SOLUTION OF A BOUNDARY VALUE PROBLEM FOR HELMHOLTZ EQUATION VIA VARIATION OF THE BOUNDARY INTO THE COMPLEX DOMAIN

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Solution of a boundary value problem for Helmholtz equation via variation of the boundary into the complex domain

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Abstract

In this paper we deal with the problem of diffraction of electromagnetic waves by a periodic interface between two materials. This corresponds to a quasi-periodic boundary value problem for Helmholtz equation. We prove that solutions behave analytically with respect to variations of the interface. The interest of this result is both theoretical –the legitimacy of power series expansions in the parameters of the problem has indeed been questioned– and, perhaps more importantly, practical: we have found that the solution can be *computed* on the basis of this observation. The simple algorithm that results from such boundary variations is described. To establish the property of analyticity of the solution for the grating

$$f_\delta(x) = \delta f(x)$$

with respect to the height δ we present a *holomorphic* formulation of the problem using surface potentials. We show that the densities entering in the potential theoretic formulation are analytic with respect to variations of the boundary, or, in other words, that the integral operator that results from the transmission conditions at the interface is invertible in a space of holomorphic functions of the variables (x, y, δ). This permits us to conclude, in particular, that the partial derivatives of u with respect to δ at $\delta = 0$ satisfy certain boundary value problems for Helmholtz equation, in regions with plane boundaries, which can be solved in closed form.

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1 Introduction

In this paper we deal with the problem of diffraction of electromagnetic waves by a periodic interface between two materials. This corresponds to a quasi-periodic boundary value problem for Helmholtz equation. Our main theorem states that solutions behave analytically with respect to variations of the interface. The interest of this result is both theoretical –the legitimacy of power series expansions in the parameters of the problem has indeed been questioned– and, perhaps more importantly, practical: we have found that the solution can be *computed* on the basis of this observation. The simple algorithm that results from such boundary variations will be briefly described in section 3. A detailed investigation of its interesting numerical properties together with some necessary refinements will be presented elsewhere. Here we concentrate on the theoretical aspects of our approach, which we treat by varying the boundary into the complex domain.

Beginning with the famous study by Lord Rayleigh [10], the diffraction problem we consider has been extensively studied in the literature (see [1,15,9] and the references contained therein). Various methods based on Rayleigh expansions, ODE's and integral equations have been proposed. Our approach yields, in particular, a power series representation for the solution about the flat interface, which is reminiscent of the work of Meecham [7]. Meecham used an iterated kernel to obtain an expansion whose first term is a well known approximation due to Kirchoff. This expansion does have a power–series like behavior (it is dominated by a geometric series) but it is not a power series. Unlike ours, its n-th term is given by the iterated integrals of a Neumann series which depend on the height of the grating in a complicated fashion, and which have not been used to evaluate the solution numerically. Meecham's expansion was criticized by Uretsky [13], who conjectured that the solution for the sinusoidal profile does not continue analytically to the one for the flat interface, in clear disagreement with the results we present.

To establish the property of analyticity of the solution for the grating

$$f_\delta(x) = \delta f(x)$$

with respect to the height δ we present a *holomorphic* formulation of the problem using surface potentials. We show that the densities entering in the potential theoretic formulation are analytic with respect to variations of the boundary, or, in other words, that the integral operator that results from the transmission conditions at the interface is invertible in a space of holomorphic functions of the variables (x, y, δ). Once the densities have been shown to be analytic, we proceed to generalize a result of Millar [8], and show that for sufficiently small δ, the function $u(x, y, \delta)$ extends analytically for (x, y) on a set $W = \{y > y_0\}$, $y_0 < 0$. This permits us to conclude that the partial derivatives of u with respect to δ at $\delta = 0$ satisfy certain boundary value problems for Helmholtz equation, in regions with plane boundaries, which can be solved in closed form.

2 Preliminaries

2.1 The Helmholtz equation

Consider an interface $\mathcal{P}$ between either two dielectrics or a dielectric and a perfect conductor. In the rectangular coordinate system (x, y, z) the surface will be assumed to be given by

$$y = f(x)$$

for a certain periodic function f. Let L denote the period of f, and assume the regions Ω^+ and Ω^- above and below the grating f are filled with materials of dielectric constants ϵ^+ and ϵ^- respectively. Alternatively, one can consider the case in which the region Ω^- is filled by a perfect conductor.

Assume the grating is illuminated by a monochromatic plane wave

$$\vec{E}^i = \vec{A}e^{i\alpha x - i\beta y}e^{(-i\omega t)}$$
$$\vec{H}^i = \vec{B}e^{i\alpha x - i\beta y}e^{(-i\omega t)}.$$

Here, the complex amplitude vector $\vec{A}$ is perpendicular to the wave vector $\vec{k} = (\alpha, -\beta, 0)$, and $\vec{B} = \frac{\vec{k}\times\vec{A}}{\omega\mu_0}$. The permeability of both dielectrics is assumed to be equal to μ_0, the permeability of vacuum.

The incident wave $(\vec{E}^i, \vec{H}^i)$ will be diffracted by the grating. In the case of a grating between two dielectrics, the total fields will be given by

$$\vec{E}^{up} = \vec{E}^i + \vec{E}^{refl}$$
$$\vec{H}^{up} = \vec{H}^i + \vec{H}^{refl} \tag{1}$$

in the region Ω^+ and by

$$\vec{E}^{down} = \vec{E}^{refr}$$
$$\vec{H}^{down} = \vec{H}^{refr} \tag{2}$$

in the region Ω^-. In the case that Ω^+ is filled by a dielectric and Ω^- is filled by a perfect conductor, the electromagnetic field is given by (1) in Ω^+, and it vanishes in Ω^-.

Dropping the factor $e^{-i\omega t}$, the incident, diffracted and total fields satisfy the time harmonic Maxwell equations

$$\nabla \times \vec{E} = i\omega\mu_0\vec{H} \ , \ \nabla \cdot \vec{E} = 0 \ ,$$
$$\nabla \times \vec{H} = -i\omega\epsilon\vec{E} \ , \ \nabla \cdot \vec{H} = 0 \ , \tag{3}$$

where $\epsilon = \epsilon^{\pm}$ in the regions $\Omega^{\pm}$, respectively. Furthermore, across the interface the *total* field satisfies

$$n \times (E^{up} - E^{down}) = 0 \ , \ n \times (H^{up} - H^{down}) = 0 \ \text{on}\ y = f(x) \ , \tag{4}$$

where n is the unit normal vector to the interface.

The periodicity of the interface together with the form of the incident wave imply that the physical solutions $\vec{E}$ and $\vec{H}$ must be α quasi-periodic, i.e. $\vec{E}(x+L,y) = e^{i\alpha L}\vec{E}(x,y)$ and $\vec{H}(x+L,y) = e^{i\alpha L}\vec{H}(x,y)$. Furthermore, because the fields $\vec{E}$, $\vec{H}$ are independent of z, the system of equations (3),(4) can be reduced to pairs of equations for a single unknown (see [6]). Explicitly, if $\vec{E} = (E_x, E_y, E_z)$ and $\vec{H} = (H_x, H_y, H_z)$ solve equations (3), then the pairs of vector fields

$$(0,0,E_z), \quad (H_x, H_y, 0) \ \text{and}$$
$$(E_x, E_y, 0), \quad (0,0,H_z)$$

are also solutions. Moreover, the components of the fields in the xy-plane can be computed from the transverse components by using the relations

$$H_x = \frac{1}{i\omega\mu_0}\frac{\partial E_z}{\partial y} \ , \quad H_y = -\frac{1}{i\omega\mu_0}\frac{\partial E_z}{\partial x}$$

and

$$E_x = -\frac{1}{i\omega\epsilon}\frac{\partial H_z}{\partial y} \ , \quad E_y = \frac{1}{i\omega\epsilon}\frac{\partial H_z}{\partial x}.$$

Then, the α quasi-periodic function $u(x,y)$ equal to either E_z (Field Transverse Electric, TE) or H_z (Field Transverse Magnetic, TM), is easily seen to satisfy, in either case, the Helmholtz equation

$$\Delta u + (k^{\pm})^2 u = 0 \ , \ \text{in}\ \Omega^{\pm} , \tag{5}$$

where $k^{\pm} = \omega\sqrt{\mu_0\epsilon^{\pm}}$. The boundary conditions (4) are then translated into Dirichlet, Neumann, or transmission conditions for the unknown u in the following way:

(i) *Dielectric-Perfect Conductor interface: TE mode.*

Here $u = E_z^{refl}$ and the boundary conditions become

$$u = -e^{i\alpha x - i\beta f(x)} \quad , \quad \text{on } \; y = f(x). \tag{6}$$

(ii) *Dielectric-Perfect Conductor interface: TM mode.*

In this case $u = H_z^{refl}$ and the condition is

$$\frac{\partial u}{\partial n} = -\frac{\partial}{\partial n}(e^{i\alpha x - i\beta y}) \quad , \quad \text{on } \; y = f(x). \tag{7}$$

