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Solute Transfer in a Power-law Fluid Flow through Permeable Tube

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Abstract

The steady mass transfer in a fluid flowing through a rigid cylindrical tube with permeable wall has been considered. The fluid under consideration is a power-law fluid. The governing equations are coupled due to the boundary condition on velocity which obeys Starling's hypothesis. The coupled equations are solved numerically by assuming an initial approximation for concentration. The resulting ordinary differential equations are solved using fourth order Runge-Kutta method. The species transport equation is solved using Crank-Nicolson finite difference scheme. The results are graphically depicted. $n = 1$ refers to Newtonian fluid. Concentration is higher for $n = 1.1$ and lower for $n = 0.9$ compared to the case of Newtonian fluid.

Keywords: Power-law fluid, Starlings hypothesis, Concentration polarization

1. Introduction

A hollow fibre module consists of a bundle of semi permeable capillaries sealed inside a tubular cartridge. The polymeric fibers are arranged parallel to one another and are plotted in a epoxy support their lumina opening into upstream and downstream manifolds. The space between the fibers is referred to as the shell side or extra capillary space (ECS).

Hollow fibre modules are used as hollow-fibre bioreactors and as artificial organs like pancreas, liver, kidneys (see [17]).

In haemodialysis and haemodiafiltration [25, 3 & 23], the usual flow configurations are counter-current with all four parts open and blood flowing in the lumen side and dialysate in the shell side.

In the artificial organs the basic operational procedures are dialysis (diffusive means transfer) and ultrafiltration (convective mass transfer) (see [19]). It is generally accepted that the ultrafiltration is governed by Starling's law (filtration velocity at the tube wall is proportional to the net trans-boundary pressure drop) and the solutes are transported across the wall by both diffusion and convection.

Cross flow membrane filtration has also been employed in protein purification, purification of drinking water from particulates [14] and removal of organics from water by pervaporation [2].

Earlier models of artificial kidneys have considered only dialysis. [12] have carried out both experimental and theoretical studies on mass transfer in parallel-plate artificial kidneys. Cooney et.al [6] and Walker and Davis [27] have obtained the blood channel concentration distribution in the terms of particular hyper geometric functions. In these models dialysate concentration is assumed to be either zero or constant. Cooney ad Davis [7] have considered the variable dialysate concentration in parallel plate artificial kidney.

Popenfuss et.al [21] have considered a one-dimensional model with crosssectionally uniform flow velocity and protein concentration to study the ultrafiltration in a hollow-fibre artificial kidney.

Jaffrin et. al [15] have modeled simultaneous diffusive and convective mass transfer coefficient to predict the variation of clearances of ultrafiltration rates. Bruining [4] has presented a general hydrodynamic analysis of hollow fibre devices. This study includes laminar and turbulent lumen flows with a constant shell pressure in various filtration models as well as laminar lumen flow with a pressure drop on the shell side in the closed-shell mode. An extension of Bruining's analysis for power-law fluid.

The clinical study conducted by [24], [16] etc suggest the importance of combined dialysis and ultrafiltration as well as the effect of haematocrit on dialysis. Pangrle et. al [20] analysed laminar fluid flow in a porous tube and concentric shell system analogous to a single hollow fibre. A model developed by [28] treated the shell side of a cross-flow hollow fibre membrane plasma separator as a porous medium. Labecki et al [17] have developed a porous medium model to the investigation of hollow fibers systems. Chaturani and Ranganatha [5] have considered convective and diffusive mass transfer of solute in Newtonian fluid flow through a tube with permeable wall along with diffusive and convective transfer of solutes across the wall.

It is well known that blood consists of gel-like formed elements in aqueous plasma, red blood cells, some white blood cells, platelets and a variety of lipoproteins. Plasma is an aqueous solution of various proteins, including clotting factors and various ions [18]. Red blood cells are very numerous and morphologically very simple, it contains hemoglobin, which transports oxygen around the body [11].

The assumption of Newtonian behavior of blood is acceptable for a high shear rate of blood flow through larger arteries [1]. But it is well known that, blood exhibits non-Newtonian behavior while flowing through narrow arteries at low shear rate. Sankar and Lee [10] have studied two fluid non-linear model for blood. Indira et al [22] have considered Casson and Quemeda model to analyze solute transfer in permeable tubes following [5].

