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SOLUTION OF A VECTOR VARIABLE BI-ADDITIVE FUNCTIONAL EQUATION

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Abstract. We investigate the relation between the vector variable bi-additive functional equation $f(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i, y_j)$ and the multi-variable quadratic functional equation

$$g(\sum_{i=1}^{n} x_i) + \sum_{1\le i<j\le n} g(x_i - x_j) = n\sum_{i=1}^{n} g(x_i).$$

Furthermore, we find out the general solution of the above two functional equations.

1. Introduction

Throughout this paper, let n be a positive integer greater than 1 and let X and Y be vector spaces.

Definition 1. A mapping $f : X \times X \to Y$ is called *bi-additive* if f satisfies the system of equations

$$\begin{aligned} f(x+y,z) &= f(x,z) + f(y,z), \\ f(x,y+z) &= f(x,y) + f(x,z) \end{aligned} \tag{1}$$

for all $x, y, z \in X$.

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x,y) := cxy$ is a solution of (1). In particular, letting $x = y$, we get a quadratic function $g : \mathbb{R} \to \mathbb{R}$ in one variable given by $g(x) := f(x,x) = cx^2$.

For a mapping $f : X \times X \to Y$, consider the bi-additive functional equation:

$$f\left(\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i, y_j). \tag{2}$$

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For a mapping $g : X \to Y$, consider the quadratic functional equation:

$$g\Big(\sum_{i=1}^{n} x_i\Big) + \sum_{1\le i<j\le n} g\Big(x_i - x_j\Big) = n\sum_{i=1}^{n} g(x_i). \tag{3}$$

J.-H. Bae and K.-W. Jun [2] proved the stability in Banach spaces of the equation (3). Recently, J.-H. Bae and W.-G. Park [3] proved the stability in Banach Modules over a C^*-algebra of the same equation. There are numerous results about various functional equations ([1, 4, 5, 6, 7]).

In this paper, we investigate the relation between (2) and (3). And we find out the general solution of (2) and (3).

2. Results

Theorem 2.1. *Let $f : X \times X \to Y$ be a mapping satisfying* (2) *and let $g : X \to Y$ be the mapping given by*

$$g(x) := f(x, x) \tag{4}$$

for all $x \in X$. If

$$f(x, y) = \frac{1}{4}[g(x + y) - g(x - y)] \tag{5}$$

for all $x, y \in X$, then g satisfies (3).

Proof. Letting $x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0$ in (2) and then using (4), we have $g(0) = 0$. Putting $y = x$ in (5) and then using (4), we get

$$g(2x) = 4g(x) \tag{6}$$

for all $x \in X$. Setting $y_1 = \cdots = y_n = 0$ in (2), we have

$$f\left(\sum_{i=1}^{n} x_i, 0\right) = n\sum_{i=1}^{n} f(x_i, 0)$$

for all $x_1, \ldots, x_n \in X$. Taking $x_2 = \cdots = x_n = 0$ in the above equality, we get $f(x_1, 0) = 0$ for all $x_1 \in X$. Similarly, $f(0, y_1) = 0$ for all $y_1 \in X$. Letting $x_1 = x$, $x_2 = y$, $x_3 = \cdots = x_n = 0$ and $y_1 = z$, $y_2 = w$, $y_3 = \cdots = y_n = 0$ in (2), we have

$$f(x + y, z + w) = f(x, z) + f(x, w) + f(y, z) + f(y, w) \tag{7}$$

for all $x, y, z, w \in X$. By (7) and (5), we obtain

$$\begin{aligned} & g(x + y + z + w) - g(x + y - z - w) \\ = \; & g(x + z) - g(x - z) + g(x + w) - g(x - w) \\ & + g(y + z) - g(y - z) + g(y + w) - g(y - w) \end{aligned} \tag{8}$$

for all $x, y, z, w \in X$. Putting $x = y = z = 0$ and then replacing w by x in (8), we see that

$$g(-x) = g(x) \tag{9}$$

for all $x \in X$. Setting $z = x$ and $w = y$ in (8) and then using (6) and (9), we see that