(iii) *Interface between two dielectrics: TE mode.*

We let $u^+ = E_z^{refl}$, $u^- = E_z^{refr}$ and the conditions at the interface become

$$\begin{aligned} u^+ - u^- &= -e^{i\alpha x - i\beta f(x)} \quad , \quad \text{on } \; y = f(x) \,, \\ \frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} &= -\frac{\partial}{\partial n}(e^{i\alpha x - i\beta y}) \quad , \quad \text{on } \; y = f(x). \end{aligned} \tag{8}$$

(iv) *Interface between two dielectrics: TM mode.*

Here u^+ and u^- are defined as in (iii) and the transmission conditions take the form

$$\begin{aligned} u^+ - u^- &= -e^{i\alpha x - i\beta f(x)} \quad , \quad \text{on } \; y = f(x) \,, \\ \frac{\partial u^+}{\partial n} - \frac{1}{\nu_0^2}\frac{\partial u^-}{\partial n} &= -\frac{\partial}{\partial n}(e^{i\alpha x - i\beta y}) \quad , \quad \text{on } \; y = f(x) \,, \end{aligned} \tag{9}$$

where

$$\nu_0^2 = \frac{\epsilon^-}{\epsilon^+} = \left(\frac{k^-}{k^+}\right)^2 . \tag{10}$$

2.2 The Rayleigh Expansion and the Radiation Condition

The Helmholtz equation (5) coupled with any one of the boundary conditions (6)-(9) are not sufficient to determine the fields due to the lack of conditions at $y = \infty$.

However, the physics of the problem imposes obvious conditions as $y \to \infty$, namely that the diffracted field u remain bounded and that it should be representable as a superposition of *outgoing waves.*

To make this statement more precise, set

$$K = \frac{2\pi}{L} \ , \ \alpha_n = \alpha + nK \ , \ \alpha_n^2 + (\beta_n^\pm)^2 = (k^\pm)^2 \, , \tag{11}$$

where $\beta_n^\pm$ is determined by $\Im(\beta_n^\pm) > 0$ or $\beta_n^\pm \geq 0$. Any quasi-periodic solution of the Helmholtz equation must be a superposition of waves, as indicated by the following elementary lemma.

Lemma 1 (Rayleigh Expansion)

Assume that

$$k^+ \neq \pm(\alpha + nK) \ , \ k^- \neq \pm(\alpha + nK) \tag{12}$$

for all integers n. Then any α quasi-periodic solution to the Helmholtz equation in the region Ω^+ is given, for $y > y_M = \max f$, by

$$u^+ = \sum_{n=-\infty}^{\infty} A_n^+ e^{i\alpha_n x - i\beta_n^+ y} + \sum_{n=-\infty}^{\infty} B_n^+ e^{i\alpha_n x + i\beta_n^+ y}. \tag{13}$$

Analogously, any solution u^- to the Helmholtz equation in Ω^- is given, for $y < y_m =$ $\min f$, *by*

$$u^- = \sum_{n=-\infty}^{\infty} A_n^- e^{i\alpha_n x - i\beta_n^- y} + \sum_{n=-\infty}^{\infty} B_n^- e^{i\alpha_n x + i\beta_n^- y}. \tag{14}$$

The case $k^\pm = \pm(\alpha + nK)$, which corresponds to physically anomalous behavior first observed by Wood [16,11], will not be considered here as we shall always assume relation (12) to hold.

Now, the physical conditions lead us to the following definition [9].

Definition 1 *A solution u of (5) in $\Omega^{\pm}$ is said to verify a radiation condition at infinity if*

$$A_n^+ = 0 \text{ for all } n \ , \ B_n^- = 0 \text{ for all } n. \tag{15}$$

3 A formula for the solution

Besides its theoretical interest, our investigation of the properties of analyticity of u is motivated by its application to the numerical computation of the solution. In this section we describe the straightforward algorithm for the evaluation of the solution that results from the variation of the boundary. We start by considering diffraction problems for gratings with variable "height". The analytic dependence of the diffracted field on the height of the grating allows us to expand it in a power series whose coefficients can be computed by solving (recursively) diffraction problems for a flat interface (i.e. "zero height"). The fact that these coefficients can indeed be computed in this manner is a consequence of, as well as a further motivation for proving, the results in sections 4-7.

Let us first consider the TE mode of polarization for the diffraction problem in the case in which Ω^- is filled by a perfect conductor, i.e. problem (5),(6). Let $f(x)$ be an L-periodic function and set

$$f_\delta(x) = \delta f(x).$$

Then if $u(x, y, \delta)$, $y > f_\delta(x)$, denotes the reflected field for the grating $f_\delta(x)$, we have from (6)

$$u(x, \delta f(x), \delta) = -u_i(x, \delta f(x)) = -e^{i\alpha x - i\beta\delta f(x)}. \tag{16}$$

In order to represent the solution as a power series in δ around $\delta = 0$, we shall compute the successive *total* δ-derivatives of the equation (16) and recursively find the *partial* δ-derivatives of u at $y = 0$, $\delta = 0$. From (16) we obtain

$$\frac{1}{n!}\frac{\partial^n u}{\partial \delta^n}(x,0,0) = -\frac{(-i\beta)^n}{n!} f(x)^n e^{i\alpha x} - \sum_{k=0}^{n-1} \frac{f(x)^{n-k}}{(n-k)!}\frac{\partial^{n-k}}{\partial y^{n-k}}\left(\frac{1}{k!}\frac{\partial^k u}{\partial \delta^k}\right)(x,0,0). \quad (17)$$

Once these boundary values are known, the function

$$\frac{1}{n!}\frac{\partial^n u}{\partial \delta^n}(x,y,0) \quad (18)$$

is readily obtained using the fact that it is an α quasi-periodic solution of Helmholtz equation in $\{y > 0\}$ (consisting of outgoing waves). Thus, its partial y-derivatives can be computed and we can proceed to evaluate the successive δ-derivatives using the relation (17). The problem for the grating f_δ can therefore be handled by solving diffraction problems in the upper half-plane, the solution of which is immediate (e.g. the representation (13) is valid up to the boundary).

For the case of a perfect conductor in TM mode (Neumann boundary conditions) equation (17) is replaced by

$$\begin{aligned} \frac{\partial}{\partial y}\left(\frac{1}{n!}\frac{\partial^n u}{\partial \delta^n}\right)(x,0,0) &= -\frac{1}{n!}\left[i\alpha n(-i\beta)^{n-1}(f'(x)f(x)^{n-1}) - (-i\beta)^{n+1}f(x)^n\right]e^{i\alpha x} \\ &+ \sum_{k=0}^{n-1}\frac{(f'(x)f(x)^{(n-k)-1})}{(n-k-1)!}\frac{\partial^{n-k}}{\partial x \partial y^{n-k-1}}\left(\frac{1}{k!}\frac{\partial^k u}{\partial \delta^k}\right)(x,0,0) \\ &- \sum_{k=0}^{n-1}\frac{f(x)^{n-k}}{(n-k)!}\frac{\partial^{n-k+1}}{\partial y^{n-k+1}}\left(\frac{1}{k!}\frac{\partial^k u}{\partial \delta^k}\right)(x,0,0), \end{aligned} \quad (19)$$

which follows upon differentiation of equation (7) n times with respect to δ. Analogous relations can be derived from (8) and (9) to treat the case of two dielectrics (cf. (67), (68)).

As can be seen from formulas (17) and (19), the implementation of a numerical method based on variation of the boundary is extremely simple. A future paper will address the convergence properties of the induced numerical scheme.

4 The fundamental solution and its dependence on the parameters of the problem

It is well known (see e.g. [12] Ch. 4) that a fundamental solution for the two-dimensional Helmoltz operator $\Delta + k^2$ is given by

$$\Phi_k(x,y) = \frac{i}{4} H_0^{(1)}(k|(x,y)|)$$

where $H_0^{(1)}(z)$ is the first Hankel function of order zero and $|(x,y)| = \sqrt{x^2+y^2}$. Thus [13,3], an α quasi-periodic fundamental solution is obtained by the formula

$$G_k(x,y) = \frac{i}{4} \sum_{n=-\infty}^{\infty} e^{-i\alpha n L} H_0^{(1)}(k|(x+nL,y)|). \tag{20}$$

This section is devoted to the study of the convergence properties of the series (20). The results that follow will be used in section 6 to deal with the operators involved in the integral formulation of the problem, when viewed as operators between spaces of holomorphic functions.

Let us define a regular part of G_k as the series

$$R_k(x,y) = \frac{i}{4} \sum_{n \neq -1,0,1} e^{-i\alpha n L} H_0^{(1)}(k|(x+nL,y)|). \tag{21}$$

whose terms are well-defined for $(x,y) \in \mathcal{B} \equiv \{ (x,y) \in \mathbb{C}^2 : -L-\gamma \leq \Re(x) \leq L+\gamma, |\Im(x)| < \frac{L}{2}, |\Im(y)| < \frac{L}{2} \}$ for some small $\gamma > 0$.