In smaller diameter tubes with very high yield stress a power-law fluid [9]. In the present study an effort is made to analyze solute transfer in power-law fluid flowing through permeable tubes.

2 Mathematical formulation

The steady mass transfer in a fluid flowing through a rigid cylindrical tube with permeable wall has been considered. A cylindrical co-ordinate system(r^*, θ^*, z^*) in which the z-axis is along the axis of the tube has been used. The flow is axi-symmetric and the fluid is assumed to be dilute.

The governing equations for the flow with the assumption that the tube length to diameter ratio is very large and end effects are neglected, are

$$\frac{\partial p}{\partial r} = 0 \tag{1}$$

$$-\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(\sigma \tau_{rz}) = 0 \tag{2}$$

where $\tau_{rz} = \frac{1}{\mu}\left(\frac{\partial u}{\partial r}\right)^n$ for the power law fluid

Continuity equation is given by

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0 \tag{3}$$

The species transport equation for such a flow is given by

$$u\frac{\partial c}{\partial z} + v\frac{\partial c}{\partial r} = D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) \tag{4}$$

where ‘c’ is the mass concentration of the solute and ‘D’ is the diffusion coefficient.

The approximate initial and boundary conditions of the problem are:

On velocity: $\frac{\partial u}{\partial r} = 0,\ v = 0$ at $r = 0$

$$u = 0,\ v = v_R = L_p(z)[\Delta p - \sigma\Delta\pi] \quad \text{at} \quad r = R \tag{5}$$

$$u = 2u_0\left[1 + \left(\frac{r}{R}\right)^2\right],\ v = v_R\left[\frac{2r}{R} - \left(\frac{r}{R}\right)^3\right] \quad \text{at} \quad z = 0 \tag{6}$$

Where $\Delta p = p(z) - p_T$ and $\Delta\pi = \pi(z) - \pi_T$

On concentration: $\frac{\partial c}{\partial r} = 0$ at $r = 0$

$$\frac{\partial c}{\partial r} = h(c - c_T) + (T_R - 1)v_R \begin{cases} c, & v_R > 0 \\ c_T, & v_R < 0 \end{cases} \quad \text{at } r = R \quad (7)$$

$$c = c_0,\ p = p_a \quad \text{at} \quad z = 0 \quad (8)$$

where R is the tube radius, L_p is the hydraulic wall permeability, v_R is the ultrafilration velocity, σ is reflection coefficient, h is the diffusive conductivity of the wall, T_R is transmittance coefficient, π and π_T are respectively the osmotic pressure inside and outside the tube. u_0, c_0 and $p_a$are respectively the average axial velocity, solute concentration and hydrostatic pressure at tube entrance.

$L_{p_0}$and $L_{p_L}$are hydraulic wall permeability at $z = 0$ and $z = L$. The osmotic pressure is related to the solute concentration by

$$\pi(c) = b_1 c + b_2 c^2 + b_3 c^3 \quad (9)$$

where b_1 , b_2 and b_3 are constants given by $b_1 = 2.1\ \frac{mmHg}{g/100ml}$, $b_2 = 0.16\ \frac{mmHg}{(g/100ml)^2}$

$$b_3 = 0.16\ \frac{mmHg}{(g/100ml)^3}$$

The total mass flux of solute across the wall J_s is given by

$$J_s = \begin{cases} h[c(R,z) - c_T] + T_R v_R c(R,z), & v_R > 0 \\ h[c(R,z) - c_T] + T_R v_R c_T, & v_R < 0 \end{cases} \quad (10)$$

The total solute clearance of the capillary $J_{cl} = \int_0^L 2\pi R J_s\, dz$ (11)

