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Research Article

Solution of Fractional Kinetic Equations Associated with the (p, q)-Mathieu-Type Series

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In this paper, our aim is to finding the solutions of the fractional kinetic equation related with the (p, q)-Mathieu-type series through the procedure of Sumudu and Laplace transforms. The outcomes of fractional kinetic equations in terms of the Mittag-Leffler function are presented.

1. Introduction and Preliminaries

Fractional calculus (FC) can be a valuable mathematical method for considering the integrals and derivatives in a fractional order. The fractional calculus has been evolved and utilized in numerous engineering and analysis areas. In several disparate sectors, along with applied research, material science, mathematical physics, chemistry, and architecture, the theory of fractional differential equations and their implementations has played a key role. The complex conditions program at a basic stem of differential equations, which illustrates the amount of modification of a star's chemical composition with each configuration in terms of generation and annihilation reaction levels. The expansion and sweeping statement of fractional kinetic equations related with different special functions were established (see [1–11] for details). Nowadays, several scholars are developing a simplified structure of the fractional kinetic equation involving the Mathieu-type series to make the dynamic state extremely relevant and acceptable in a few astrophysical problems.

The first Mathieu series was explored by Mathieu in his book Elasticity of Solid Bodies [12], which is represented as an infinite series of the following form:

$$S(\vartheta) = \sum_{\ell=1}^{\infty} \frac{2\ell}{\left(\ell^2 + \vartheta^2\right)^2}, \quad (\vartheta > 0). \tag{1}$$

An integral representation of (1) is defined as (see [13])

$$S(\vartheta) = \frac{1}{\vartheta} \int_0^{\infty} \frac{x \sin(\vartheta x)}{e^x - 1} dx. \tag{2}$$

A few curiously special cases and their solutions deal with integral representations, their another account with a fractional image power characterized by Cerone and Lenard ([14], p. 2, Equation (16)), Milovanovic and Pogány ([15], p. 181):

$$S_\mu(\vartheta) = \sum_{\ell=1}^{\infty} \frac{2\ell}{\left(\ell^2 + \vartheta^2\right)^{\mu+1}}, \quad (\mu > 0, \vartheta > 0). \tag{3}$$

Inspired fundamentally by the works of Cerone and Lenard [14] (see also [16]), Srivastava and Tomovski established a generalized Mathieu series family in [17].

$$S_\mu^{(\alpha,\beta)}(\vartheta, a) = S_\mu^{(\alpha,\beta)}\left(\vartheta, \{a_\ell\}_{\ell=1}^{\infty}\right) - \sum_{\ell=1}^{\infty} \frac{2a_\ell^\beta}{\left(a_\ell^\alpha + \vartheta^2\right)^\mu}, \tag{4}$$
$$(\alpha, \beta, \mu > 0, \vartheta > 0),$$

where it is tacitly presumed that the positive sequence $a = \{a_\ell\} = \{a_1, a_2, \ldots\}$ such that $\lim_{\ell\longrightarrow\infty} a_\ell = \infty$ and so, taken that, perhaps, the (4) infinite series converges, which is to say the preceding auxiliary series $\sum_{\ell=1}^{\infty}(1/a_\ell^{\alpha,\mu-\beta})$ is convergent.

Within the continuation, in terms of the following power series, Tomovski and Mehrez [18] introduced a more generalized form of the series (4):

$$S_{\mu,\nu}^{(\alpha,\beta)}(\vartheta, a; z) = S_{\mu,\nu}^{(\alpha,\beta)}(\vartheta, \{a_\ell\}_{\ell=1}^{\infty}; z) = \sum_{\ell=1}^{\infty} \frac{2a_\ell^{\beta}(\nu)_\ell}{\left(a_\ell^{\alpha} + \vartheta^2\right)^{\mu}} \frac{z^\ell}{\ell!},$$
$$(\alpha, \beta, \vartheta, a, \mu > 0, |z| \le 1), \tag{5}$$

where

$$(\nu)_\ell = \begin{cases} 1, & (\ell = 0; \nu \in \mathbb{C}\backslash\{0\}), \\ \nu(\nu+1)\ldots(\nu+\ell-1), & (\nu = \ell \in \mathbb{N}; \nu \in \mathbb{C}). \end{cases} \tag{6}$$