Theorem 1 *The series in (21) converges uniformly for (x,y,k) in compact subsets of $\mathcal{B} \times \mathcal{K}$ where*

$$\mathcal{K} \equiv \{ k \in \mathbb{C} : \Im(k) \geq 0, k \neq 0 \text{ and } k \neq \pm\alpha_n, \forall n \}. \tag{22}$$

In order to prove Theorem 1 we need the following lemma. Let s and μ be complex valued functions of bounded variation defined on the interval $[0,1] \subset \mathbb{R}$ such that

$$s(t)\,,\ \mu(t) \to 0 \text{ as } t \to 0.$$

Write

$$\begin{aligned} s(t) &= a(t) + b(t) \\ \mu(t) &= c(t) + d(t) \end{aligned} \tag{23}$$

where the real and imaginary parts of a, b, c and d are monotone functions of t and

$$\begin{aligned} &\Re(a)\,,\ \Im(a) \uparrow 0 \quad , \quad \Re(b)\,,\ \Im(b) \downarrow 0\,, \\ &\Re(c)\,,\ \Im(c) \uparrow 0 \quad , \quad \Re(d)\,,\ \Im(d) \downarrow 0\,. \end{aligned} \tag{24}$$

Lemma 2 *Let*

$$r_n = (nA - r) + s(\frac{1}{n}) \tag{25}$$

with $\Im(A) \geq 0$, $A \neq 2\pi l$ $(l \in \mathbb{Z})$, $r \in \mathbb{C}$ and s defined as above. Then,

(i) *there exists a constant C such that*

$$|\sum_{n=1}^{N} e^{ir_n}| \leq C \quad \textit{for all } N \geq 1\,.$$

(ii) *If $\mu_n = \mu(\frac{1}{n})$ then the series*

$$\sum_{n=1}^{\infty} \mu_n e^{ir_n} \tag{26}$$

converges.

(iii) *If s and μ depend on an additional parameter $\lambda \in \mathbb{R}^p$ and the convergence in (24) is uniform for $\lambda \in I \subset \mathbb{R}^p$, then the convergence of the series (26) is uniform for $\lambda \in I$.*

Proof: Because of (24) we may assume, without loss of generality, that the real and imaginary parts of a, b, c and d lie in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We shall show that *if*

$$|\sum_{n=1}^{N} e^{i(An-r)+\gamma_n}| \leq M \text{ for all } N$$

and $\epsilon_n \in \mathbf{R}$ or $i\epsilon_n \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\epsilon_n \to 0$ monotonically, *then*

$$|\sum_{n=1}^{N} e^{i(An-r)+\gamma_n+\epsilon_n}| \leq 3\, M \max(1, |e^{\epsilon_1}|) \text{ for all } N. \tag{27}$$

Repeated application of this fact together with the decomposition (23) yield (i), since one has

$$|\sum_{n=1}^{N} e^{i(An-r)}| \leq \frac{2e^{\Im(r)}}{1-e^{iA}}.$$

To establish (27) we use Abel's summation formula

$$\sum_{n=1}^{N} \kappa_n \mu_n = \sum_{n=1}^{N-1} (\sum_{j=1}^{n} \kappa_j)(\mu_n - \mu_{n+1}) + \mu_N \sum_{j=1}^{N} \kappa_j\,, \tag{28}$$

and compute

$$\begin{aligned} D_N &\equiv \sum_{n=1}^{N} e^{i(An-r)+\gamma_n+\epsilon_n} = \sum_{n=1}^{N} \Gamma_n e^{\epsilon_n} \\ &= \sum_{n=1}^{N-1} (\sum_{j=1}^{n} \Gamma_j)(e^{\epsilon_n} - e^{\epsilon_{n+1}}) + e^{\epsilon_N} \sum_{j=1}^{N} \Gamma_j \end{aligned}$$

where $\Gamma_n \equiv e^{i(An-r)+\gamma_n}$. The real and imaginary parts of the D_N's are sums or differences of quantities of the form

$$\sum_{n=1}^{N-1} P(\sum_{j=1}^{N} \Gamma_j)(g(\epsilon_n) - g(\epsilon_{n+1})) + g(\epsilon_N) P(\sum_{j=1}^{N} \Gamma_j)\,, \tag{29}$$

where $P(z)$ stands for the real or imaginary parts of z, and $g(z)$ is either $\sin(\Im(z))$ or $\cos(\Im(z))$ if z is purely imaginary, and it equals e^z if z is real. The absolute value of (29) is less than

$$\sum_{n=1}^{N-1} M |g(\epsilon_n) - g(\epsilon_{n+1})| + |g(\epsilon_N)| M.$$

Since $\epsilon_n \in \mathbf{R}$ or $i\epsilon_n \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the sequence $\{g(\epsilon_n)\}$ is monotone, and the last quantity equals

$$M(|g(\epsilon_1) - g(\epsilon_N)| + |g(\epsilon_N)|).$$

This, in turn, is smaller than

$$3M \max_n |g(\epsilon_n)|$$

and (27) follows. Statement (ii) follows from (i) and (28) by considering the decomposition (23). Finally, under the assumptions in (iii) the above argument is easily seen to be uniform in λ. The proof of the lemma is now complete.

Remark 1 : *The hypothesis on s and μ in the previous lemma cannot be dropped. For example, the series*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e^{ir_n}$$

with $r_n = n\pi + \frac{(-1)^n + 1}{n^{\frac{1}{4}}}$ does not converge.

We now turn to the proof of the theorem.

Proof of Theorem 1: The proof is based on Lemma 2 and the following asymptotic formula (see [14];p.197):

$$H_0^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z-\frac{\pi}{4})} \left(1 + O\left(\frac{1}{z}\right)\right) \quad \text{as } z \to \infty. \tag{30}$$

We shall only prove that the part of the series in (21) with $n \geq 2$ converges uniformly for (x, y, k) in compact subsets of $B \times K$, since the part with $n \leq -2$ is handled in a similar way. Furthermore, by (30) it suffices to show that the series

$$\sum_{n=2}^{\infty} \frac{e^{i(k\rho_n - \alpha n L)}}{\sqrt{k\rho_n}}$$

converges uniformly, where $\rho_n = ((x + nL)^2 + y^2)^{\frac{1}{2}}$.

Define

$$\rho(t) = ((x + \frac{L}{t})^2 + y^2)^{\frac{1}{2}},$$

and notice that $\rho(t)$, as well as $\rho(t)^{1/2}$ are well defined in B for $t \in (0, \frac{1}{2}]$ (provided γ is small enough).

Now, the functions

$$s(x,y,k,t) \equiv k\rho(t) - k(x + \frac{L}{t}) = \frac{ky^2}{((x+\frac{L}{t})^2 + y^2)^{\frac{1}{2}} + (x + \frac{L}{t})}$$

and

$$\mu(x,y,k,t) = \frac{1}{\sqrt{k\rho(t)}}$$

are smooth functions of t and vanish for $t = 0$. In particular, they are functions of bounded variation. Furthermore, we have

$$k\rho_n - \alpha n L = nL(k - \alpha) + kx + s(\frac{1}{n})$$

and the condition $k \neq \pm\alpha_n$ tells us that $A = L(k - \alpha) \notin 2\pi\mathbb{Z}$. Therefore Lemma 2 can be applied, and the theorem follows. .

It is well known [9] that a fundamental solution of (5) is given by

$$\frac{i}{2L} \sum_{n=-\infty}^{\infty} \frac{e^{i\beta_n|y|}}{\beta_n} e^{i\alpha_n x}.$$

To end this section we show that this fundamental solution is equal to G_k (see also [3]).

Theorem 2 *Let $(nL, 0) \neq (x,y) \in \mathbb{R}^2$ $(n \in \mathbb{Z})$ and let $k \in K$. Then,*

$$G_k(x,y) = \frac{i}{2L} \sum_{n=-\infty}^{\infty} \frac{e^{i\beta_n|y|}}{\beta_n} e^{i\alpha_n x}. \tag{31}$$

The series in (31) converges uniformly in compact sets but it *cannot* be termwise differentiated with respect to y at $y = 0$.

Proof: First assume that $\Im(k) > 0$. Then, by (30), if $k = a + ib$

$$\begin{aligned} H_0^{(1)}(k\rho_n) &= \left(\frac{2}{\pi k\rho_n}\right)^{\frac{1}{2}} e^{i(k\rho_n - \frac{\pi}{4})}\left(1 + O\left(\frac{1}{k\rho_n}\right)\right) \\ &= e^{-b\rho_n} f(k\rho_n) \end{aligned}$$

where $|f(k\rho_n)| < 1$ for $0 \leq x \leq L$ provided $|y|$ is sufficiently large. It follows that

$$|G_k(x,y)| \to 0 \text{ as } y \to \pm\infty \text{ uniformly for } x \in [0,L]. \tag{32}$$

Let $y \neq 0$ and consider

$$F_k(x,y) = e^{-i\alpha x} G_k(x,y)$$

which is a periodic function of x. Its n-th Fourier coefficient is then given by

$$a_n(y) = \frac{1}{L}\int_0^L e^{-i\alpha x} G_k(x,y) e^{-in\frac{2\pi}{L}x}\, dx.$$

Since G_k is a fundamental solution of (5) it follows that a_n satisfies

$$a_n'' + \beta_n^2 a_n = -\frac{1}{L}\delta(y) \tag{33}$$

where $\delta(\cdot)$ is the Dirac measure. In particular using (32), we conclude

$$a_n(y) = \begin{cases} a_n^+ e^{i\beta_n y} & \text{if } y > 0, \\ a_n^- e^{-i\beta_n y} & \text{if } y < 0 \end{cases}$$

for some constants a_n^+ and a_n^-. Now equation (33) can only be satisfied if

$$a_n^+ = a_n^- = \frac{i}{2\beta_n L},$$

thereby completing the proof for the case $\Im(k) > 0$.