Solving the equations (2) and (3) we get the radial and axial velocities in the form,

$$u(R,z) = \left(\frac{1}{2}\frac{d\Delta p}{dz}\right)^{\frac{1}{n}} \frac{n(5-n)}{2(n+1)^2}\mu\left[1 - r^{1+\frac{1}{n}}\right] \quad (12)$$

and

$$v(R,z) = \frac{n(5-n)}{2^{\frac{1}{n}}(n+1)^2(3n+1)} \frac{d^2\Delta p}{dz^2}\left[\frac{r}{2} - n\, r^{2+\frac{1}{n}}\right] \quad (13)$$

Using equations (13) and (5) we get

$$\frac{1}{\mu}\frac{d^2\Delta p}{dz^2} = L_p(z)\,\frac{n(5-n)}{2^{\frac{1}{n}}\,(n+1)^2}\,[\Delta p - \sigma\Delta\pi] \tag{14}$$

The above equations are non- dimensionalised using

$$\left.\begin{aligned}(r^*, z^*) = \frac{(r,z)}{R}, \quad (u^*, v^*, v_R{}^*) = \frac{(u, v, v_R)}{u_0}, \quad (c^*, c_T{}^*) = \frac{(c, c_T)}{c_0} \\ b_1{}^* = \frac{b_1 c_0}{\Delta p_a}, \quad b_2{}^* = \frac{b_2 c_0^2}{\Delta p_a}, \quad b_3{}^* = \frac{b_3 c_0^3}{\Delta p_a} \\ (\Delta p^*, \Delta\pi^*) = \frac{(\Delta p, \Delta\pi)}{\Delta p_a}, Sh_w = \frac{hR}{D}, \epsilon = \frac{L_{p_0}\mu}{R}, Pe = \frac{u_0 R}{D}\end{aligned}\right\} \tag{15}$$

where the parameters Sh_w, Pe and ϵ are respectively wall Sherwood number, Peclet number and ultrafiltration parameter.

After non-dimesionalisation the governing equations and the velocity take the following form

Species equation, $Pe\left[u\frac{\partial c}{\partial z} + v\frac{\partial c}{\partial r}\right] = \frac{1}{r}\frac{\partial}{\partial r}\left(Dr\frac{\partial c}{\partial r}\right)$ (16)

The equation $\frac{d^2\Delta p}{dz^2} - 16R_p\epsilon\frac{(5-n)n}{2^{\frac{1}{n}}(n+1)^2}(\Delta p - \sigma\Delta\pi) = 0$ (17)

gives pressure gradient.

$$u(r,z) = \left(\frac{1}{2}\frac{d\Delta p}{dz}\right)^{\frac{1}{n}} R_p\,\frac{n(5-n)}{2(n+1)^2}\left[R^{1+\frac{1}{n}} - r^{1+\frac{1}{n}}\right] \tag{18}$$

$$v(R,z) = \frac{n(5-n)}{2^{\frac{1}{n}}\,(n+1)^2(3n+1)}\frac{d^2\Delta p}{dz^2}\left[\frac{rR^{1+\frac{1}{n}}}{2} - n\,r^{2+\frac{1}{n}}\right] \tag{19}$$

are the axial and radial velocities respectively.

The boundary conditions take the form

at $z = 0, \Delta p = 1, \frac{d\Delta p}{dz} = \frac{-8}{R_p}$ (20)

at $r = 1, \frac{\partial c}{\partial r} = Sh_w(c - c_T) + Pe(1 - T_R)v_R\begin{cases} c, & if\ v_R > 0 \\ c_T, & if\ v_R < 0\end{cases}$ (21)

at $r = 0, \frac{\partial c}{\partial r} = 0, v = \frac{\partial u}{\partial r} = 0$ (22)

$$J_s = \begin{cases} Sh_w[c(1,z) - c_T] + PeT_R v_R c(1,z), & v_R > 0 \\ Sh_w[c(1,z) - c_T] + PeT_R v_R c_T \quad , & v_R < 0\end{cases} \tag{23}$$

and total clearance $J_{cl} = \frac{2}{Pe}\int_0^{\frac{L}{R}} J_0\,dz$ (24)

where $v_R = \epsilon L_p(z)[\Delta p - \sigma \Delta\pi], R_p = \frac{\Delta p_a R}{\mu u_0}, \frac{d\Delta p}{dz} = \frac{dp}{dz}, L_p(z) = \left(1 - \frac{\theta z}{L}\right)$

$$\Delta\pi = b_1[c(1,z) - c_T] + b_2[c^2(1,z) - c_T^2] + b_3[c^3(1,z) - c_T^3]$$