Quite recently, Mehrez and Tomovski [19] found the more conventional version of the so-called (p, q)-Mathieu power series in the following version:

$$S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}(\vartheta, a; p, q; z) = \sum_{\ell=1}^{\infty} \frac{2a_\ell^{\beta}(\nu)_\ell \mathscr{B}_{p,q}(\tau+\ell, \omega-\tau) z^\ell}{\left(a_\ell^{\alpha} + \vartheta^2\right)^{\mu} \mathscr{B}(\tau, \omega-\tau)\ell!},$$
$$(\vartheta, a, \nu, \mu, \tau, \omega, \alpha, \beta \in \mathfrak{R}^+, |z| \le 1), \mathfrak{R}(q) \ge 0, \mathfrak{R}(p) > 0, \tag{7}$$

where $\mathscr{B}(\sigma, \rho; p, q)$ is the (p, q)-extended beta function provided by Choi et al. [20],

$$\mathscr{B}(\sigma, \rho; p, q) = \mathscr{B}_{p,q}(\sigma, \rho) = \int_0^1 x^{\sigma-1}(1-x)^{\rho-1} e^{-(p/x)-(q/(1-x))} \mathrm{d}x, \tag{8}$$

when $\min\{\mathfrak{R}(\sigma), \mathfrak{R}(\rho)\} > 0$; $\min\{\mathfrak{R}(p), \mathfrak{R}(q)\} \ge 0$. This (p, q)-Mathieu-type series contains, as limited cases, different aspects of the Mathieu-type series:

(i) When $p = q = 0$, then the generalized Mathieu-type power series is defined by

$$S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}(\vartheta, a; z) = S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}(\vartheta; a; 0, 0; z) = \sum_{\ell=1}^{\infty} \frac{2a_\ell^{\beta}(\nu)_\ell(\tau)_\ell z^\ell}{\left(a_\ell^{\alpha} + \vartheta^2\right)^{\mu}(\omega)_\ell \ell!},$$
$$(\vartheta, \alpha, \beta, \mu, a, \nu, \tau, \omega \in \mathfrak{R}^+, |z| \le 1). \tag{9}$$

(ii) By setting $\tau = \omega$ in (9), we obtain ([21], Equation (5), p. 974)

$$S_{(\mu,\nu)}^{(\alpha,\beta)}(\vartheta, a; z) = \sum_{\ell=1}^{\infty} \frac{2a_\ell^{\beta}(\nu)_\ell z^\ell}{\left(a_k^{\alpha} + \vartheta^2\right)^{\mu}\ell!}, \tag{10}$$
$$(\vartheta, \alpha, \beta, \mu, \nu, a \in \mathfrak{R}^+, |z| \le 1).$$

Furthermore, in the special cases when $\nu = z = 1$, we get the generalized Mathieu series (4).

2. Fractional Kinetic Equations

As of late, a startling interest has emerged in learning regarding the solution of fractional kinetic equations owing to their importance in astronomy and scientific material science. The kinetic equations of fractional order have been effectively utilized to decide certain physical wonders overseeing dissemination in permeable media and response and unwinding forms in complex frameworks. Subsequently, a large body of research into the application of these equations has been spread by publishing.