Since $\frac{x}{L} \notin \mathbb{Z}$ we can apply Lemma 2 to conclude that the series in the right hand side of (31) converges uniformly on compact subsets of $\{\frac{x}{L} \notin \mathbb{Z},\ y \in \mathbb{R},$

$k \neq 0$, $\Im(k) \geq 0$, $k \neq \pm\alpha_n\}$ and is therefore a continuous function of (x, y, k). Since we know that for $\Im(k) > 0$ the equality in (31) holds, and since G_k is also continuous in k, it follows that the equality remains valid for $\Im(k) = 0$ (away from the "Wood anomalies", $k = \pm\alpha_n$).

5 Uniqueness results

Several uniqueness results for the problem of diffraction by a periodic grating are known, but the list of such results is still, regrettably, incomplete [15]. Uniqueness for the TE case of polarization when one of the materials is perfectly conducting was proved in [1,2], while the result for the TM mode ($k^+ \in \mathbf{R}$) can be found in [2]. This last paper also contains a proof of uniqueness for the diffracted field between two dielectrics, in TE mode with $k^\pm \in \mathbf{R}$.

In this section we present an additional elementary uniqueness result. We shall show that equation (5) with condition (8) or (9) admits a unique solution verifying the radiation condition (15), provided the imaginary part of one of the dielectric constants is positive. We will treat both the TE and TM modes simultaneously, and for this we write (8) and (9) as the single pair of conditions

$$\begin{aligned} u^+ - u^- &= F_1(x) \;\;,\;\; \text{on } \; y = f(x)\,, \\ \frac{\partial u^+}{\partial n} - C_0 \frac{\partial u^-}{\partial n} &= F_2(x) \;\;,\;\; \text{on } \; y = f(x)\,, \end{aligned} \tag{34}$$

where

$$F_1 = -e^{i\alpha x - i\beta f(x)} \;\;,\;\; F_2 = -\frac{\partial}{\partial n}(e^{i\alpha x - i\beta y})\Big|_{y=f(x)} \;\;, \tag{35}$$

and

$$C_0 = \begin{cases} 1 & \text{for TE polarization,} \\ \frac{1}{\nu_0^2} & \text{for TM polarization.} \end{cases} \tag{36}$$

We shall further write

$$k = k^+ \text{ so that } k^- = k\nu_0. \tag{37}$$

and assume that

$$k \text{ is real and } \nu_0^2 \text{ has positive imaginary part.} \tag{38}$$

Since ν_0^2 has a positive imaginary part, it is clear that the expansion (14) for u^- contains only evanescent modes. In other words, if

$$U^\pm = \{n : \beta_n^\pm > 0\}$$

then U^- is empty.

We will need the following two lemmas (see also [1,2]), the first of which results from a simple calculation.

Lemma 3 *Assume u is a solution to (5) verifying the radiation conditions as in Definition 1. Then, if $l > y_M$, we have*

$$\frac{1}{2iL} \left(\int_0^L \overline{u^+}(x,l)\frac{\partial u^+}{\partial y}(x,l)dx - \int_0^L u^+(x,l)\overline{\frac{\partial u^+}{\partial y}}(x,l)dx \right) = \sum_{n\in U^+} \beta_n^+ \mid B_n^+ \mid^2 . \tag{39}$$

Lemma 4 *Let $u = u^\pm$ be a solution to (5) satisfying (15) and (34) with $F_1 = F_2 = 0$. Then $B_n^+ = 0$ for $n \in U^+$.*

Proof: From (5) we have

$$-2ik^2\Im(C_0\nu_0^2)\int_{\Omega^-\cap\{|y|<l\}} \mid u^- \mid^2 dx \;=$$

$$\int_{\Omega^+\cap\{|y|<l\}} (\overline{u}^+\Delta u^+ - u^+\overline{\Delta u^+})dx \;+\; \int_{\Omega^-\cap\{|y|<l\}} (C_0\overline{u}^-\Delta u^- - \overline{C_0}u^-\overline{\Delta u^-})dx.$$

Because of the boundary conditions in (34), integration by parts shows that this equality now yields, for l large enough,

$$-2ik^2\Im(C_0\nu_0^2)\int_{\Omega^-\cap\{|y|<l\}} | u^- |^2\, dx \;=$$

$$\left[\int_0^L (\overline{u^+}\frac{\partial u^+}{\partial y} - u^+\overline{\frac{\partial u^+}{\partial y}})dx\right]_{y=l} \;-\; \left[\int_0^L (C_0\overline{u^-}\frac{\partial u^-}{\partial y} - \overline{C_0}u^-\overline{\frac{\partial u^-}{\partial y}})dx\right]_{y=-l}.$$

As $l \to \infty$, the last integral drops out, since u^- decays to zero. By the previous lemma the first integral equals $2iL\sum_{n\in U^+}\beta_n^+ | B_n^+ |^2$, so that, as $l \to \infty$ we obtain

$$-2ik^2\Im(C_0\nu_0^2)\int_{\Omega^-} | u^- |^2\, dx = 2iL\sum_{n\in U^+}\beta_n^+ | B_n^+ |^2 .$$

Since β_n^+ is positive for $n \in U^+$, we conclude that $B_n^+ = 0$ for $n \in U^+$ as claimed.

Finally we establish the following uniqueness result.

Theorem 3 *Let $u^\pm$ be a solution of (5) (with $\Im((k^-)^2) > 0$) satisfying the radiation conditions and the transmission conditions (34) with $F_1 = F_2 = 0$. Then $u \equiv 0$.*

Proof: We first show that $u^- = 0$. From Lemma 4 and the fact that $\Im\beta_n^- > 0$ we know that $u^\pm$ and $\frac{\partial u^\pm}{\partial y}$ tend to zero exponentially as $y \to \pm\infty$. Then, integration by parts in Ω^+ and Ω^- yield, respectively,

$$\int_{y=f(x)} \overline{u^+}\frac{\partial u^+}{\partial n}d\sigma = k^2\int_{\Omega^+} | u^+ |^2\, dxdy - \int_{\Omega^+} | \nabla u^+ |^2\, dxdy, \tag{40}$$

and

$$C_0\int_{y=f(x)} \overline{u^-}\frac{\partial u^-}{\partial n}d\sigma = C_0\int_{\Omega^-} | \nabla u^- |^2\, dxdy - C_0k^2\nu_0^2\int_{\Omega^-} | u^- |^2\, dxdy. \tag{41}$$

Using the boundary conditions on $y = f(x)$, we see that the left hand sides in (40) and (41) coincide, so that we have

$$k^2\int_{\Omega^+} | u^+ |^2\, dxdy - \int_{\Omega^+} | \nabla u^+ |^2\, dxdy \;=$$

$$C_0\int_{\Omega^-} | \nabla u^- |^2\, dxdy \;-\; C_0\nu_0^2k^2\int_{\Omega^-} | u^- |^2\, dxdy. \tag{42}$$

The imaginary part of the right hand side in (42) equals

$$-\Im(k^2\nu_0^2)\int_{\Omega^-} | u^- |^2 \, dxdy \quad \text{if} \quad C_0 = 1$$

and

$$\Im(\frac{1}{\nu_0^2})\int_{\Omega^-} | \nabla u^- |^2 \, dxdy \quad \text{if} \quad C_0 = \frac{1}{\nu_0^2}.$$

Since the left hand side in (42) is real and the imaginary part of ν_0^2 is positive, we conclude, in any case, that u^- is constant. Since u^- decays to zero as $y \to -\infty$, we see that

$$u^- = 0. \tag{43}$$

Finally, it follows from (43) and the boundary conditions in (34) that

$$u^+ = \frac{\partial u^+}{\partial n} = 0 \text{ on } y = f(x)$$

and therefore, $u^+ = 0$.

6 Complex analytic formulation via surface potentials

In this section, we establish that the operator that results from the potential theoretic formulation of the diffraction problem is compact in a space of analytic functions. This will be used in the next section to show that, given an L-periodic and analytic interface $f(x)$, the solution $u(x, y, \delta)$ corresponding to the grating $\delta f(x)$ can be extended holomorphically to complex values of (x, y, δ).

In subsection 6.1 we study the kernels entering the integral equations. In 6.2 we show that the logarithmic branch point in the kernels is not an obstacle for a holomorphic formulation of the problem. This complex analytic setup together with the main compactness result are presented in subsection 6.3. For the sake of brevity, we will deal only with the case of two dielectric materials, since the simpler case of a dielectric and a perfect conductor can be treated in a similar way.