3. Numerical procedure

The coupled system of equations (16) and (17) can be solved simultaneously using numerical methods, but a relatively simpler iterative procedure has been used to obtain the numerical solution. The equations are decoupled by using an approximate initial value for concentration at wall (say $c^{(0)}$). Using fourth order Runge-Kutta method equation (17) has been solved equations (18) and (19). A finite difference scheme of Crank- Nicholson type has been used to solve the parabolic differential equation (16). The difference equation may be written as

$$Pe(u_{i,j+1} + u_{i,j})c_{i,j+1} - \lambda L_{i,j+1} = Pe(u_{i,j+1} + u_{i,j})c_{i,j} - \lambda L_{i,j} \quad (25)$$

where, for a given j

$$L_{i,j+1} = \begin{cases} (D_{i,j} - X_{i,j+1})c_{i+1,j+1} - 2D_{i,j}c_{i,j} + (2D_{i,j} + X_{i,j+1})c_{i-1,j+1} & for\ k \le i \le N-1 \\ D_{i.j}c_{i+1,j+1} - 2D_{i,j}c_{i,j} + D_{i,j}c_{i-1,j+1} & for\ 1 \le i \le k \\ 4D_{i,j}(c_{i+1,j+1} - c_{i,j+1}) & for\ i+1 \end{cases}$$

where $k = \frac{r_c}{\Delta r}, X_{i,j+1} = D_{i,j+1} = D_{i+1,j} - \frac{D_{i,j}}{2i} - \frac{Pe}{2}\Delta r v_{i,j+1}, \lambda = \frac{\Delta z}{\Delta r^2}$

4. Results and Discussions

The analysis or the results has been carried out by varying one flow parameter of a typical hollow fibre data and fixing the value of the remaining parameters at a time. The values of various flow variables such as ultra filtration, velocity, hydrostatic pressure, osmotic pressure, velocity profiles, concentration profiles and solute clearance have been evaluated for different values of power-law index, ultra filtration parameter, transmittance coefficient etc have been computed.

The Numerical results have been cross checked by comparing with existing results for a particular case. The values of the constants used are listed in the following table 1. The radial and axial concentration profiles, velocity profiles are depicted graphically.

Quantity	**Values**	**Units**
R	$30x10^{-4}$	cm
L	$6x10^{-2}$	cm
D	$3.3x10^{-7} – 3.3x10^{-6}$	cm^2/sec
L_{po}	$1.0x10^{-6} – 7 x 10^{-5}$	cm/sec
h	$1.0x10^{-6} – 7x10^{-5}$	cm^3/dyn sec
ΔPa	$40x10^3 – 60x10^3$	cm/sec
U_0	0.1-0.4	dyn /cm^2
c_o	6.0	g/100ml
c_T	0.7	g/100ml
μ	$3x10^{-2}$	dyn sec/cm^2

Table 1. Physiological data used in the analysis

The radial concentration profile for different values of power law index is plotted in figures 2, 3 and 4. Power law index n = 1 represents Newtonian fluid, n < 1 represent non-Newtonian nature and n > 1 represents dilatants nature of the fluid. The concentration remains constant in the middle of the tube up to a radius of about 0.58 and increases very fast and again decreases at the wall. This increase is due to the osmotic pressure present at the wall.

Comparison of the figures 2 and 3 shows that effect of non-Newtonian nature is more visible in case of smaller permeability. Figure shows that concentration for n < 1 is less compared to n = 1 which is due to the increase in viscosity. The concentration for n > 1 is lesser than the other to cases due to the fact that the fluid becomes dilatants.