$$g(x+y) + g(x-y) = 2g(x) + 2g(y) \tag{10}$$

for all $x, y \in X$. By (2) and (5), we obtain

$$\begin{aligned} &g\left(\sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j\right) - g\left(\sum_{i=1}^{n} x_i - \sum_{j=1}^{n} y_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \Big[g(x_i + y_j) - g(x_i - y_j)\Big] \end{aligned} \tag{11}$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Taking $y_1 = x_1, \ldots, y_n = x_n$ in (11) and then using (6) and (9), we see that

$$2g\left(\sum_{i=1}^{n} x_i\right) = 2\sum_{i=1}^{n} g(x_i) + \sum_{1 \le i < j \le n} g(x_i + x_j) - \sum_{1 \le i < j \le n} g(x_i - x_j)$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. By (10) and the above equality, we obtain that

$$\begin{aligned} 2g\left(\sum_{i=1}^{n} x_i\right) = 2\sum_{i=1}^{n} g(x_i) &+ \sum_{1 \le i < j \le n} [\, 2g(x_i) + 2g(x_j) - g(x_i - x_j) \,] \\ &- \sum_{1 \le i < j \le n} g(x_i - x_j) \end{aligned}$$

and thus

$$g\left(\sum_{i=1}^{n} x_i\right) + \sum_{1 \le i < j \le n} g(x_i - x_j) = \sum_{i=1}^{n} g(x_i) + \sum_{1 \le i < j \le n} [\, g(x_i) + g(x_j)]$$

for all $x_1, \ldots, x_n \in X$. Hence g satisfies (3). □

Theorem 2.2. *Let $g : X \to Y$ be a mapping satisfying* (3) *and let $f : X \times X \to Y$ be the mapping given by* (5) *for all $x, y \in X$. Then f satisfies* (2) *and* (4).

Proof. Letting $x_1 = \cdots = x_n = 0$ in (3), we have $g(0) = 0$. Putting $x_1 = x$, $x_2 = y$ and $x_3 = \cdots = x_n = 0$ in (3), we obtain that g satisfies (10) and so satisfies (6) and (9). Setting $y = x$ in (5) and then using (6), the equality (4) holds. By (3), we see that

$$g\left[\sum_{i=1}^{n}(x_i + y_i)\right] = n\sum_{i=1}^{n} g(x_i + y_i) - \sum_{1 \le i < j \le n} g[(x_i + y_i) - (x_j + y_j)] \tag{12}$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. By (10), we have that

$$g(x+y) - g(x-y) = 2[g(x+y) - g(x) - g(y)] \tag{13}$$

for all $x, y \in X$. By (10), (12) and (13),

$$
\begin{aligned}
(14) \qquad & g\left(\sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j\right) - g\left(\sum_{i=1}^{n} x_i - \sum_{j=1}^{n} y_j\right) \\
= \; & g\left[\sum_{i=1}^{n}(x_i + y_i)\right] - g\left[\sum_{i=1}^{n}(x_i - y_i)\right] \\
= \; & n\sum_{i=1}^{n}[g(x_i + y_i) - g(x_i - y_i)] \\
& - \sum_{1 \le i < j \le n} \Big(g[(x_i + y_i) - (x_j + y_j)] - g[(x_i - y_i) - (x_j - y_j)]\Big) \\
= \; & n\sum_{i=1}^{n}[g(x_i + y_i) - g(x_i - y_i)] \\
& - \sum_{1 \le i < j \le n} \Big(g[(x_i + y_i) - (x_j + y_j)] - g[(x_i + y_j) - (x_j + y_i)]\Big) \\
= \; & 2n\sum_{i=1}^{n}[g(x_i + y_i) - g(x_i) - g(y_i)] \\
& - \sum_{1 \le i < j \le n} \Big[\Big(2g(x_i + y_i) + 2g(x_j + y_j) - g[(x_i + y_i) + (x_j + y_j)]\Big) \\
& - \Big(2g(x_i + y_j) + 2g(x_j + y_i) - g[(x_i + y_j) + (x_j + y_i)]\Big)\Big] \\
= \; & 2n\sum_{i=1}^{n}[g(x_i + y_i) - g(x_i) - g(y_i)] \\
& -2\sum_{1 \le i < j \le n} [g(x_i + y_i) + g(x_j + y_j) - g(x_i + y_j) - g(x_j + y_i)]
\end{aligned}
$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Note that

