} -null set.

“ \Rightarrow ”: Suppose $(\aleph_1)^{\mathbf{L}[a]} = \aleph_1$ for some a . Then there is in $\mathbf{L}[a]$ an almost disjoint family \mathcal{A} with $|\mathcal{A}| = (2^\omega)^{\mathbf{L}[a]} = (\aleph_1)^{\mathbf{L}[a]} = \aleph_1$. We know that the sets X_A are HECHLER-null sets in $\mathbf{L}[a]$. Because of that $\mathbf{N}_{\mathbb{D}}(\mathbf{L}[a])$ contains all of the X_A and hence more than countably many of these sets. 5.9 shows that $\mathbf{N}_{\mathbb{D}}(\mathbf{L}[a])$ is not HECHLER-null.

q.e.d.

6 MILLER Measurability

The main goal of this section is the proof of the following characterization:

6.1 Theorem

The following are equivalent:

- (i). $\forall a \in \omega^\omega : \omega^\omega \cap \mathbf{L}[a]$ is not dominating in ω^ω
- (ii). $\Delta_2^1(\mathbb{M})$
- (iii). $\Sigma_2^1(\mathbb{M})$

From [BRENDLE–HJORTH–SPINAS 1995] we introduce the following notion:

6.2 Definition

A set $B \subseteq \omega^\omega$ is called *w-regular* if either B contains the set of branches through a superperfect tree or B is not dominating.

An old result from [KECHRIS 1977] (*cf.* Theorem 4) yields (*cf.* Proposition 2.3 in [BRENDLE–HJORTH–SPINAS 1995]):

6.3 Theorem (KECHRIS/SPINAS)

The following are equivalent:

- (i). Every Σ_2^1 set is *w-regular*
- (ii). $\forall a \in \omega^\omega : \omega^\omega \cap \mathbf{L}[a]$ is not dominating in ω^ω

Proof of 6.1 :

(i) \Rightarrow (iii): With Theorem 6.3 we immediately get $w\Sigma_2^1(\mathbb{M})$ and with that via 2.1 $\Sigma_2^1(\mathbb{M})$.

(ii) \Rightarrow (i): Let $\langle \sigma_n : n < \omega \rangle$ be an enumeration of $\omega^{<\omega}$. Let $\text{code} : \omega^{<\omega} \rightarrow \omega$ be defined by

$$\text{code}(\sigma) = n \iff \sigma = \sigma_n$$

With a given superperfect tree $T \subseteq \omega^{<\omega}$ we associate a function $f_T \in \omega^{(\omega^{<\omega})}$ and a sequence $\langle \tau_\sigma^T : \sigma \in \omega^{<\omega} \rangle$ of elements of T using the following recursion:

$$f_T(\langle \rangle) := \text{code}(\text{stem}(T))$$

$$\tau_{\langle \rangle}^T := \text{stem}(T)$$

$$f_T(\sigma) := \min\{n : \text{stem}(T) \hat{\ } \tau_{\langle \sigma(0) \rangle}^T \hat{\ } \dots \hat{\ } \tau_{\langle \sigma(|\sigma|-1) \rangle}^T \hat{\ } \sigma_n \in \text{Split}(T) \\ \text{and } \sigma_n(0) > \sigma(|\sigma| - 1)\}$$

$$\tau_\sigma^T := \sigma_{f_T(\sigma)}$$

Call $f \in \omega^{(\omega^{<\omega})}$ *fast* iff for all $\sigma \in \omega^{<\omega}$ there is $\tau \in \omega^{<\omega}$ with $\tau(0) > \sigma(|\sigma| - 1)$ and $\text{code}(\tau) < f(\sigma)$. Given $f \in \omega^{(\omega^{<\omega})}$ fast, $g \in \omega^\omega$ and a natural number $m \in \omega$, we define recursively the tree $T = T(f, g, m)$:

$$T_0 := \{\sigma_i | \ell : i < m, \ell \leq |\sigma_i|\}$$

$$\tilde{g}(0) := m$$

$$T_1 := \{p \hat{\ } (\sigma_i | \ell) : p \in T_0, i < f(\langle \tilde{g}(0) \rangle), \ell \leq |\sigma_i|, \sigma_i(0) > \tilde{g}(0)\}$$

Note that $T_1 \setminus T_0 \neq \emptyset$. In the n th step we put:

$$\tilde{g}(n) := \max\{g(j) : j \leq \text{height}(T_n)\}$$

$$T_{n+1} := \{p \hat{\ } (\sigma_i | \ell) : p \in T_n \setminus T_{n-1}, i < f(\langle \tilde{g}(0), \dots, \tilde{g}(n) \rangle), \ell \leq |\sigma_i|, \sigma_i(0) > \tilde{g}(n)\}$$

Let $T = \bigcup_{n \in \omega} T_n$. Notice that T is a finitely branching tree, that is $[T]$ is compact. Obviously, no branch of $T(f, g, m)$ is eventually dominated by g .