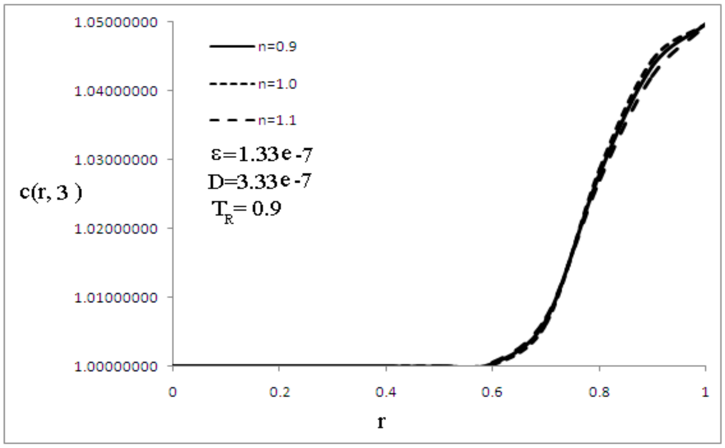

Figure. 2 Radial Concentration profile for different values of power law index

The effect of variation in n is more predominant in figure 3, where the permeability parameter is very low. Due to low permeability the concentration of the fluid initially remains constants and start increasing only after r = 6.3. The increase is similar to previous case. There is not much difference in concentration for n = 1 and n < 1 and follow same pattern as in figure 2 In case of dilatants fluid the concentration starts increasing at a lesser radius and reaches a constant value at r = 0.8 due to lesser viscosity.

Figure 4 shows concentration profile for different values of n with higher diffusion coefficient. The concentration for n = 1 is higher than that of n < 1 and n > 1. Due to increase in D more solute is attracted towards the wall and is permeated through. Due to this fact there is increase in concentration between radius r = 0.2 and r = 0.8 and then concentration decreases. As the permeability and diffusivity increases more solute is permeated and removed from the wall there by decreasing the concentration polarization.