6.1 Integral equations

We seek a solution $u^\pm$ in the form

$$\begin{aligned} u^+(x,y,\delta) &= \mathcal{S}_\delta^+(\mu) + \mathcal{D}_\delta^+(\eta), \\ u^-(x,y,\delta) &= \mathcal{S}_\delta^-(\mu) + \frac{1}{C_0}\mathcal{D}_\delta^-(\eta) \end{aligned} \tag{44}$$

where the operators $\mathcal{S}_\delta^\pm$ and $\mathcal{D}_\delta^\pm$ are defined by

$$\mathcal{S}_\delta^\pm(\mu)(x,y) \equiv \int_{\mathcal{P}_\delta} G_{k^\pm}((x,y)-Q)\mu(Q)\,d\sigma(Q)$$

and

$$\mathcal{D}_\delta^\pm(\eta)(x,y) \equiv \int_{\mathcal{P}_\delta} \frac{\partial G_{k^\pm}}{\partial n_\delta(Q)}((x,y)-Q)\eta(Q)\,d\sigma(Q).$$

Here,

$$\mathcal{P}_\delta = \{\,(x,\delta f(x)) \,:\, 0 \le x \le L\,\}$$

and

$$n_\delta(Q) = \text{unit normal to } \mathcal{P}_\delta \text{ at } Q \text{ directed towards } \Omega^+,$$

that is, if $Q = (x', \delta f(x'))$,

$$n_\delta(Q) = \frac{1}{(1+(\delta f'(x'))^2)^{1/2}}(-\delta f'(x'), 1).$$

Notice that, by Theorem 2, $u^\pm$ as defined by (44) satisfy the radiation conditions (15). The factor of $\frac{1}{C_0}$ in (44) was introduced to produce a cancellation in formula (51) below.

From (20) and (53) it follows that the fundamental solution G_k is given by

$$G_k(x,y) = -\frac{1}{2\pi}\log(k|(x,y)|) + S_k(x,y)$$

where S_k is continuous together with its gradient for $0 < x < L$. Therefore, the transmission conditions (34) and the well known *jump relations* for the logarithmic potentials imply

$$(\frac{1}{2}I - \mathcal{A}_\delta)(g)(x) = F_\delta(x),\ 0 \le x \le L, \tag{45}$$

where

$$\mathcal{A}_\delta = \begin{pmatrix} \frac{C_0}{C_0+1}(\frac{1}{C_0}\mathcal{D}_\delta^- - \mathcal{D}_\delta^+) & \frac{C_0}{C_0+1}(\mathcal{S}_\delta^- - \mathcal{S}_\delta^+) \\ -\frac{1}{C_0+1}\mathcal{T}_\delta & -\frac{1}{C_0+1}(C_0\mathcal{R}_\delta^- - \mathcal{R}_\delta^+) \end{pmatrix}, \tag{46}$$

$$g(x') = \begin{pmatrix} \eta(x') \\ \mu(x') \end{pmatrix}$$

and

$$F_\delta(x) = \frac{1}{C_0+1}\begin{pmatrix} -C_0 e^{i(\alpha x - \beta\delta f(x))} \\ -i\frac{\alpha\delta f'(x)+\beta}{(1+(\delta f'(x))^2)^{1/2}} e^{i(\alpha x - \beta\delta f(x))} \end{pmatrix}. \tag{47}$$

In (46) the operators $\mathcal{S}_\delta^\pm$, $\mathcal{D}_\delta^\pm$, $\mathcal{R}_\delta^\pm$ and $\mathcal{T}_\delta$ are defined by

$$\mathcal{S}_\delta^\pm(\mu)(x) = \int_0^L G_{k\pm}(x - x', \delta(f(x) - f(x')))(1 + (\delta f'(x'))^2)^{1/2}\mu(x')\,dx', \tag{48}$$

$$\begin{aligned} \mathcal{D}_\delta^\pm(\eta)(x) &= \int_0^L \Big[\delta f'(x')\frac{\partial G_{k\pm}}{\partial x}(x - x', \delta(f(x) - f(x'))) \\ &- \frac{\partial G_{k\pm}}{\partial y}(x - x', \delta(f(x) - f(x')))\Big]\eta(x')\,dx', \end{aligned} \tag{49}$$

$$\begin{aligned} \mathcal{R}_\delta^\pm(\mu)(x) &= \int_0^L \Big[-\delta f'(x)\frac{\partial G_{k\pm}}{\partial x}(x - x', \delta(f(x) - f(x'))) \\ &+ \frac{\partial G_{k\pm}}{\partial y}(x - x', \delta(f(x) - f(x')))\Big]\frac{(1 + (\delta f'(x'))^2)^{1/2}}{(1 + (\delta f'(x))^2)^{1/2}}\mu(x')\,dx' \end{aligned} \tag{50}$$

and

$$\mathcal{T}_\delta(\eta)(x) = \int_0^L \Big[-\delta^2 f'(x)f'(x')\frac{\partial^2(G_{k-} - G_{k+})}{\partial x^2}(x - x', \delta(f(x) - f(x'))) \tag{51}$$

$$+ \quad \delta(f'(x)+f'(x'))\frac{\partial^2(G_{k^-}-G_{k^+})}{\partial x\partial y}(x-x',\delta(f(x)-f(x')))$$

$$- \quad \frac{\partial^2(G_{k^-}-G_{k^+})}{\partial y^2}(x-x',\delta(f(x)-f(x')))\Big]\frac{1}{(1+(\delta f'(x))^2)^{1/2}}\eta(x')\,dx'.$$

Regarding the form of these operators, we shall now prove

Theorem 4 *Let $\delta_0 \in \mathbf{R}$. Then there exist $\nu > 0$, $\epsilon > 0$ and $\gamma > 0$ such that the kernels $\kappa(x,x',\delta)$ of the operators in (48)-(51) have the form*

$$\kappa(x,x',\delta) = \sum_{n=-1}^{1} e^{-i\alpha nL}\log|x-x'+nL|B(x,x'-nL,\delta) + C(x,x',\delta) \qquad (52)$$

for $(x,x',\delta) \in \mathcal{E} \cap \{\Im(x)=0\} \cap \{\Im(x')=0\} \cap \{\Im(\delta)=0\}$ where $B(x,x',\delta)$ is an analytic function in $\{\,|\Im(x)|<\epsilon,\ |\Im(x')|<\epsilon,\ |\delta-\delta_0|<\nu\,\}$ and $C(x,x',\delta)$ is an analytic function in $\mathcal{E}$. Here we have put $\mathcal{E} \equiv \{\,|\Im(x)|<\epsilon,\ |\Im(x')|<\epsilon,\ |\delta-\delta_0|<\nu,\ -\gamma<\Re(x)<L+\gamma,\ -\gamma<\Re(x')<L+\gamma\,\}$.

Proof: We first recall the formula (see e.g. [14];pp.40,60,64,73)

$$H_0^{(1)}(z) = J_0(z) + iY_0(z) \qquad (53)$$

where J_0 and Y_0 are the Bessel functions

$$J_0(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^{2m}$$

and

$$\pi Y_0(z) = 2\left(\gamma_0+\log\left(\frac{z}{2}\right)\right)J_0(z) - 2\sum_{m=1}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^{2m}\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right).$$

Thus, in particular, we can write

$$H_0^{(1)}(z) = h_1(z^2) + h_2(z^2)\log(z) = h_1(z^2) + (\gamma_0 + z^2h_3(z^2))\log(z)$$

for some entire functions h_1, h_2 and h_3.

Then, from (20),(21), we obtain

$$
\begin{aligned}
G_k(x-x' \quad , \quad y-y') &= R_k(x-x',y-y') \qquad\qquad (54)\\
&+ \frac{i}{4}\sum_{n=-1}^{1} e^{-i\alpha nL} H_0^{(1)}(k|(x-x'+nL,y-y')|)\\
&= R_k(x-x',y-y') + \frac{i}{4}\sum_{n=-1}^{1} e^{-i\alpha nL}(h_1(k^2\rho_n^2) + h_2(k^2\rho_n^2)\log(k\rho_n))
\end{aligned}
$$

where $\rho_n = ((x-x'+nL)^2 + (y-y')^2)^{1/2}$. Thus, if we set

$$
\begin{aligned}
a(x-x',y-y') &\equiv R_k(x-x',y-y') + \frac{i}{4}\Big(\sum_{n=-1}^{1} e^{-i\alpha nL} h_1(k^2\rho_n^2)\\
&+ \sum_{n=-1}^{1} e^{-i\alpha nL} h_2(k^2\rho_n^2)\log(k)\Big) \ ,
\end{aligned}
$$

then (54) becomes

$$
\begin{aligned}
G_k(x-x',y-y') &= a(x-x',y-y') + \frac{i}{4}\sum_{n=-1}^{1} e^{-i\alpha nL}(\gamma_0 + (k^2\rho_n^2)h_3(k^2\rho_n^2))\log(\rho_n)\\
&= a(x-x',y-y') + \frac{i}{4}\sum_{n=-1}^{1} e^{-i\alpha nL}(\gamma_0 + (k^2\rho_n^2)h_3(k^2\rho_n^2))\,\cdot\\
&\Big(\log|x-x'+nL| + \frac{1}{2}\log(1+(\frac{y-y'}{x-x'+nL})^2)\Big).
\end{aligned}
$$

Using Theorem 1, we see that the function a and its derivatives evaluated at $y = \delta f(x)$, $y' = \delta f(x')$ are analytic in the set $\mathcal{E}$. Therefore, it now suffices to show that the kernels of the operators in (48)-(51) with $G_{k^\pm}(x-x',y-y')$ replaced by

$$
W_{k^\pm}(x-x' \quad , \quad y-y') = \sum_{n=-1}^{1} e^{-i\alpha nL}(\gamma_0 + ((k^\pm)^2\rho_n^2)h_3((k^\pm)^2\rho_n^2))\log(\rho_n)
$$

$$= \sum_{n=-1}^{1} e^{-i\alpha nL}(\gamma_0 + ((k^{\pm})^2\rho_n^2)h_3((k^{\pm})^2\rho_n^2))\Big(\log|x - x' + nL|$$

$$+ \frac{1}{2}\log(1 + (\frac{y - y'}{x - x' + nL})^2)\Big) \tag{55}$$

are of the form (52).

