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Solution of elasto-statics problems using the element free Galerkin method with local maximum entropy shape functions

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ABSTRACT

The element free Galerkin method (EFGM) [1] is one of the most robust meshless methods for the solution of elasto-statics problems. In the EFGM, moving least squares (MLS) shape functions are used for the approximation of the field variable. The essential boundary conditions cannot be implemented directly as in the case of Finite Element Method (FEM), because the MLS shape functions do not possess the Kronecker-delta property and use Lagrange multipliers instead. In this paper the recently developed local maximum entropy shape functions are used in the EFGM for the approximation of the field variable instead of MLS. As the local maximum entropy shape functions possess the Kronecker-delta property at the boundaries so the essential boundary conditions are enforced directly as in the case of FEM. Two benchmark problems, a cantilever beam subjected to parabolic traction at the free end and an infinite plate with circular hole subjected to unidirectional tension are solved to show the implementation and performance of the current approach. The displacement and stresses calculated by the current approach show good agreement with the analytical results.

1 INTRODUCTION

The finite element method (FEM) is the most prominent tool for the solution of boundary value problems in solid mechanics. However there are certain classes of problems for which the FEM is not an ideal choice, e.g. crack growth and large deformation problems, since elements can become distorted affecting solution accuracy and remeshing is required at different stages, which is computationally very expensive. Meshless methods are therefore a suitable choice for these problems, because only a set of nodes is required for the problem discretization. However, to compete with the FEM some technical problems mentioned in the literature must be addressed. One of the main being the imposition of the essential boundary conditions. The EFGM in [1] uses moving least squares (MLS) shape functions for the approximation of the field variable. These shape functions do not possess the Kronecker-delta property and essential boundary condition are imposed using the method of Lagrange multipliers. The dimension of the final system of equations is increased and the stiffness matrix is no longer positive definite [2]. Here the EFGM is reformulated using the local maximum entropy shape functions recently presented in [3]. As the local maximum entropy shape functions possess the Kronecker-delta property at the boundary so essential boundary conditions can be implemented directly as in the FEM.

2 MAXIMUM ENTROPY SHAPE FUNCTION

Consider mutually independent events $x_1, x_2, ..., x_n$ within a sample space Ω with unknown probabilities $p_1, p_2, ..., p_n$ respectively. The quantity for measuring the amount of information or uncertainty of the finite scheme is termed as information entropy [4] and is given as

$$H(p_1, ..., p_n) = -\sum_{i=1}^{N} p_i \log p_i. \tag{1}$$

The most likely probability distribution with constraints $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} p_i g_r(x_i) = \langle g_r(x) \rangle$, where $\langle g_r(x) \rangle$ is known as the expectation of $g_r(x)$ can be determined using Jaynes' principle of maximum entropy.

$$Maximize\ [H(p_1, p_2, ..., p_n)] = -\sum_{i=1}^{n} p_i \log p_i. \tag{2}$$

We can consider the probabilities above to be the unknown shape functions we wish to determine which will, however, be highly non-local and non-interpolating (termed global maximum entropy shape functions). The local maximum entropy shape function formulation is summarized in [3] as

$$H(p, m) = -\sum_{i=1}^{n} p_i \log\left(\frac{p_i}{m_i}\right). \tag{3}$$

Here m_i is the prior distribution, which is used in the calculation of p_i. Using the analogy between probabilities and shape functions and using the above strategy

$$maximize\ \left(H(\phi, w) = -\sum_{i=1}^{n} \phi_i \log\left(\frac{\phi_i}{w_i}\right)\right), \tag{4}$$

subject to the linear reproducing constraints

$$\sum_{i=1}^{n} \phi_i = 1,\ \ \sum_{i=1}^{n} \phi_i \widetilde{x}_i = 0,\ \ \sum_{i=1}^{n} \phi_i \widetilde{y}_i = 0. \tag{5}$$

Where w_i is the prior distribution, i.e. any weight function, and ϕ_i are the shape functions. The shape functions are found by the method of Lagrange multipliers [1]

$$\phi_i = \frac{Z_i}{Z},\ \ Z_i = w_i e^{-\lambda_1 \widetilde{x}_i - \lambda_2 \widetilde{y}_i},\ \ Z = \sum_{j=1}^{n} Z_j. \tag{6}$$