Haubold and Mathai [22] study the fractional differential equation between reaction rate $\mathfrak{J} = \mathfrak{J}(t)$, destruction rate $d = d(\mathfrak{J})$, and production rate $p = p(\mathfrak{J})$ as follows:

$$\frac{\mathrm{d}(\mathfrak{J})}{\mathrm{d}t} = -d(\mathfrak{J}_t) + p(\mathfrak{J}_t), \tag{11}$$

where $\mathfrak{J}_t$ is the function represented by $\mathfrak{J}_t(t^*) = \mathfrak{J}(t - t^*), t^* > 0$. Undermining the inhomogeneity in the number $\mathfrak{J}(t)$, (11) is given a special case as follows:

$$\frac{\mathrm{d}\mathfrak{J}_i}{\mathrm{d}t} = -c_i \mathfrak{J}_i(t), \tag{12}$$

where the primary condition $\mathfrak{J}_i(t = 0) = \mathfrak{J}_0$ is the number of density of species i at time $t = 0$. Neglecting index i and integrating, (12) becomes

$$\mathfrak{J}(t) - \mathfrak{J}_0 = c_0 D_t^{-1} \mathfrak{J}(t). \tag{13}$$

We keep in mind that the standard fractional integral operator is ${}_0D_t^{-1}$.

In fact, the fractional sweep argument of the standard kinetic equation (13) is defined by Haubold and Mathai [22] inside the equation:

$$\mathfrak{J}(t) - \mathfrak{J}_0 = c^{\varsigma}{}_0D_t^{-\varsigma} \mathfrak{J}(t), \tag{14}$$

where ${}_0D_t^{-\varsigma}$ is the most common Riemann–Liouville (R–L) fractional integral operator. More details of R–L in [23] are defined as

$${}_0D_t^{-\varsigma} f(x) = \frac{1}{\Gamma(\varsigma)} \int_0^t (t-u)^{\varsigma-1} f(u) \mathrm{d}u, \quad (x > 0, \mathfrak{R}(\varsigma) > 0). \tag{15}$$

The solution for (14), fractional equation, is given by (see [22])

$$\mathfrak{J}(t) = \mathfrak{J}_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\varsigma k + 1)} (ct)^{\varsigma k}. \tag{16}$$

Further, Saxena et al. [24, 25] explored the generalized type solution of (14) in terms of a generalized Mittag-Leffler function (see [26–28] for details),

$$E_{\varsigma,\ell}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\varsigma k + \ell)} \quad (\varsigma, \ell, z \in \mathbb{C}; \mathfrak{R}(\varsigma) > 0; \mathfrak{R}(\ell) > 0), \tag{17}$$

and the function $E_{\varsigma,\ell}$ is now called the two-parameter Mittag-Leffler function (also known as the Wiman function). The extension of (17) is called three-parameter Mittag-Leffler function (or else Prabhakar's function), and Garra and Garrappa [29] introduced this in terms of a series representation.

$$E^{\varepsilon}_{\varsigma,\ell}(z) = \sum_{k=0}^{\infty} \frac{(\varepsilon)_k}{(\varsigma k + \ell)k!} z^k, \quad (\varsigma, \ell, \varepsilon, z \in \mathbb{C}, \mathfrak{R}(\varsigma) > 0). \tag{18}$$

For the effectiveness and significance of the fractional kinetic equations in specific astronomy issues, the authors establish a modern and encourage generalized form of the fractional kinetic equation pertaining to the (p,q)-Mathieu-type power series utilizing the strategy of Laplace transform. Furthermore, the findings obtained here are very capable of generating a large range of established and (presumably) novel outcomes.

3. Solution of Generalized Fractional Kinetic Equations

In this section, we obtain a fractional kinetic equation pertaining to the (p,q)-Mathieu-type power series using the Laplace transforms technique.