$$
(15) \qquad \sum_{1 \le i < j \le n} (a_i + a_j) = (n-1)\sum_{i=1}^{n} a_i
$$

for all $a_1, \ldots, a_n \in Y$. By (10), (13) and (15),

$$
\begin{aligned}
(16) \qquad & 2n\sum_{i=1}^{n}[g(x_i + y_i) - g(x_i) - g(y_i)] \\
& -2\sum_{1 \le i < j \le n} [g(x_i + y_i) + g(x_j + y_j) - g(x_i + y_j) - g(x_j + y_i)]
\end{aligned}
$$

$$= 2n\sum_{i=1}^{n}[g(x_i+y_i)-g(x_i)-g(y_i)]$$

$$+2\sum_{1\le i<j\le n}[g(x_i+y_j)+g(x_j+y_i)]-2(n-1)\sum_{i=1}^{n}g(x_i+y_i)$$

$$= n\sum_{i=1}^{n}[g(x_i+y_i)-g(x_i-y_i)]$$

$$+2\sum_{1\le i<j\le n}[g(x_i+y_j)+g(x_j+y_i)]-2(n-1)\sum_{i=1}^{n}g(x_i+y_i)$$

$$= \sum_{i=1}^{n}[g(x_i+y_i)-g(x_i-y_i)]+(n-1)\sum_{i=1}^{n}[g(x_i+y_i)-g(x_i-y_i)]$$

$$+2\sum_{1\le i<j\le n}[g(x_i+y_j)+g(x_j+y_i)]-2(n-1)\sum_{i=1}^{n}g(x_i+y_i)$$

$$= \sum_{i=1}^{n}[g(x_i+y_i)-g(x_i-y_i)]-(n-1)\sum_{i=1}^{n}[g(x_i+y_i)+g(x_i-y_i)]$$

$$+2\sum_{1\le i<j\le n}[g(x_i+y_j)+g(x_j+y_i)]$$

$$= \sum_{i=1}^{n}[g(x_i+y_i)-g(x_i-y_i)]-2(n-1)\sum_{i=1}^{n}[g(x_i)+g(y_i)]$$

$$+2\sum_{1\le i<j\le n}[g(x_i+y_j)+g(x_j+y_i)]$$

$$= \sum_{i=1}^{n}[g(x_i+y_i)-g(x_i-y_i)]$$

$$+2\sum_{1\le i<j\le n}\Big([g(x_i+y_j)-g(x_i)-g(y_j)]+[g(x_j+y_i)-g(x_j)-g(y_i)]\Big)$$

$$= \sum_{i=1}^{n}[g(x_i+y_i)-g(x_i-y_i)]$$

$$+\sum_{1\le i<j\le n}\Big([g(x_i+y_j)-g(x_i-y_j)]+[g(x_j+y_i)-g(x_j-y_i)]\Big)$$

$$= \sum_{i=1}^{n}g(x_i+y_i)+\sum_{1\le i<j\le n}g(x_i+y_j)+\sum_{1\le i<j\le n}g(x_j+y_i)$$

$$-\sum_{i=1}^{n}g(x_i-y_i)-\sum_{1\le i<j\le n}g(x_i-y_j)-\sum_{1\le i<j\le n}g(x_j-y_i)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} g(x_i + y_j) - \sum_{i=1}^{n}\sum_{j=1}^{n} g(x_i - y_j)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \Big[g(x_i + y_j) - g(x_i - y_j)\Big]$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. By (14) and (16), we obtain that g satisfies (11). By (5) and (11), we see that f satisfies (2). □

Next we obtain the solutions of the equations (2) and (3).