6.4 Lemma

If S is a superperfect tree, f is fast and $f_S <^* f$, then there is an $m \in \omega$ such that $[S] \cap [T(f, g, m)] \neq \emptyset$.

Proof :

Choose $m \in \omega$ such that $f_S(\langle \rangle) < m$ and $f_S(\langle m \rangle \hat{\ } \sigma) < f(\langle m \rangle \hat{\ } \sigma)$ for all σ . We construct a branch belonging to both trees as follows:

$$\tau_0 := \text{stem}(S)$$

$$\tau_1 := \tau_{\langle m \rangle}^S$$

$$\tau_n := \tau_{\langle m, \tilde{g}(1), \dots, \tilde{g}(n-1) \rangle}^S,$$

where \tilde{g} is constructed from g as above. Then $\tau_0 \in T_0$ (by $f_S(\langle \rangle) < m$), $\tau_0 \hat{\ } \tau_1 \in T_1$ (by $f(\langle m \rangle) > f_S(\langle m \rangle)$) and $\tau_1(0) = \tau_{\langle m \rangle}^S(0) > m$, and so on.

Thus $x := \tau_0 \hat{\ } \tau_1 \hat{\ } \dots \hat{\ } \tau_n \hat{\ } \dots \in [S] \cap [T(f, g, m)]$.

q.e.d.

We are now ready to complete the proof of 6.1.

Suppose the reals of $\mathbf{L}[a]$ were dominating. Let $\langle g_\alpha : \alpha < \omega_1 \rangle$ and $\langle g'_\alpha : \alpha < \omega_1 \rangle$ be the $\Sigma_2^1(a)$ -enumerations of $\omega^\omega \cap \mathbf{L}[a]$ and $\omega^{(\omega^{<\omega})} \cap \mathbf{L}[a]$, respectively. We construct recursively $\langle f_\alpha : \alpha < \omega_1 \rangle \subseteq \omega^\omega \cap \mathbf{L}[a]$ and auxiliary $\langle h_\alpha : \alpha < \omega_1 \rangle \subseteq \omega^{(\omega^{<\omega})} \cap \mathbf{L}[a]$, such that for $\alpha < \beta$:

- (i). h_β eventually dominates g'_β and h_α
- (ii). f_α is fast
- (iii). $f_{\alpha+1}$ eventually dominates all branches of all trees $T(h_\alpha, f_\alpha, m)$ for $m < \omega$ (possible by compactness)
- (iv). f_β eventually dominates f_α and g_β
- (v). f_α and h_α are $<_{\mathbf{L}[a]}$ -minimal with these properties

We form

$$A := \{y \in \omega^\omega : \min\{\alpha : y <^* f_\alpha\} \text{ is even}\}$$

$$B := \{y \in \omega^\omega : \min\{\alpha : y <^* f_\alpha\} \text{ is odd}\}$$

Then both A and B have $\Sigma_2^1(a)$ -definitions, $A \cap B = \emptyset$ and $A \cup B = \omega^\omega$, because we worked through the $\langle g_\alpha : \alpha < \omega_1 \rangle$.

Since the reals in $\mathbf{L}[a]$ are dominating, $A \cup B = \omega^\omega$ still holds in the real world \mathbf{V} . Thus A and B are both $\Delta_2^1(a)$ in \mathbf{V} . We now show that A and B are super-BERNSTEIN-sets, *i.e.* for each superperfect tree T we have $A \cap [T] \neq \emptyset \neq B \cap [T]$. For this purpose let $T \in \mathbf{V}$ be superperfect. There is an $\alpha < \omega_1$ such that $f_T <^* h_\beta$ for all $\beta \geq \alpha$, because $\omega^\omega \cap \mathbf{L}[a]$ is dominating and we worked through the $\langle g'_\alpha : \alpha \in \omega_1 \rangle$.

Thus there are (by 6.4) m and $m' \in \omega$ with

$$[T] \cap [T(h_\alpha, f_\alpha, m)] \neq \emptyset$$

and

$$[T] \cap [T(h_{\alpha+1}, f_{\alpha+1}, m')] \neq \emptyset$$

Without loss of generality α is even. Let y be an element of the first set and y' an element of the second set. By (iii) $y <^* f_{\alpha+1}$ and, by the remark after the construction of $T(f, g, m)$, $y \not<^* f_\alpha$. Thus $y \in B$. Similarly $y' \in A$. Hence the $\Delta_2^1(a)$ -set A is not \mathbb{M} -measurable, a contradiction. This completes the proof.

q.e.d.

7 SACKS Measurability

In this last section we will prove no new theorem, but apply well-known results to get an analogous characterization for SACKS measurability.