We recall the Laplace transform of $f(x)$ as defined by Sneddon [30]:

$$\mathcal{F}(s) = \mathfrak{L}\{f(x); s\} = \int_0^{\infty} e^{-sx} f(x) \mathrm{d}x, \quad (\mathfrak{R}(s) > 0). \tag{19}$$

Theorem 1. *If $\gamma, \varsigma, d > 0$; $d \neq \gamma$; $a, \vartheta, \alpha, \beta, \nu, \mu, \tau, \omega \in \mathfrak{R}^+$, $\mathfrak{R}(p) > 0$, $\mathfrak{R}(q) \geq 0$, $|\gamma t| \leq 1$, then the equation solution,*

$$\mathfrak{I}(t) - \mathfrak{I}_0 S^{(\alpha,\beta)}_{(\mu,\nu,\tau,\omega)}(\vartheta, a; p, q; \gamma t) = -d^{\varsigma}\, {}_0D_t^{-\varsigma} \mathfrak{I}(t), \tag{20}$$

holds the formula

$$\mathfrak{I}(t) = \mathfrak{I}_0 \sum_{\ell=1}^{\infty} \frac{2a_{\ell}^{\beta}(\nu)_{\ell} \mathscr{B}_{p,q}(\tau + \ell, \omega - \tau)(\gamma t)^{\ell}}{\left(a_{\ell}^{\alpha} + \vartheta^2\right)^{\mu} \mathscr{B}(\tau, \omega - \tau)} E_{\varsigma,\ell+1}(-d^{\varsigma} t^{\varsigma}). \tag{21}$$

Proof. The Laplace transform of the R–L fractional integral operator given by the authors in [23] is as follows:

$$\mathscr{L}\{{}_0D_t^{-\varsigma} f(t); s\} = s^{-\varsigma} \mathfrak{I}(s), \tag{22}$$

where in (19), $\mathfrak{I}(s)$ is defined. Then, using the Laplace transform on both sides of the (20) and using (7) and (22) order, we get

$$\begin{aligned} \mathfrak{L}[f(t); s] &= \mathfrak{I}_0 \mathfrak{L}\left[S^{(\alpha,\beta)}_{(\mu,\nu,\tau,\omega)}(\vartheta, a; p, q; \gamma t); s\right] - d^{\varsigma} \mathfrak{L}\left[{}_0D_t^{-\varsigma} f(t); s\right], \\ \mathfrak{I}(s) &= \mathfrak{I}_0 \int_0^{\infty} e^{-st} \sum_{\ell=1}^{\infty} \frac{2a_{\ell}^{\beta}(\nu)_{\ell} \mathscr{B}_{p,q}(\tau + \ell, \omega - \tau)(\gamma t)^{\ell}}{\left(a_{\ell}^{\alpha} + \vartheta^2\right)^{\mu} \mathscr{B}(\tau, \omega - \tau)\ell!} \mathrm{d}t - d^{\varsigma} s^{-\varsigma} \mathfrak{I}(s), \end{aligned} \tag{23}$$

$$\mathfrak{I}(s) + d^{\varsigma} s^{-\varsigma} \mathfrak{I}(s) = \mathfrak{I}_0 \sum_{\ell=1}^{\infty} \frac{2a_{\ell}^{\beta}(\nu)_{\ell} \mathscr{B}_{p,q}(\tau + \ell, \omega - \tau)(\gamma)^{\ell}}{\left(a_{\ell}^{\alpha} + \vartheta^2\right)^{\mu} \mathscr{B}(\tau, \omega - \tau)\ell!} \int_0^{\infty} e^{-st} t^{\ell} \mathrm{d}t. \tag{24}$$

Under these conditions, calculating the integral in (24) term by term and using $\mathfrak{L}\{t^{\ell}; s\} = s^{-(\ell+1)}\Gamma(\ell+1)$, we have

$$\mathfrak{I}(s)\left\{1 + \left(\frac{d}{s}\right)^{\varsigma}\right\} = \mathfrak{I}_0 \sum_{\ell=1}^{\infty} \frac{2a_{\ell}^{\beta}(\nu)_{\ell} \mathscr{B}_{p,q}(\tau + \ell, \omega - \tau)(\gamma)^{\ell}}{\left(a_{\ell}^{\alpha} + \vartheta^2\right)^{\mu} \mathscr{B}(\tau, \omega - \tau)\ell!} \frac{\Gamma(\ell+1)}{s^{(\ell+1)}}. \tag{25}$$

