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Solution of Cubic and Quartic Equations

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Summary. In this article, the principal n-th root of a complex number is defined, the Vieta's formulas for polynomial equations of degree 2, 3 and 4 are formalized. The solution of quadratic equations, the Cardan's solution of cubic equations and the Descartes-Euler solution of quartic equations in terms of their complex coefficients are also presented [5].

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The articles [11], [1], [4], [2], [10], [6], [8], [9], [12], [7], and [3] provide the notation and terminology for this paper.

1. Preliminaries

In this paper a, b are complex numbers.

The following propositions are true:

(1) $a \cdot a = a^2$.

(2) $a \cdot a \cdot a = a^3$.

(3) $a \cdot a \cdot a \cdot a = a^4$.

(4) $(a - b)^2 = (a^2 - 2 \cdot a \cdot b) + b^2$.

(5) $(a - b)^3 = ((a^3 - 3 \cdot a^2 \cdot b) + 3 \cdot b^2 \cdot a) - b^3$.

(6) $(a - b)^4 = (((a^4 - 4 \cdot a^3 \cdot b) + 6 \cdot a^2 \cdot b^2) - 4 \cdot b^3 \cdot a) + b^4$.

Let n be a natural number and let r be a real number. We introduce $r^{1/n}$ as a synonym of $\sqrt[n]{r}$.

Let n be a non zero natural number and let z be a complex number. The functor $\sqrt[n]{z}$ yields a complex number and is defined by:

(Def. 1) $\sqrt[n]{z} = |z|^{1/n} \cdot (\cos(\frac{\operatorname{Arg} z}{n}) + \sin(\frac{\operatorname{Arg} z}{n}) \cdot i)$.

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In the sequel z denotes a complex number and n_0 denotes a non zero natural number.

The following propositions are true:

(7) $\sqrt[n_0]{z^{n_0}} = z$.

(8) For every real number r such that $r \geq 0$ holds $\sqrt[n_0]{r} = r^{1/n_0}$.

(9) For every real number r such that $r > 0$ holds $\sqrt[n_0]{\frac{z}{r}} = \frac{\sqrt[n_0]{z}}{\sqrt[n_0]{r}}$.

(10) $z^2 = a$ iff $z = \sqrt[2]{a}$ or $z = -\sqrt[2]{a}$.

2. Solution of Quadratic Equations

In the sequel a_0, a_1, a_2, s_1, s_2 are complex numbers.

Next we state two propositions:

(11) If $a_1 = -(s_1 + s_2)$ and $a_0 = s_1 \cdot s_2$, then $z^2 + a_1 \cdot z + a_0 = 0$ iff $z = s_1$ or $z = s_2$.

(12) If $a_2 \neq 0$, then $a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ iff $z = -\frac{a_1}{2 \cdot a_2} + \frac{\sqrt[2]{\Delta(a_0,a_1,a_2)}}{2 \cdot a_2}$ or $z = -\frac{a_1}{2 \cdot a_2} - \frac{\sqrt[2]{\Delta(a_0,a_1,a_2)}}{2 \cdot a_2}$.

3. Solution of Cubic Equations

In the sequel a_3, x, q, r, s, s_3 are complex numbers.

The following four propositions are true:

(13) Suppose $z = x - \frac{a_2}{3}$ and $q = \frac{3 \cdot a_1 - a_2{}^2}{9}$ and $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2{}^3 - 27 \cdot a_0}{54}$. Then $z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ if and only if $(x^3 + 3 \cdot q \cdot x) - 2 \cdot r = 0$.

(14) If $a_2 = -(s_1 + s_2 + s_3)$ and $a_1 = s_1 \cdot s_2 + s_1 \cdot s_3 + s_2 \cdot s_3$ and $a_0 = -s_1 \cdot s_2 \cdot s_3$, then $z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ iff $z = s_1$ or $z = s_2$ or $z = s_3$.