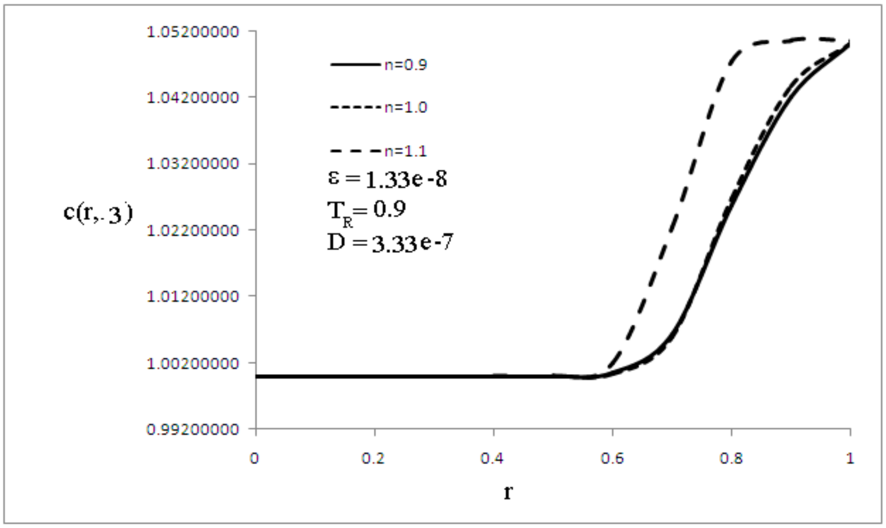

Figure 3. Radial concentration for different values of power law index

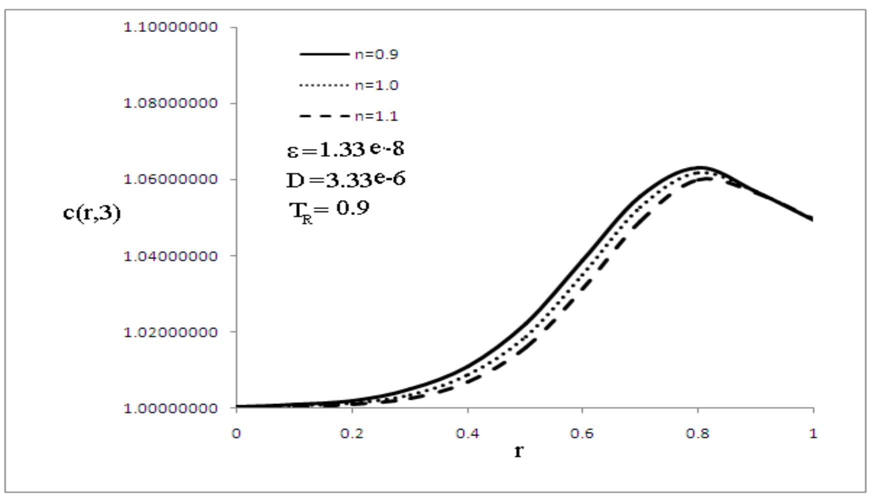

Figure 4. Radial concentration Profile for different values of power law index

Figures 5 and 6 show axial and radial velocity profiles for different values of n. The axial velocity is higher for Newtonian than the non Newtonian cases and radial velocity for Newtonian fluid is lower for other two cases. As the fluid becomes dilatants more fluid is convected to the wall which increases the radial velocity and decreases the axial velocity. For $n < 1$ the viscosity of the fluid is higher due to which there is resistance to flow and hence the axial velocity is lower than the Newtonian.

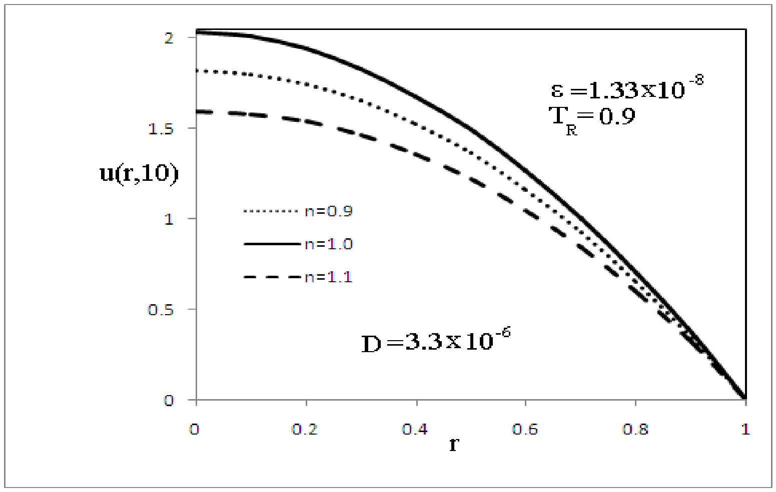

Figure 5. Axial velocity profile for different values of power law index

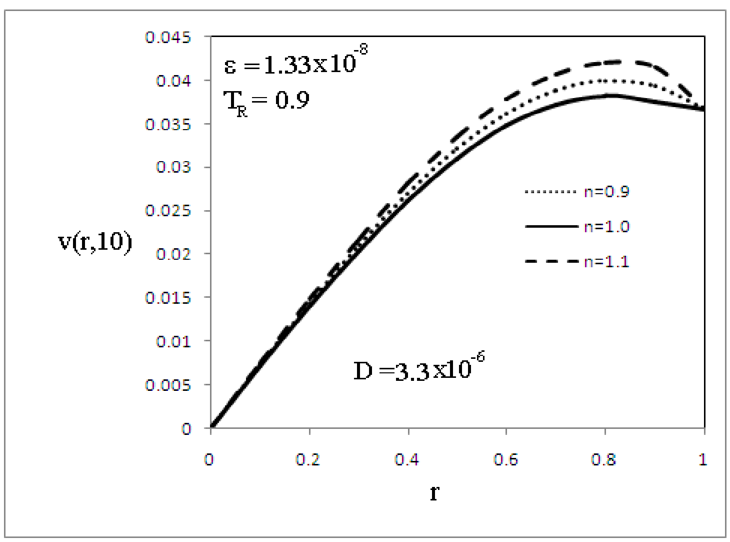

Figure 6. Radial velocity profile for different values of power law index

5. Conclusions

The present study concentrates on effects of power law fluid on concentration polarization in a flow through permeable tube. The effect of power law index is more dominant when permeability and diffusivity are low. The results reduces to that of Newtonian fluid for n = 1. The concentration polarization is more in case of dilatants fluid compared to a pseudo plastic fluid due to increased convection towards the wall.

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