Using $(1 + (d/S)^{\varsigma})^{-1}$ geometric series expansion for $d < |s|$, we have

$$\mathfrak{I}(s) = \mathfrak{I}_0 \sum_{\ell=1}^{\infty} \frac{2a_{\ell}^{\beta}(\nu)_{\ell} \mathscr{B}_{p,q}(\tau + \ell, \omega - \tau)(\gamma)^{\ell}}{\left(a_{\ell}^{\alpha} + \vartheta^2\right)^{\mu} \mathscr{B}(\tau, \omega - \tau)s^{(\ell+1)}} \sum_{m=0}^{\infty} \frac{(1)_m}{m!} \left[-\left(\frac{d}{S}\right)^{\varsigma}\right]^m. \tag{26}$$

Taking inverse Laplace transform on both sides of (26) and using $\mathfrak{L}^{-1}\{s^{-\varsigma}; t\} = (t^{\varsigma-1}/\Gamma(\varsigma))$ for $\mathfrak{R}(\varsigma) > 0$, we obtain

$$\begin{aligned} \mathfrak{I}(t) = \mathfrak{I}_0 \sum_{\ell=1}^{\infty} &\frac{2a_{\ell}^{\beta}(\nu)_{\ell} \mathscr{B}_{p,q}(\tau + \ell, \omega - \tau)(\gamma t)^{\ell}}{\left(a_{\ell}^{\alpha} + \vartheta^2\right)^{\mu} \mathscr{B}(\tau, \omega - \tau)} \\ &\cdot \left\{\sum_{m=0}^{\infty} (-1)^m \frac{(dt)^{m\varsigma}}{\Gamma(m\varsigma + \ell + 1)}\right\}. \end{aligned} \tag{27}$$

Interpreting the result in (27) in the view of (17), the necessary result is (21). □

Corollary 1. *If $\gamma, d, \varsigma > 0$; $d \neq \gamma$; $a, \vartheta, \alpha, \beta, \mu, \tau, \nu, \omega \in \mathfrak{R}^+$, $|\gamma t| \leq 1$, then equation*

$$\mathfrak{I}(t) - \mathfrak{I}_0 S^{(\alpha,\beta)}_{(\mu,\nu,\tau,\omega)}(\vartheta, a; \gamma t) = -d^{\varsigma}\, {}_0D_t^{-\varsigma} \mathfrak{I}(t), \tag{28}$$

has the solution

$$\mathfrak{I}(t)=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^{\beta}(\nu)_\ell(\tau)_\ell(\gamma t)^\ell}{\left(a_\ell^{\alpha}+\vartheta^2\right)^{\mu}(\omega)_\ell}E_{\varsigma,\ell+1}\left(-d^{\varsigma}t^{\varsigma}\right). \tag{29}$$

Corollary 2. *If $\gamma, d, \varsigma>0$; $\gamma\neq d$; $\vartheta, a, \alpha, \beta, \mu, \nu\in\Re^{+}$, $|\gamma t|\leq 1$, then equation*

$$\mathfrak{I}(t)-\mathfrak{I}_0 S_{(\mu,\nu)}^{(\alpha,\beta)}(\vartheta, a;\gamma t)=-d^{\varsigma}{}_0D_t^{-\varsigma}\mathfrak{I}(t), \tag{30}$$

has the solution

$$\mathfrak{I}(t)=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^{\beta}(\nu)_\ell(\gamma t)^\ell}{\left(a_\ell^{\alpha}+\vartheta^2\right)^{\mu}}E_{\varsigma,\ell+1}\left(-d^{\varsigma}t^{\varsigma}\right). \tag{31}$$