(15) Suppose $q = \frac{3 \cdot a_1 - a_2{}^2}{9}$ and $q \neq 0$ and $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2{}^3 - 27 \cdot a_0}{54}$ and $s = \sqrt[2]{q^3 + r^2}$ and $s_1 = \sqrt[3]{r + s}$ and $s_2 = -\frac{q}{s_1}$. Then $z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ if and only if one of the following conditions is satisfied:

- (i) $z = (s_1 + s_2) - \frac{a_2}{3}$, or
- (ii) $z = (-\frac{s_1 + s_2}{2} - \frac{a_2}{3}) + \frac{(s_1 - s_2) \cdot \sqrt[2]{3} \cdot i}{2}$, or
- (iii) $z = -\frac{s_1 + s_2}{2} - \frac{a_2}{3} - \frac{(s_1 - s_2) \cdot \sqrt[2]{3} \cdot i}{2}$.

(16) Suppose $q = \frac{3 \cdot a_1 - a_2{}^2}{9}$ and $q = 0$ and $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2{}^3 - 27 \cdot a_0}{54}$ and $s_1 = \sqrt[3]{2 \cdot r}$. Then $z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ if and only if one of the following conditions is satisfied:

- (i) $z = s_1 - \frac{a_2}{3}$, or
- (ii) $z = (-\frac{s_1}{2} - \frac{a_2}{3}) + \frac{s_1 \cdot \sqrt[2]{3} \cdot i}{2}$, or
- (iii) $z = -\frac{s_1}{2} - \frac{a_2}{3} - \frac{s_1 \cdot \sqrt[2]{3} \cdot i}{2}$.

Let a_0, a_1, a_2 be complex numbers. The functor $\rho_1(a_0, a_1, a_2)$ yielding a complex number is defined by:

(Def. 2)(i) There exist r, s_1 such that $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2{}^3 - 27 \cdot a_0}{54}$ and $s_1 = \sqrt[3]{2 \cdot r}$ and $\rho_1(a_0, a_1, a_2) = s_1 - \frac{a_2}{3}$ if $3 \cdot a_1 - a_2{}^2 = 0$,

(ii) there exist q, r, s, s_1, s_2 such that $q = \frac{3 \cdot a_1 - a_2{}^2}{9}$ and $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2{}^3 - 27 \cdot a_0}{54}$ and $s = \sqrt[2]{q^3 + r^2}$ and $s_1 = \sqrt[3]{r + s}$ and $s_2 = -\frac{q}{s_1}$ and $\rho_1(a_0, a_1, a_2) = (s_1 + s_2) - \frac{a_2}{3}$, otherwise.

The functor $\rho_2(a_0, a_1, a_2)$ yields a complex number and is defined by:

(Def. 3)(i) There exist r, s_1 such that $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2{}^3 - 27 \cdot a_0}{54}$ and $s_1 = \sqrt[3]{2 \cdot r}$ and $\rho_2(a_0, a_1, a_2) = (-\frac{s_1}{2} - \frac{a_2}{3}) + \frac{s_1 \cdot \sqrt[2]{3} \cdot i}{2}$ if $3 \cdot a_1 - a_2{}^2 = 0$,

(ii) there exist q, r, s, s_1, s_2 such that $q = \frac{3 \cdot a_1 - a_2{}^2}{9}$ and $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2{}^3 - 27 \cdot a_0}{54}$ and $s = \sqrt[2]{q^3 + r^2}$ and $s_1 = \sqrt[3]{r + s}$ and $s_2 = -\frac{q}{s_1}$ and $\rho_2(a_0, a_1, a_2) = (-\frac{s_1 + s_2}{2} - \frac{a_2}{3}) + \frac{(s_1 - s_2) \cdot \sqrt[2]{3} \cdot i}{2}$, otherwise.

The functor $\rho_3(a_0, a_1, a_2)$ yields a complex number and is defined as follows:

(Def. 4)(i) There exist r, s_1 such that $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2{}^3 - 27 \cdot a_0}{54}$ and $s_1 = \sqrt[3]{2 \cdot r}$ and $\rho_3(a_0, a_1, a_2) = -\frac{s_1}{2} - \frac{a_2}{3} - \frac{s_1 \cdot \sqrt[2]{3} \cdot i}{2}$ if $3 \cdot a_1 - a_2{}^2 = 0$,

(ii) there exist q, r, s, s_1, s_2 such that $q = \frac{3 \cdot a_1 - a_2{}^2}{9}$ and $r = \frac{9 \cdot a_2 \cdot a_1 - 2 \cdot a_2{}^3 - 27 \cdot a_0}{54}$ and $s = \sqrt[2]{q^3 + r^2}$ and $s_1 = \sqrt[3]{r + s}$ and $s_2 = -\frac{q}{s_1}$ and $\rho_3(a_0, a_1, a_2) = -\frac{s_1 + s_2}{2} - \frac{a_2}{3} - \frac{(s_1 - s_2) \cdot \sqrt[2]{3} \cdot i}{2}$, otherwise.