Upon replacing y by $\delta f(x)$ and y' by $\delta f(x')$ in (55), we immediately obtain the desired result for the kernels of the single layer potentials $\mathcal{S}_\delta^{\pm}$ provided ν, ϵ and γ are sufficiently small. In order to handle the remaining kernels, we first set

$$\mathcal{N}_\delta \equiv (f'(x)\delta\frac{\partial}{\partial x} - \frac{\partial}{\partial y}),$$

$$\mathcal{N}_\delta' \equiv (f'(x')\delta\frac{\partial}{\partial x} - \frac{\partial}{\partial y})$$

and observe that the functions

$$\mathcal{N}_\delta(\log(\rho_n))$$

and

$$\mathcal{N}_\delta'(\log(\rho_n))$$

evaluated at $(y, y') = (\delta f(x), \delta f(x'))$ are analytic in (x, x', δ). Thus, the parts of the kernels in $\mathcal{D}_\delta^{\pm}$ and $\mathcal{R}_\delta^{\pm}$ that arise from the term $\gamma_0 \log(\rho_n)$ in (55) are analytic; the corresponding part in the expression for $\mathcal{T}_\delta$ cancels out. (The factor $\frac{1}{C_0}$ in (44) was introduced precisely to produce this last cancellation).

Finally we must consider the terms of the form

$$\rho_n^2 h_3((k^{\pm})^2\rho_n^2)\log(\rho_n) \tag{56}$$

in (55). Since

$$\mathcal{N}_\delta'\Big(\rho_n^2 \quad h_3((k^{\pm})^2\rho_n^2) \quad \log(\rho_n)\Big) = \mathcal{N}_\delta'\Big(\rho_n^2\Big)h_3((k^{\pm})^2\rho_n^2)\log(\rho_n) +$$

$$+ \qquad \rho_n^2 \mathcal{N}_\delta'\Big(h_3((k^\pm)^2\rho_n^2)\Big)\log(\rho_n) + \rho_n^2 h_3((k^\pm)^2\rho_n^2)\mathcal{N}_\delta'\Big(\log(\rho_n)\Big) \tag{57}$$

it follows that the kernels defining $\mathcal{D}_\delta^\pm$ are of the form (52). The corresponding result for the kernels in $\mathcal{R}_\delta^\pm$ follows analogously.

Finally, applying the operator $\mathcal{N}_\delta$ to (57), it is easily checked that the terms in the kernel for $\mathcal{T}_\delta$ arising from (56) are also of the form (52). All contributions have now been considered and the theorem follows.

6.2 Continuation of logarithmic integrals

Here we establish that integrals with kernels of the type (52) define complex analytic functions of the spatial variable x and the height parameter δ. A closely related observation was used by Millar [8] in his study of the validity of the Rayleigh hypothesis.

Lemma 5 *Let $\eta(x', \delta)$ be an α quasi-periodic function of x' which is complex analytic in $\mathcal{U} \times \mathcal{V}$ where $\mathcal{U} = \{\, x' \in \mathbb{C} : |\Im(x')| < \epsilon \,\}$ and $\mathcal{V}$ is a complex neighborhood of $\delta = \delta_0$. Let $B(x, x', \delta)$ be analytic in $\mathcal{U} \times \mathcal{U} \times \mathcal{V}$. Then, the function*

$$\begin{aligned} F(x,\delta) &= \int_0^L \{\log|x - x'|B(x,x',\delta) + e^{-i\alpha L}\log|x - x' + L|B(x, x' - L, \delta) \\ &+ e^{i\alpha L}\log|x - x' - L|B(x, x' + L, \delta)\}\eta(x', \delta)\, dx' \end{aligned}$$

defined for $x \in \mathbb{R}$, can be extended to a complex analytic function on $\mathcal{U} \times \mathcal{V}$. Furthermore, if $C_1 = \{-\gamma \le \Re(x') \le L + \gamma,\, |\Im(x')| \le \gamma\}$ with $0 < \gamma < \epsilon$ and $C_2 \subset \mathcal{V}$ is compact, then

$$\sup_{C_1 \times C_2} |F(x,\delta)| \le \kappa \sup_{C_1 \times C_2} |\eta(x,\delta)| \le \kappa \sup_{(|\Im(x)| \le \gamma) \times C_2} |\eta(x,\delta)|. \tag{58}$$

for some constant $\kappa > 0$.

Proof: By a change of variables we get

$$F(x,\delta) = \int_{-L}^{2L} \log|x - x'| B(x,x',\delta)\eta(x',\delta)\, dx'. \tag{59}$$

Then, defining

$$N(x,x',\delta) = \int_{x}^{x'} B(x,x',\delta)\eta(x',\delta)\, dx' ,$$

integration by parts in (59) yields

$$\begin{aligned} F(x,\delta) &= \log(2L - x)N(x,2L,\delta) - \log(x + L)N(x,-L,\delta) \\ &+ \int_{-L}^{2L} \frac{1}{(x-x')} N(x,x',\delta)\, dx'. \end{aligned} \tag{60}$$

Clearly, the right hand side of (60) can be extended to a complex analytic function in $\mathcal{U} \times \mathcal{V}$ thereby showing that F can be extended to $\mathcal{U} \times \mathcal{V}$, and that it satisfies (58).

Remark 2 *Using a triangular contour to integrate by parts the last term in (60) one obtains the formula*

$$\begin{aligned} F(x,\delta) &= \int_{-L}^{2L} \log|x - x'| B(x,x',\delta)\eta(x',\delta)\, dx' + i\arg(x + L) \int_{-L}^{x} B(x,x',\delta)\eta(x',\delta)\, dx' \\ &+ i\arg(x - 2L) \int_{x}^{2L} B(x,x',\delta)\eta(x',\delta)\, dx' \mp i\pi \int_{x}^{2L} B(x,x',\delta)\eta(x',\delta)\, dx'. \end{aligned} \tag{61}$$

The sign in the last integral is negative if $\Im(x) \geq 0$ and it is positive if $\Im(x) \leq 0$.

6.3 Continuity and Compactness

In order to discuss the analyticity properties of $u^{\pm}$, we need to consider several operators related to A_δ. In this section we define these operators and establish the basic compactness result.

Definition 2 *We define the Banach spaces* $\mathcal{H}_\nu$ *and* $\mathcal{J}_{\epsilon,\delta}(\delta_0)$ $(\delta_0 \in \mathbf{R})$ *by*

$$\mathcal{H}_\nu = \{g(x) = \begin{pmatrix} \eta(x) \\ \mu(x) \end{pmatrix} : g \text{ is } \alpha \text{ quasi-periodic;}$$
$$\text{analytic for } |\Im(x)| < \nu \text{ and continuous for } |\Im(x)| \leq \nu\}$$

and

$$\mathcal{J}_{\epsilon,\nu}(\delta_0) = \{g(x,\delta) = \begin{pmatrix} \eta(x,\delta) \\ \mu(x,\delta) \end{pmatrix} : g(\cdot,\delta) \text{ is } \alpha \text{ quasi-periodic,}$$
$$\text{analytic for } |\Im(x)| < \nu\,,\ |\delta - \delta_0| < \epsilon \text{ and}$$
$$\text{continuous for } |\Im(x)| \leq \nu\,,\ |\delta - \delta_0| \leq \epsilon\}$$

with norms

$$\|\cdot\|_{\mathcal{H}} = \sup_{|Im(x)| \leq \nu} |\cdot|\ ,\quad \|\cdot\|_{\mathcal{J}} = \sup_{(|Im(x)| \leq \nu) \times (|\delta - \delta_0| \leq \epsilon)} |\cdot|.$$

Further, for a function $g(x,\delta)$ *we define the operators*

$$(\widetilde{\mathcal{A}_{\delta_0}} g)(x,\delta) = (\mathcal{A}_{\delta_0} g(\cdot,\delta))(x) \tag{62}$$

and

$$(\mathcal{K}g)(x,\delta) = (\mathcal{A}_\delta g(\cdot,\delta))(x). \tag{63}$$

Lemma 6 *Fix* $\delta_0 \in \mathbf{R}$. *Then, there exist* $\nu > 0\,,\ \epsilon > 0$ *such that:*

(i) *The operator* $\mathcal{A}_{\delta_0}$ *maps* $\mathcal{H}_\nu$ *into itself, continuously in the norm* $\|\cdot\|_{\mathcal{H}}$.