Theorem 2. *If $d, \eta, \varsigma>0$; $\vartheta, \alpha, \beta, \mu, \nu, \tau, \omega, a\in\Re^{+}$, $\Re(p)>0$, $\Re(q)\geq 0$, $|\eta t|\leq 1$, then equation*

$$\mathfrak{I}(t)-\mathfrak{I}_0 S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}(\vartheta, a; p, q;\eta t^{\varsigma})=-\left[\sum_{k=1}^{n}\binom{n}{k}d^{-\varsigma k}{}_0D_t^{-\varsigma k}\right]\mathfrak{I}(t), \tag{32}$$

has the solution

$$\mathfrak{I}(t)=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^{\beta}(\nu)_\ell\mathscr{B}_{p,q}(\tau+\ell,\omega-\tau)\Gamma(\ell\varsigma+1)\left(\eta t^{\varsigma}\right)^\ell}{\left(a_\ell^{\alpha}+\vartheta^2\right)^{\mu}\mathscr{B}(\tau,\omega-\tau)\ell!}E_{\varsigma,\ell\varsigma+1}^{n}\left(-d^{\varsigma}t^{\varsigma}\right). \tag{33}$$

Proof. Now, using the Laplace transform on both sides of (32) and using (7) and (22) lead to

$$\begin{aligned}\mathfrak{L}[\mathfrak{I}(t);s]&=\mathfrak{I}_0\mathfrak{L}\left[S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}(\vartheta, a; p, q;\eta t^{\varsigma});s\right]\\&-\mathfrak{L}\left[\left\{\sum_{k=1}^{n}\binom{n}{k}d^{-\varsigma k}{}_0D_t^{-\varsigma k}\right\}\mathfrak{I}(t);s\right],\end{aligned} \tag{34}$$

which upon solving for $\mathfrak{I}(s)$ yields

$$\begin{aligned}\mathfrak{I}(s)&=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^{\beta}(\nu)_\ell\mathscr{B}_{p,q}(\tau+\ell,\omega-\tau)(\eta)^\ell}{\left(a_\ell^{\alpha}+\vartheta^2\right)^{\mu}\mathscr{B}(\tau,\omega-\tau)\ell!}\frac{\Gamma(\ell\varsigma+1)}{s^{(\ell\varsigma+1)}}\\&\cdot\left\{1+\left(\frac{d}{s}\right)^{\varsigma}\right\}^{-n}.\end{aligned} \tag{35}$$

Employing the binomial formula $(1-x)^{-\delta}=\sum_{k=0}^{\infty}((\delta)_k/k!)x^k$, which converges for $|x|<1$, we have

$$\begin{aligned}\mathfrak{I}(s)&=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^{\beta}(\nu)_\ell\mathscr{B}_{p,q}(\tau+\ell,\omega-\tau)\Gamma(\ell\varsigma+1)(\eta)^\ell}{\left(a_\ell^{\alpha}+\vartheta^2\right)^{\mu}\mathscr{B}(\tau,\omega-\tau)\ell!s^{(\ell\varsigma+1)}}\\&\cdot\sum_{k=0}^{\infty}\frac{(n)_k}{k!}\left[\left(\frac{-d^{\varsigma}}{s^{\varsigma}}\right)^k\right].\end{aligned} \tag{36}$$

Taking inverse Laplace transform on both sides of (36), we obtain

$$\begin{aligned}\mathfrak{I}(t)&=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^{\beta}(\nu)_\ell\mathscr{B}_{p,q}(\tau+\ell,\omega-\tau)\Gamma(\ell\varsigma+1)\left(\eta t^{\varsigma}\right)^\ell}{\left(a_\ell^{\alpha}+\vartheta^2\right)^{\mu}\mathscr{B}(\tau,\omega-\tau)\ell!}\\&\times\left\{\sum_{k=0}^{\infty}(-1)^k\frac{(n)_k\left((\mathrm{d}t)^{\varsigma}\right)^k}{\Gamma(\varsigma k+\ell\varsigma+1)}\right\}.\end{aligned} \tag{37}$$

Interpreting the result (37) in the view of (18), we get the necessary result (33). □