We now state four propositions:

(17) $\rho_1(a_0, a_1, a_2) + \rho_2(a_0, a_1, a_2) + \rho_3(a_0, a_1, a_2) = -a_2$.

(18) $\rho_1(a_0, a_1, a_2) \cdot \rho_2(a_0, a_1, a_2) + \rho_1(a_0, a_1, a_2) \cdot \rho_3(a_0, a_1, a_2) + \rho_2(a_0, a_1, a_2) \cdot \rho_3(a_0, a_1, a_2) = a_1$.

(19) $\rho_1(a_0, a_1, a_2) \cdot \rho_2(a_0, a_1, a_2) \cdot \rho_3(a_0, a_1, a_2) = -a_0$.

(20) If $a_3 \neq 0$, then $a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ iff $z = \rho_1(\frac{a_0}{a_3}, \frac{a_1}{a_3}, \frac{a_2}{a_3})$ or $z = \rho_2(\frac{a_0}{a_3}, \frac{a_1}{a_3}, \frac{a_2}{a_3})$ or $z = \rho_3(\frac{a_0}{a_3}, \frac{a_1}{a_3}, \frac{a_2}{a_3})$.

4. Solution of Quartic Equations

In the sequel a_4, p, s_4 are complex numbers.

One can prove the following propositions:

(21) Suppose $z = x - \frac{a_3}{4}$ and $p = \frac{8 \cdot a_2 - 3 \cdot a_3{}^2}{32}$ and $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3{}^3}{64}$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3{}^2 \cdot a_2) - 3 \cdot a_3{}^4}{1024}$. Then $z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ if and only if $x^4 + 4 \cdot p \cdot x^2 + 8 \cdot q \cdot x + 4 \cdot r = 0$.

(22) Suppose $a_3 = -(s_1 + s_2 + s_3 + s_4)$ and $a_2 = s_1 \cdot s_2 + s_1 \cdot s_3 + s_1 \cdot s_4 + s_2 \cdot s_3 + s_2 \cdot s_4 + s_3 \cdot s_4$ and $a_1 = -(s_1 \cdot s_2 \cdot s_3 + s_1 \cdot s_2 \cdot s_4 + s_1 \cdot s_3 \cdot s_4 + s_2 \cdot s_3 \cdot s_4)$ and $a_0 = s_1 \cdot s_2 \cdot s_3 \cdot s_4$. Then $z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ if and only if $z = s_1$ or $z = s_2$ or $z = s_3$ or $z = s_4$.

(23) Suppose $q \neq 0$ and $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$ and $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$ and $s_3 = -\frac{q}{s_1 \cdot s_2}$. Then $z^4 + 4 \cdot p \cdot z^2 + 8 \cdot q \cdot z + 4 \cdot r = 0$ if and only if $z = s_1 + s_2 + s_3$ or $z = s_1 - s_2 - s_3$ or $z = (-s_1 + s_2) - s_3$ or $z = (-s_1 - s_2) + s_3$.

(24) Suppose that $p = \frac{8 \cdot a_2 - 3 \cdot {a_3}^2}{32}$ and $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + {a_3}^3}{64}$ and $q \neq 0$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot {a_3}^2 \cdot a_2) - 3 \cdot {a_3}^4}{1024}$ and $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$ and $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$ and $s_3 = -\frac{q}{s_1 \cdot s_2}$. Then $z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ if and only if $z = (s_1 + s_2 + s_3) - \frac{a_3}{4}$ or $z = s_1 - s_2 - s_3 - \frac{a_3}{4}$ or $z = (-s_1 + s_2) - s_3 - \frac{a_3}{4}$ or $z = ((-s_1 - s_2) + s_3) - \frac{a_3}{4}$.