7.1 Theorem

The following are equivalent:

- (i). $\forall a \in \omega^\omega : \omega^\omega \cap \mathbf{L}[a] \neq \omega^\omega$
- (ii). $\Delta_2^1(\mathbb{S})$
- (iii). $\Sigma_2^1(\mathbb{S})$

Proof :

For the direction (ii) \Rightarrow (i) we use the well-known construction of a $\Delta_2^1(a)$ -BERNSTEIN set in $\mathbf{L}[a]$. This leaves only the direction (i) \Rightarrow (iii) to be proved. For this we need the theorem of MANSFIELD and SOLOVAY (*cf.* [SOLOVAY 1969] or [JECH 1978], p. 533*sq.*):

7.2 Theorem (MANSFIELD-SOLOVAY)

If A is a $\Sigma_2^1(a)$ -set of reals, then either it is in $\mathbf{L}[a]$ or it contains the branches through a perfect tree.

Now we take a $\Sigma_2^1(a)$ -set A and show that it is \mathbb{S} -measurable. As we have a real which is not constructible from a we can even find a perfect set of reals P in $\omega^\omega \setminus \mathbf{L}[a]$. MANSFIELD-SOLOVAY tells us that either A contains a perfect set or A is in $\mathbf{L}[a]$ in which case P lies completely in the complement of A . Therefore we have $w\Sigma_2^1(\mathbb{S})$ and with 2.1 even $\Sigma_2^1(\mathbb{S})$.

q.e.d.

8 Summary and Questions

We summarize our results and the older results of SOLOVAY and [JUDAH-SHELAH 1989] in the following table, where \mathbb{B} denotes Random Forcing, $\text{Ran}(\mathfrak{M})$ the set of all random reals over \mathfrak{M} and λ the LEBESGUE measure:

Forcing	$\Delta_2^1(\mathbb{P})$	$\Sigma_2^1(\mathbb{P})$
\mathbb{B}	$\forall a \in \omega^\omega : \text{Ran}(\mathbf{L}[a]) \neq \emptyset$	$\forall a \in \omega^\omega : \lambda(\text{Ran}(\mathbf{L}[a])) = 1$
\mathbb{C}	$\forall a \in \omega^\omega : \text{Coh}(\mathbf{L}[a]) \neq \emptyset$	$\forall a \in \omega^\omega : \text{Coh}(\mathbf{L}[a]) \in (c^1)$
\mathbb{D}	$\forall a \in \omega^\omega : \text{Hech}(\mathbf{L}[a]) \neq \emptyset$ $\iff \Sigma_2^1(\mathbb{C})$	$\forall a \in \omega^\omega : \text{Hech}(\mathbf{L}[a]) \in (d^1)$ $\iff \forall a \in \omega^\omega : \aleph_1^{\mathbf{L}[a]} < \aleph_1$
\mathbb{L}	$\forall a \in \omega^\omega : \omega^\omega \cap \mathbf{L}[a]$ is σ -bounded	
\mathbb{M}	$\forall a \in \omega^\omega : \omega^\omega \cap \mathbf{L}[a]$ is not dominating	
\mathbb{S}	$\forall a \in \omega^\omega : \omega^\omega \cap \mathbf{L}[a] \neq \omega^\omega$	

By the characterizations in the table and by well-known forcing arguments, none of the arrows in Corollary 3.5 reverses (in ZFC) for Γ being either Δ_2^1 or Σ_2^1 . However, for $\Gamma = \Sigma_2^1$, the diagram gets simpler because we then have $\Gamma(\mathbb{C}) \implies \Gamma(\mathbb{L})$.

A few comments about full projective measurability in each of our cases are in order. First, standard arguments show that $\Sigma_n^1(\mathbb{P})$ holds, for all n and all \mathbb{P} considered in this work, in SOLOVAY's model which is gotten by collapsing an inaccessible (*cf.* [SOLOVAY 1970] or [JECH 1978], p. 537*sqq.*). Hence the consistency strength of full projective measurability is at most an inaccessible. In the HECHLER case, it is exactly an inaccessible by 5.11.

Furthermore, $\Sigma_n^1(\mathbb{S})$ holds for all n in the model gotten by adding \aleph_1 COHEN reals. To see this, simply note that COHEN forcing adds a perfect set of COHEN reals, and then use homogeneity of COHEN forcing. Finally, $\Sigma_n^1(\mathbb{M})$ holds for all n in SHELAH's model for the projective BAIRE property (*cf.* [SHELAH 1984] or [BARTOSZYŃSKI–JUDAH 1995], p. 495*sqq.*). This is true by Corollary 3.5. Hence in both cases the consistency strength of full projective measurability is ZFC alone. However, we do not know the answer to the following

8.1 Question

Can one prove the consistency of “all projective sets are \mathbb{L} -measurable” on the basis of the consistency of ZFC alone?

Since LAVER forcing is closely related to MATHIAS forcing, this question has a flavour similar to the famous open problem about the consistency strength of “all projective sets are completely RAMSEY”.

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