Here λ_1 and λ_2 are the Lagrange multipliers determined using the dual formulation, i.e. to minimize F

$$F = \log Z(\lambda_1, \lambda_2). \tag{7}$$

F is a convex function and Newton's method is used to solve Equation (7). The expression for the derivatives of the shape functions is given as

$$\nabla\phi_i = \phi_i \left(\nabla f_i - \sum_{i=1}^{n} \phi_i \nabla f_i\right) \tag{8}$$

where

$$\nabla f_i = \frac{\nabla w_i}{w_i} + \lambda + \widetilde{x}^i \left[H^{-1} - H^{-1}A\right], \quad A = \sum_{k=1}^{n} \phi_k \widetilde{x}^k \otimes \frac{\nabla w_k}{w_k}. \tag{9}$$

H is the Hessian matrix and $\otimes$ is the dyadic product of two vectors.

3 IMPLEMENTATION

To demonstrate the use of these shape functions in the EFGM two benchmark problems are now presented. A cantilever beam subjected to parabolic traction at the free end and an infinite plate with circular hole subjected to unidirectional tension.

[1] Not the same use as for essential boundary conditions

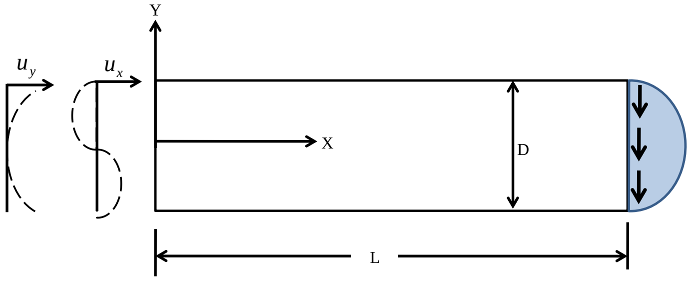


Figure 1: Geometry and coordinate system the cantilever beam problem

3.1 Cantilever Beam

The behaviour of a cantilever beam subjected to parabolic traction at the free end is examined. The geometry, coordinate system and boundary conditions for the problem, which are more complicated than is often appreciated, are given in Figure 1. The exact solutions for the displacement and stress fields are given in [5] as

$$u(x,y) = \frac{Py}{6EI}\left[(6L-3x)\,x + (2+\nu)\,y^2 - \frac{3D^2}{2}(1+\nu)\right], \tag{10a}$$

$$v(x,y) = -\frac{P}{6EI}\left[3\nu y^2 (L-x) + (3L-x)\,x^2\right] \tag{10b}$$

and

$$\sigma_{xx} = \frac{P(L-x)\,y}{I}, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -\frac{P}{2I}\left[\frac{D^2}{4} - y^2\right]. \tag{11}$$

E is the modulus of elasticity, ν is the Poisson's ratio and I is the second moment of area. The problem is solved for the plane stress case with $P = 1000$, $\nu = 0.3$, $E = 3 \times 10^7$, $D = 12$, $L = 48$ and unit thickness, all in compatible units. $99\,(11 \times 9)$ nodes, $40\,(10 \times 4)$ background cells and 4×4 gauss quadrature per cell is used, while 4 gauss quadrature per line cell is used for the integration of the force on the traction boundary. Figure: 2(a) shows the normal stress σ_{xx} and Figure: 2(b) shows the shear stress σ_{xy} at $x = L/2$ vs y. The numerical solution for the stresses can be seen to be almost the same as the exact solution.