(25) Suppose $p = \frac{8 \cdot a_2 - 3 \cdot {a_3}^2}{32}$ and $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + {a_3}^3}{64}$ and $q = 0$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot {a_3}^2 \cdot a_2) - 3 \cdot {a_3}^4}{1024}$ and $s_1 = \sqrt[2]{p^2 - r}$. Then $z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ if and only if one of the following conditions is satisfied:
  (i) $z = \sqrt[2]{-2 \cdot (p - s_1)} - \frac{a_3}{4}$, or
  (ii) $z = -\sqrt[2]{-2 \cdot (p - s_1)} - \frac{a_3}{4}$, or
  (iii) $z = \sqrt[2]{-2 \cdot (p + s_1)} - \frac{a_3}{4}$, or
  (iv) $z = -\sqrt[2]{-2 \cdot (p + s_1)} - \frac{a_3}{4}$.

Let a_0, a_1, a_2, a_3 be complex numbers. The functor $\rho_1(a_0, a_1, a_2, a_3)$ yielding a complex number is defined by:

(Def. 5)(i) There exist p, r, s_1 such that $p = \frac{8 \cdot a_2 - 3 \cdot {a_3}^2}{32}$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot {a_3}^2 \cdot a_2) - 3 \cdot {a_3}^4}{1024}$ and $s_1 = \sqrt[2]{p^2 - r}$ and $\rho_1(a_0, a_1, a_2, a_3) = \sqrt[2]{-2 \cdot (p - s_1)} - \frac{a_3}{4}$ if $(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + {a_3}^3 = 0$,
  (ii) there exist p, q, r, s_1, s_2, s_3 such that $p = \frac{8 \cdot a_2 - 3 \cdot {a_3}^2}{32}$ and $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + {a_3}^3}{64}$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot {a_3}^2 \cdot a_2) - 3 \cdot {a_3}^4}{1024}$ and $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$ and $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$ and $s_3 = -\frac{q}{s_1 \cdot s_2}$ and $\rho_1(a_0, a_1, a_2, a_3) = (s_1 + s_2 + s_3) - \frac{a_3}{4}$, otherwise.

The functor $\rho_2(a_0, a_1, a_2, a_3)$ yields a complex number and is defined as follows:

(Def. 6)(i) There exist p, r, s_1 such that $p = \frac{8 \cdot a_2 - 3 \cdot {a_3}^2}{32}$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot {a_3}^2 \cdot a_2) - 3 \cdot {a_3}^4}{1024}$ and $s_1 = \sqrt[2]{p^2 - r}$ and $\rho_2(a_0, a_1, a_2, a_3) = -\sqrt[2]{-2 \cdot (p - s_1)} - \frac{a_3}{4}$ if $(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + {a_3}^3 = 0$,
  (ii) there exist p, q, r, s_1, s_2, s_3 such that $p = \frac{8 \cdot a_2 - 3 \cdot {a_3}^2}{32}$ and $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + {a_3}^3}{64}$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot {a_3}^2 \cdot a_2) - 3 \cdot {a_3}^4}{1024}$ and $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$ and $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$ and $s_3 = -\frac{q}{s_1 \cdot s_2}$ and $\rho_2(a_0, a_1, a_2, a_3) = ((-s_1 - s_2) + s_3) - \frac{a_3}{4}$, otherwise.

The functor $\rho_3(a_0, a_1, a_2, a_3)$ yielding a complex number is defined as follows:

(Def. 7)(i) There exist p, r, s_1 such that $p = \frac{8 \cdot a_2 - 3 \cdot {a_3}^2}{32}$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot {a_3}^2 \cdot a_2) - 3 \cdot {a_3}^4}{1024}$ and $s_1 = \sqrt[2]{p^2 - r}$ and $\rho_3(a_0, a_1, a_2, a_3) = \sqrt[2]{-2 \cdot (p + s_1)} - \frac{a_3}{4}$ if $(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + {a_3}^3 = 0$,
  (ii) there exist p, q, r, s_1, s_2, s_3 such that $p = \frac{8 \cdot a_2 - 3 \cdot {a_3}^2}{32}$ and

$q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3{}^3}{64}$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3{}^2 \cdot a_2) - 3 \cdot a_3{}^4}{1024}$ and $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$ and $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$ and $s_3 = -\frac{q}{s_1 \cdot s_2}$ and $\rho_3(a_0, a_1, a_2, a_3) = (-s_1 + s_2) - s_3 - \frac{a_3}{4}$, otherwise.