Theorem 2.3. *A mapping $f : X \times X \to Y$ satisfies* (1) *if and only if it satisfies* (2).

Proof. If f satisfies (1), then

$$f\left(\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} y_j\right) = \sum_{i=1}^{n} f\left(x_i, \sum_{j=1}^{n} y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i, y_j)$$

for all $x_1, \ldots, x_n \in X$.

Conversely, assume that f satisfies (2). Choosing $x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0$ in (2), $f(0,0) = 0$. Letting $x_1 = x$ and $x_2 = \cdots = x_n = y_1 = \cdots = y_n = 0$ in (2), we have $f(x, 0) = 0$ for all $x \in X$. Putting $x_1 = x$, $x_2 = y$, $y_1 = z$ and $x_3 = \cdots = x_n = y_2 = \cdots = y_n = 0$ in (2), we get

$$f(x + y, z) = f(x, z) + f(y, z)$$

for all $x, y, z \in X$. Setting $y_1 = y$ and $x_1 = \cdots = x_n = y_2 = \cdots = y_n = 0$ in (2), we obtain $f(0, y) = 0$ for all $y \in X$. Taking $x_1 = x$, $y_1 = y$, $y_2 = z$ and $x_2 = \cdots = x_n = y_3 = \cdots = y_n = 0$ in (2), we see that

$$f(x, y + z) = f(x, y) + f(x, z)$$

for all $x, y, z \in X$. □

Theorem 2.4. *A function $g : X \to Y$ satisfies* (3) *if and only if there exists a symmetric bi-additive function $S : X \times X \to Y$ such that $g(x) = S(x, x)$ for all $x \in X$.*

Proof. Define $f : X \times X \to Y$ by (5) for all $x, y \in X$. By Theorem 2.2, we obtain that f satisfies (2) and (4). Using Theorem 2.3, we see that f also satisfies (1). So f is bi-additive. Define $S : X \times X \to Y$ by

$$S(x, y) := \frac{1}{2}[f(x, y) + f(y, x)]$$

for all $x, y \in X$. Then S is symmetric and bi-additive. By (4), we obtain that $g(x) = S(x, x)$ for all $x \in X$.

Conversely, assume that there exists a symmetric bi-additive function $S : X \times X \to Y$ such that $g(x) = S(x,x)$ for all $x \in X$. Note that

$$\sum_{1\le i<j\le n} (a_i + b_j) = \sum_{i=1}^{n-1} (n-i)a_i + \sum_{j=2}^{n} (j-1)b_j$$

for all $a_1, \ldots, a_{n-1}, b_2, \ldots, b_n \in Y$. Thus

$$\begin{aligned}
& g\Big(\sum_{i=1}^{n} x_i\Big) + \sum_{1\le i<j\le n} g\Big(x_i - x_j\Big) \\
= \;& S\Big(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i\Big) + \sum_{1\le i<j\le n} S\Big(x_i - x_j, x_i - x_j\Big) \\
= \;& \sum_{i=1}^{n}\sum_{j=1}^{n} S(x_i, x_j) + \sum_{1\le i<j\le n} [S(x_i, x_i) - 2S(x_i, x_j) + S(x_j, x_j)] \\
= \;& \Bigg[\sum_{i=1}^{n} S(x_i, x_i) + 2 \sum_{1\le i<j\le n} S(x_i, x_j)\Bigg] \\
& + \Bigg[\sum_{i=1}^{n-1} (n-i) S(x_i, x_i) - 2 \sum_{1\le i<j\le n} S(x_i, x_j) + \sum_{j=2}^{n} (j-1) S(x_j, x_j)\Bigg] \\
= \;& S(x_n, x_n) + \sum_{i=1}^{n-1} (1+n-i) S(x_i, x_i) + \sum_{j=2}^{n} (j-1) S(x_j, x_j) \\
= \;& S(x_n, x_n) + \sum_{i=2}^{n-1} [(1+n-i) + (i-1)] S(x_i, x_i) \\
& + nS(x_1, x_1) + (n-1) S(x_n, x_n) \\
= \;& n \sum_{i=1}^{n} S(x_i, x_i) = n \sum_{i=1}^{n} g(x_i)
\end{aligned}$$