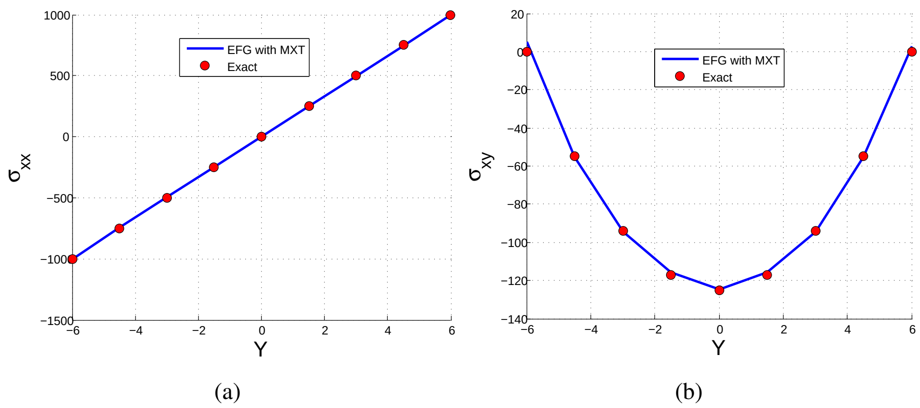


Figure 2: Normal and shear stress at x=L/2 for the cantilever beam problem

3.2 Infinite Plate with a circular hole

Consider an infinite plate with circular hole of radius 1 in the centre . Due to symmetry only a portion of the upper right quadrant is modelled as shown in Figure 3. The exact solution for the stress field is given in [5]. In this analysis a plane stress condition is assumed with $E = 1000$ and $\nu = 0.25$. $121\,(11 \times 11)$ nodes and $100\,(10 \times 10)$ background cells with (4×4) gauss quadrature per cell and 4 gauss quadrature per line cell is used for the integration on the traction boundary. Figure 4(a) shows σ_{xx} vs y while Figure 4(b) shows σ_{yy} vs x. Once again the numerical solutions match closely the analytical solutions.

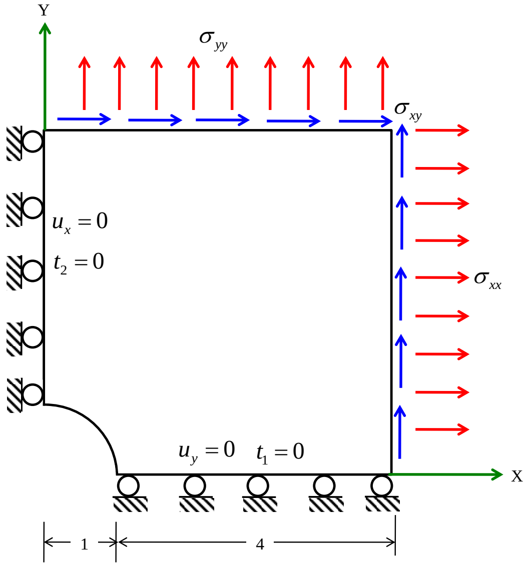
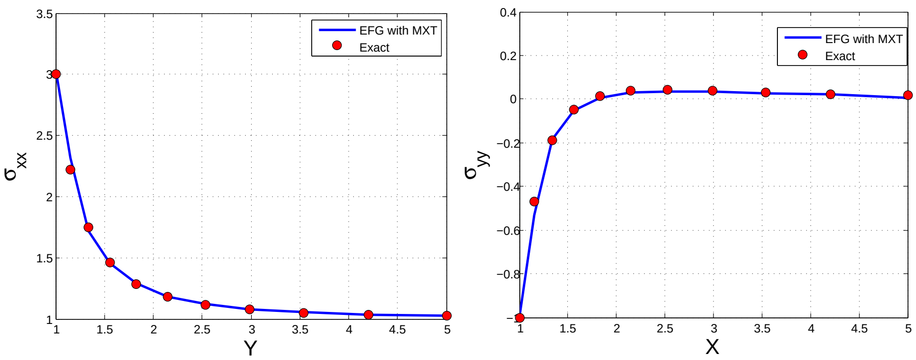


Figure 3: Geometry and coordinate system the infinite plate with a hole problem



(a) (b)

Figure 4: Normal stress at x=0 and at y=0 for the infinite plate with a hole problem

4 CONCLUSION

In this paper the recently developed local maximum entropy shape functions are used in the EFGM. These allow the imposition of essential boundary conditions directly as in the FEM. The two benchmark problems, a cantilever beam subjected to parabolic traction at the free end and an infinite plate with a circular hole subjected to unidirectional tensile load are analyzed by the current approach. The numerical results shows good agreement with the analytical solutions.

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