The functor $\rho_4(a_0, a_1, a_2, a_3)$ yields a complex number and is defined as follows:

(Def. 8)(i) There exist p, r, s_1 such that $p = \frac{8 \cdot a_2 - 3 \cdot a_3{}^2}{32}$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3{}^2 \cdot a_2) - 3 \cdot a_3{}^4}{1024}$ and $s_1 = \sqrt[2]{p^2 - r}$ and $\rho_4(a_0, a_1, a_2, a_3) = -\sqrt[2]{-2 \cdot (p + s_1)} - \frac{a_3}{4}$ if $(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3{}^3 = 0$,

(ii) there exist p, q, r, s_1, s_2, s_3 such that $p = \frac{8 \cdot a_2 - 3 \cdot a_3{}^2}{32}$ and $q = \frac{(8 \cdot a_1 - 4 \cdot a_2 \cdot a_3) + a_3{}^3}{64}$ and $r = \frac{((256 \cdot a_0 - 64 \cdot a_3 \cdot a_1) + 16 \cdot a_3{}^2 \cdot a_2) - 3 \cdot a_3{}^4}{1024}$ and $s_1 = \sqrt[2]{\rho_1(-q^2, p^2 - r, 2 \cdot p)}$ and $s_2 = \sqrt[2]{\rho_2(-q^2, p^2 - r, 2 \cdot p)}$ and $s_3 = -\frac{q}{s_1 \cdot s_2}$ and $\rho_4(a_0, a_1, a_2, a_3) = s_1 - s_2 - s_3 - \frac{a_3}{4}$, otherwise.

One can prove the following propositions:

(26) $\rho_1(a_0, a_1, a_2, a_3) + \rho_2(a_0, a_1, a_2, a_3) + \rho_3(a_0, a_1, a_2, a_3) + \rho_4(a_0, a_1, a_2, a_3) = -a_3$.

(27) $\rho_1(a_0, a_1, a_2, a_3) \cdot \rho_2(a_0, a_1, a_2, a_3) + \rho_1(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) + \rho_1(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) + \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) + \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) + \rho_3(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) = a_2$.

(28) $\rho_1(a_0, a_1, a_2, a_3) \cdot \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) + \rho_1(a_0, a_1, a_2, a_3) \cdot \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) + \rho_1(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) + \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) = -a_1$.

(29) $\rho_1(a_0, a_1, a_2, a_3) \cdot \rho_2(a_0, a_1, a_2, a_3) \cdot \rho_3(a_0, a_1, a_2, a_3) \cdot \rho_4(a_0, a_1, a_2, a_3) = a_0$.

(30) Suppose $a_4 \neq 0$. Then $a_4 \cdot z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$ if and only if $z = \rho_1(\frac{a_0}{a_4}, \frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4})$ or $z = \rho_2(\frac{a_0}{a_4}, \frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4})$ or $z = \rho_3(\frac{a_0}{a_4}, \frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4})$ or $z = \rho_4(\frac{a_0}{a_4}, \frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4})$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[3] Yuzhong Ding and Xiquan Liang. Solving roots of polynomial equation of degree 2 and 3 with complex coefficients. *Formalized Mathematics*, 12(**2**):85–92, 2004.

[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[5] G.A. Korn and T.M. Korn. *Mathematical Handbook for Scientists and Engineers*. Dover Publication, New York, 2000.

[6] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[7] Robert Milewski. Trigonometric form of complex numbers. *Formalized Mathematics*, 9(**3**):455–460, 2001.

[8] Jan Popiołek. Quadratic inequalities. *Formalized Mathematics*, 2(**4**):507–509, 1991.

[9] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.

[10] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[11] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[12] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

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