for all $x_1, \ldots, x_n \in X$. □

Let Y be complete and $\varphi : X \times X \times X \to [0, \infty)$ and $\psi : X \times X \times X \to [0, \infty)$ be two functions satisfying

$$\tilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \left[\frac{1}{2^{j+1}} \varphi(2^j x, 2^j y, z) + \frac{1}{2^j} \varphi(x, y, 2^j z)\right] < \infty \tag{17}$$

and

$$\tilde{\psi}(x, y, z) := \sum_{j=0}^{\infty} \left[\frac{1}{2^{j+1}} \psi(x, 2^j y, 2^j z) + \frac{1}{2^j} \psi(2^j x, y, z)\right] < \infty \tag{18}$$

for all $x, y, z \in X$.

Theorem 2.5. *Let $f : X \times X \to Y$ be a mapping such that*

$$\|f(x+y,z) - f(x,z) - f(y,z)\| \leq \varphi(x,y,z) \tag{19}$$

$$\|f(x,y+z) - f(x,y) - f(x,z)\| \leq \psi(x,y,z) \tag{20}$$

for all $x, y, z \in X$, and let $f(x,0) = 0$ and $f(0,y) = 0$ for all $x, y \in X$. Then there exist two bi-additive mappings $F_1, F_2 : X \times X \to Y$ such that

$$\|f(x,y) - F_1(x,y)\| \leq \tilde{\varphi}(x,x,y) \tag{21}$$

$$\|f(x,y) - F_2(x,y)\| \leq \tilde{\psi}(x,y,y) \tag{22}$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y), \quad F_2(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(x, 2^j y)$$

for all $x, y \in X$.

Proof. Letting $y = x$ in (19), we get

$$\left\| f(x,z) - \frac{1}{2} f(2x,z) \right\| \leq \frac{1}{2}\varphi(x,x,z) \tag{23}$$

for all $x, z \in X$. Thus

$$\left\| \frac{1}{2^j} f(2^j x, z) - \frac{1}{2^{j+1}} f(2^{j+1} x, z) \right\| \leq \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, z)$$

for all $x, z \in X$ and all j. Replacing z by y, we have

$$\left\| \frac{1}{2^j} f(2^j x, y) - \frac{1}{2^{j+1}} f(2^{j+1} x, y) \right\| \leq \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, y)$$

for all $x, y \in X$ and all j. For given integers l, m $(0 \leq l < m)$, we obtain

$$\left\| \frac{1}{2^l} f(2^l x, y) - \frac{1}{2^m} f(2^m x, y) \right\| \leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, y) \tag{24}$$

for all $x, y \in X$. By (17), the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ converges for all $x, y \in X$. Define $F_1 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (24), one can obtain the inequality (21). By (19) and (20), we see that

$$\left\| \frac{1}{2^j} f(2^j x + 2^j y, z) - \frac{1}{2^j} f(2^j x, z) - \frac{1}{2^j} f(2^j y, z) \right\| \leq \frac{1}{2^j} \varphi(2^j x, 2^j y, z)$$

and

$$\left\| \frac{1}{2^j} f(2^j x, y+z) - \frac{1}{2^j} f(2^j x, y) - \frac{1}{2^j} f(2^j x, z) \right\| \leq \frac{1}{2^j} \psi(2^j x, y, z)$$

for all $x, y, z \in X$ and all j. Letting $j \to \infty$ in the above two inequalities and using (18), we obtain that F_1 is bi-additive.

Next, setting $y = z$ in (20), we get

$$\left\| f(x,y) - \frac{1}{2} f(x, 2y) \right\| \le \frac{1}{2} \psi(x, y, y) \tag{25}$$

for all $x, y \in X$. By the same method as above, F_2 is bi-additive which satisfies (22), where $F_2(x,y) := \lim_{j \to \infty} \frac{1}{2^j} f(x, 2^j y)$ for all $x, y \in X$. □

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