(ii) *The operators* $\widetilde{\mathcal{A}_{\delta_0}}$ *and* $\mathcal{K}$ *map* $\mathcal{J}_{\epsilon,\nu}(\delta_0)$ *into itself, continuously in the norm* $\|\cdot\|_{\mathcal{J}}$.

Proof: Since $G_{k\pm}$ is α quasi-periodic, it is clear that so are the functions $\mathcal{A}_{\delta_0} g$ for $g \in \mathcal{H}_\nu$ and $\widetilde{\mathcal{A}_{\delta_0}} g$ and $\mathcal{K}g$ for $g \in \mathcal{J}_{\epsilon,\nu}(\delta_0)$. Thus, it suffices to prove that, for some

$\gamma > 0$ and for $-\gamma < \Re(x) < L+\gamma$, $\mathcal{A}_{\delta_0}g$ is analytic for $|\Im(x)| < \nu$ and continuous for $|\Im(x)| \leq \nu$, and that $\widetilde{\mathcal{A}_{\delta_0}}g$, $\mathcal{K}g$ are analytic for $|\Im(x)| < \nu$, $|\delta-\delta_0| < \epsilon$ and continuous for $|\Im(x)| \leq \nu$, $|\delta - \delta_0| \leq \epsilon$. But this is an immediate consequence of Theorem 4 and Lemma 5.

Finally we establish the compactness of $\mathcal{A}_{\delta_0}$.

Theorem 5 *The operator $\mathcal{A}_{\delta_0}$ is compact in $\mathcal{H}_\nu$.*

Proof: First notice that from (46), (52) and (60) we can write

$$\begin{aligned}(\mathcal{A}_{\delta_0})g(x) &= \log|2L-x|\int_x^{2L} B(x,x',\delta_0)g(x')\,dx' - \log|x+L|\int_x^{-L} B(x,x',\delta_0)g(x')\,dx' \\ &+ \int_{-L}^{2L}\frac{1}{(x-x')}\int_x^{x'} B(x,s,\delta_0)g(s)\,ds dx' + \int_0^L C(x,x',\delta_0)g(x')\,dx' ,\end{aligned}$$

where $g(x) = \begin{pmatrix} \eta(x) \\ \mu(x) \end{pmatrix}$ and B and C are matrix-valued.

Set

$$\Lambda_2(g)(x) = \int_0^L C(x,x',\delta_0)g(x')\,dx' \ , \ \ \Lambda_1 = \mathcal{A}_{\delta_0} - \Lambda_2 ,$$

and let $\{g_n\}$ be a bounded sequence in $\mathcal{H}_\nu$,

$$\|g_n\|_{\mathcal{H}} \leq M \ , n \geq 1.$$

Since the functions $\mathcal{A}_{\delta_0}g_n(x)$ are α quasi-periodic, it suffices to show that there exists a subsequence $\{g_{n_k}\}$ such that $\mathcal{A}_{\delta_0}g_{n_k}(x)$ converges uniformly for $0 \leq \Re(x) \leq L$, $|\Im(x)| \leq \nu$.

Since $\{g_n\}$ is a bounded sequence of holomorphic functions, there exists a subsequence $\{g_{n_k}\}$ such that

$$\{g_{n_k}\} \to g \qquad \text{uniformly on } \textit{compact subsets}$$
$$\text{of } D \equiv \{\,|\Im(x)| < \nu\,\}.$$

Clearly,

$$\sup_{\overline{D}} |\Lambda_2(g_{n_k} - g)| \leq \text{const.} \sup_{x' \in [0,L]} |g_{n_k} - g| \to 0 ,$$

since $[0, L] \subset D$.

Thus, it suffices to prove that

$$\sup_{0 \leq \Re(x) \leq L, |\Im(x)| \leq \nu} |\Lambda_1(g_{n_k} - g)| \to 0.$$

But this follows readily from the formula (see remark 2)

$$\begin{aligned} \Lambda_1(g)(x) &= \int_{-L}^{2L} \log|x - x'| B(x, x', \delta_0) g(x')\, dx' + i\arg(x + L) \int_{-L}^{x} B(x, x', \delta_0) g(x')\, dx' \\ &+ i\arg(x - 2L) \int_{x}^{2L} B(x, x', \delta_0) g(x')\, dx' \mp i\pi \int_{x}^{2L} B(x, x', \delta_0) g(x')\, dx' . \end{aligned}$$

7 Analyticity of the solution

In this section we shall use the results of sections 5 and 6 to prove that for any given $\delta_0 \in \mathbf{R}$ there exists a number $\lambda > 0$ such that the solution $u^{\pm}(x, y, \delta)$ is analytic for $|\delta - \delta_0| < \lambda$, $x \in \mathbf{R}$ and $|y - \delta_0 f(x)| < \lambda$. We shall assume throughout that the diffraction problem (5), (15), (34) admits a unique solution (see section 5). Under this hypothesis, the result of analyticity up to the boundary will be a consequence of the Cauchy-Kowaleski Theorem and the invertibility in $\mathcal{J}_{\epsilon,\nu}(\delta_0)$ of the operator $\mathcal{K}$ defined in (63). Furthermore, we shall show that $u^{\pm}(x, y, \delta)$ is analytic in its variables away from the interface. These facts provide the theoretical foundations for the calculations in section 3. We first show

Theorem 6 *Let $\delta_0 \in \mathbf{R}$. Then, the operator $\frac{1}{2}\mathcal{I} - \mathcal{A}_{\delta_0}$ is invertible in $\mathcal{H}_\nu$.*

Proof: Since $\mathcal{A}_{\delta_0}$ is compact (Theorem 5), it suffices to show that if $g \in \mathcal{H}_\nu$ and

$$(\frac{1}{2}\mathcal{I} - \mathcal{A}_{\delta_0}) g(x) \equiv 0 \tag{64}$$

then $g(x) = \begin{pmatrix} \eta(x) \\ \mu(x) \end{pmatrix} \equiv 0$. A standard argument (see e.g. [4] p. 101) shows that if equation (64) holds for real x then $g = 0$ for $x \in \mathbf{R}$. Being g analytic, it follows that $g \equiv 0$ as desired.

Next, we use the previous result to establish the invertibility of the operators $\frac{1}{2}\mathcal{I} - \widetilde{\mathcal{A}_{\delta_0}}$ and $\frac{1}{2}\mathcal{I} - \mathcal{K}$.

Theorem 7 *The operator $\frac{1}{2}\mathcal{I} - \widetilde{\mathcal{A}_{\delta_0}}$ is invertible in $\mathcal{J}_{\epsilon,\nu}(\delta_0)$.*

Proof: Let $\mathcal{C}$ denote the operator

$$(\mathcal{C}g)(x,\delta) = ((\frac{1}{2}\mathcal{I} - \mathcal{A}_{\delta_0})^{-1} g(\cdot,\delta))(x).$$

We first show that if $g \in \mathcal{J}_{\epsilon,\nu}(\delta_0)$ then $\mathcal{C}g \in \mathcal{J}_{\epsilon,\nu}(\delta_0)$. By Theorem 6, for each fixed δ, $\mathcal{C}g(\cdot,\delta) \in \mathcal{H}_\nu$.Thus, to prove that $\mathcal{C}g \in \mathcal{J}_{\epsilon,\nu}(\delta_0)$, it suffices to show that $\mathcal{C}g(x,\cdot)$ is analytic for each fixed x and that $\mathcal{C}g$ is continuous in (x,δ).

Let $\mathcal{L} \equiv (\frac{1}{2}\mathcal{I} - \mathcal{A}_{\delta_0})^{-1}$, let $(x_n,\delta_n) \to (x,\delta)$ and set $g_n(x) \equiv g(x,\delta_n)$. Since g is continuous, we have

$$g_n(x) \to g(x,\delta) \text{ uniformly in } |\Im(x)| \leq \nu.$$

Thus, by Theorem 6,

$$\begin{aligned} |\mathcal{L}g_n(x_n) - \mathcal{L}g(x)| &\leq |\mathcal{L}g_n(x_n) - \mathcal{L}g(x_n)| + |\mathcal{L}g(x_n) - \mathcal{L}g(x)| \\ &\leq \|\mathcal{L}\| \|g_n - g\|_{\mathcal{H}} + |\mathcal{L}g(x_n) - \mathcal{L}g(x)| \end{aligned}$$

and the continuity of $\mathcal{C}g$ in (x,δ) follows.

Next notice that, since $g \in \mathcal{J}_{\epsilon,\nu}(\delta_0)$, for each δ fixed the function $g_h = \frac{1}{h}(g(\cdot,\delta + h) - g(\cdot,\delta))$ converges uniformly in $|\Im(x)| \leq \nu$ as $h \to 0$ as seen from Cauchy's

Theorem (in the variable δ) and the quasi-periodicity of g. Thus, $\mathcal{C}g_h$ converges uniformly in $|\Im(x)| \leq \nu$ and therefore $\mathcal{C}g(x,\cdot)$ is analytic for fixed x.

Now, since $\mathcal{L}$ is continuous in $\mathcal{H}_\nu$, we have

$$\sup_{|\Im(x)|\leq\nu, |\delta-\delta_0|\leq\epsilon} |(\mathcal{C}g)(x,\delta)| \leq \|\mathcal{L}\| \sup_{|\delta-\delta_0|\leq\epsilon} \sup_{|\Im(x)|\leq\nu} |g(x,\delta)|$$

and it follows that $\mathcal{C}$ is continuous in the norm $\|\cdot\|_{\mathcal{J}}$.

Finally, it is clear that

$$(\frac{1}{2}\mathcal{I} - \widetilde{\mathcal{A}_{\delta_0}})\mathcal{C} = \mathcal{C}(\frac{1}{2}\mathcal{I} - \widetilde{\mathcal{A}_{\delta_0}}) = \mathcal{I}$$

and therefore the theorem is proved.

Theorem 8 *The operator $\frac{1}{2}\mathcal{I} - \mathcal{K}$ is invertible in $\mathcal{J}_{\epsilon,\nu}(\delta_0)$, for all sufficiently small $\epsilon > 0$.*

Proof: Set

$$\mathcal{C} = \sum_{n=0}^{\infty}[(\frac{1}{2}\mathcal{I} - \widetilde{\mathcal{A}_{\delta_0}})^{-1}(\mathcal{K} - \widetilde{\mathcal{A}_{\delta_0}})]^n(\frac{1}{2}\mathcal{I} - \widetilde{\mathcal{A}_{\delta_0}})^{-1}.$$

Clearly, it suffices to show that

$$\|\mathcal{K} - \widetilde{\mathcal{A}_{\delta_0}}\|_{\mathcal{J}} \leq \text{const.}\,\epsilon \tag{65}$$

for then

$$\mathcal{C} = (\frac{1}{2}\mathcal{I} - \mathcal{K})^{-1},$$

since

$$\frac{1}{2}\mathcal{I} - \mathcal{K} = (\frac{1}{2}\mathcal{I} - \widetilde{\mathcal{A}_{\delta_0}})(\mathcal{I} - (\frac{1}{2}\mathcal{I} - \widetilde{\mathcal{A}_{\delta_0}})^{-1}(\mathcal{K} - \widetilde{\mathcal{A}_{\delta_0}})).$$

But (65) follows immediately from Theorem 4 and Lemma 5.

Finally, we give the analyticity results for the solution of the diffraction problem.

Theorem 9 *Let $u^{\pm}$ be the solution of (5),(34) satisfying (15). Then*

(i) *If $\delta_0 \in \mathbf{R}$ and $y_0 > \delta_0 \max|f|$ ($y_0 < -\delta_0 \max|f|$), there exists $\epsilon_0 = \epsilon_0(\delta_0, y_0)$ such that $u^+(x,y,\delta)$ ($u^-(x,y,\delta)$) is analytic in (x,y,δ) for $|\delta - \delta_0| < \epsilon_0$ and $|y - y_0| < \epsilon_0$.*

(ii) *The functions $u^{\pm}(x,\delta f(x),\delta)$ and $\frac{\partial u^{\pm}}{\partial n_\delta(x,\delta f(x))}(x,\delta f(x),\delta)$ are analytic.*

Proof: Statement (i) follows from (44) and Theorem 8. The fact that $u^{\pm}(x,\delta f(x),\delta)$ is analytic is a consequence of (44), Theorem 4, Lemma 5 and Theorem 8. To show that $\frac{\partial u^{\pm}}{\partial n_\delta(x,\delta f(x))}(x,\delta f(x),\delta)$ is analytic in (x,δ) we first notice that it suffices to prove that the normal derivative of the logarithmic double layer potential is analytic. Using Green's formula we can transform the singularity in the double layer potential into a singularity in a two-dimensional integral involving the second derivatives of the density (see [5] pp. 256–260). The y-derivative of this area integral is easily seen to reduce to a line integral whose analyticity is a direct consequence of Lemma 5. Since $u^{\pm}(x,\delta f(x),\delta)$ is analytic it follows that so is the normal derivative.

Theorem 10 *Fix $\delta_0 \in \mathbf{R}$. Then there exists $\epsilon_1 > 0$ such that $u^+(x,y,\delta)$ and $u^-(x,y,\delta)$ are analytic for $|\delta - \delta_0| < \epsilon_1$, $|y - \delta_0 f(x)| < \epsilon_1$.*

Proof: Consider the Cauchy Problem for the Helmholtz equation

$$v_{xx} + v_{yy} + (k^+)^2 v = 0, \quad v = v(x,y,\delta)$$

with data $v(x,\delta f(x),\delta) = u^+(x,\delta f(x),\delta)$ and $\frac{\partial v}{\partial n_\delta}(x,\delta f(x),\delta) = \frac{\partial u^+}{\partial n_\delta(x,\delta f(x))}(x,\delta f(x),\delta)$ on the two-dimensional surface

$$S : y - \delta f(x) = 0.$$

The desired conclusion for u^+ will follow from part (ii) of Theorem 9 and the Cauchy-Kowaleski Theorem, once we show that S is non-characteristic for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. But the normal vector to S is in the direction

$$N = (-\delta f'(x), 1, -f'(x))$$

and the symbol of the operator is

$$P(\xi_1, \xi_2, \xi_3) = \xi_1^2 + \xi_2^2 ,$$

so that

$$P(N) = 1 + \delta^2 (f'(x))^2 \neq 0$$

which implies that S is nowhere characteristic.

The proof for u^- is analogous.

Corollary 1 *There exists $\epsilon > 0$ such that for all $(x, y) \in \mathbf{R}^2$*

$$u^\pm(x, y, \delta) = \sum_{n=0}^{\infty} u_n^\pm(x, y)\delta^n \tag{66}$$

where the series converges for $|\delta| < \epsilon$. The functions $u_n^\pm$ satisfy Helmoltz equation

$$\Delta u_n^\pm + (k^\pm)^2 u_n^\pm = 0 \text{ in } \{ (x, y) \, : \, \pm y > 0 \}$$

and the condition of radiation at infinity together with transmission conditions on $\{y = 0\}$ given (recursively) by

$$u_n^+ - u_n^- = -\frac{(-i\beta)^n}{n!} f^n e^{i\alpha x} - \sum_{k=0}^{n-1} \frac{f^{n-k}}{(n-k)!} \left(\frac{\partial^{n-k} u_k^+}{\partial y^{n-k}} - \frac{\partial^{n-k} u_k^-}{\partial y^{n-k}} \right) \tag{67}$$

and

$$\frac{\partial u_n^+}{\partial y} - C_0 \frac{\partial u_n^-}{\partial y} = -\frac{1}{n!}(i\alpha n(-i\beta)^n (f' f^{n-1}) - (-i\beta)^{n+1} f^n) e^{i\alpha x} \tag{68}$$

$$+ \sum_{k=0}^{n-1} \frac{(f' f^{n-k-1})}{(n-k-1)!} \left(\frac{\partial^{n-k} u_k^+}{\partial x y^{n-k-1}} - C_0 \frac{\partial^{n-k} u_k^-}{\partial x y^{n-k-1}} \right)$$

$$- \sum_{k=0}^{n-1} \frac{f^{n-k}}{(n-k)!} \left(\frac{\partial^{n-k} u_k^+}{\partial y^{n-k+1}} - C_0 \frac{\partial^{n-k} u_k^-}{\partial y^{n-k+1}} \right).$$

Proof: Formula (66) follows from Theorems 9 and 10. On the other hand, since

$$u_n^{\pm}(x,y) = \frac{1}{n!} \frac{\partial^n u^{\pm}}{\partial \delta^n} \Big|_{\delta=0}$$

it is clear that $u_n^{\pm}$ satisfy the corresponding Helmholtz equation. Finally, the conditions (67) and (68) are obtained by differentiating with respect to δ the equalities

$$u^+(x, \delta f(x), \delta) - u^-(x, \delta f(x), \delta) = -e^{i\alpha x - i\beta\delta f(x)}$$

and

$$-\delta f'(x) \left(\frac{\partial u^+}{\partial x}(x, \delta f(x), \delta) - C_0 \frac{\partial u^-}{\partial x}(x, \delta f(x), \delta) \right) + \left(\frac{\partial u^+}{\partial y}(x, \delta f(x), \delta) \right.$$

$$\left. - C_0 \frac{\partial u^-}{\partial y}(x, \delta f(x), \delta) \right) = (i\alpha\delta f(x) + i\beta) e^{i\alpha x - i\beta\delta f(x)}$$

This last step is justified by Theorem 10.

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