Corollary 3. *If $d, \varsigma, \eta>0$; $a, \vartheta, \alpha, \beta, \mu, \nu, \tau, \omega\in\Re^{+}$, $|\eta t|\leq 1$, then equation*

$$\mathfrak{I}(t)-\mathfrak{I}_0 S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}(\vartheta, a;\eta t^{\varsigma})=-\left[\sum_{k=1}^{n}\binom{n}{k}d^{-\varsigma k}{}_0D_t^{-\varsigma k}\right]\mathfrak{I}(t), \tag{38}$$

is given by

$$\mathfrak{I}(t)=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^{\beta}(\nu)_\ell(\tau)_\ell\Gamma(\ell\varsigma+1)\left(\eta t^{\varsigma}\right)^\ell}{\left(a_\ell^{\alpha}+\vartheta^2\right)^{\mu}(\omega)_\ell\ell!}\mathrm{E}_{\varsigma,\ell\varsigma+1}^{n}\left(-d^{\varsigma}t^{\varsigma}\right). \tag{39}$$

Corollary 4. *If $d, \varsigma, \eta>0$; $a, \vartheta, \alpha, \beta, \mu, \nu\in\Re^{+}$, $|\eta t|\leq 1$, then equation*

$$\mathfrak{I}(t)-\mathfrak{I}_0 S_{(\mu,\nu)}^{(\alpha,\beta)}(\vartheta, a;\eta t^{\varsigma})=-\left[\sum_{k=1}^{n}\binom{n}{k}d^{-\varsigma k}{}_0D_t^{-\varsigma k}\right]\mathfrak{I}(t), \tag{40}$$

gives a solution

$$\mathfrak{I}(t)=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^{\beta}(\nu)_\ell\Gamma(\ell\varsigma+1)\left(\eta t^{\varsigma}\right)^\ell}{\left(a_\ell^{\alpha}+\vartheta^2\right)^{\mu}\ell!}\mathrm{E}_{\varsigma,\ell\varsigma+1}^{n}\left(-d^{\varsigma}t^{\varsigma}\right). \tag{41}$$

Theorem 3. *If $\varsigma, d>0$; $a, \vartheta, \alpha, \beta, \nu, \mu, \omega, \tau\in\Re^{+}$, $\Re(p)>0$, $\Re(q)\geq 0$, $|t|\leq 1$, then the solution of the equation*

$$\mathfrak{I}(t)-\mathfrak{I}_0 S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}(\vartheta, a; p, q; d^{\varsigma}t^{\varsigma})=-d^{\varsigma}{}_0D_t^{-\varsigma}\mathfrak{I}(t), \tag{42}$$

holds the formula

$$\mathfrak{I}(t)=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^{\beta}(\nu)_\ell\mathscr{B}_{p,q}(\tau+\ell,\omega-\tau)\Gamma(\varsigma\ell+1)(dt)^\ell}{\left(a_\ell^{\alpha}+\vartheta^2\right)^{\mu}\mathscr{B}(\tau,\omega-\tau)\ell!}E_{\varsigma,\varsigma\ell+1}\left(-d^{\varsigma}t^{\varsigma}\right). \tag{43}$$

Proof. The thorough proof of Theorem 3 is similar to that of Theorem 1, so we omit the details. □

Corollary 5. *If $\varsigma, d>0$; $a, \vartheta, \alpha, \beta, \nu, \mu, \omega, \tau\in\Re^{+}$, $|t|\leq 1$, then equation*

$$\mathfrak{I}(t)-\mathfrak{I}_0 S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}(\vartheta, a; d^{\varsigma}t^{\varsigma})=-d^{\varsigma}{}_0D_t^{-\varsigma}\mathfrak{I}(t), \tag{44}$$

has the solution

$$\mathfrak{I}(t)=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^\beta(\nu)_\ell(\tau)_\ell\Gamma(\varsigma\ell+1)(dt)^\ell}{\left(a_\ell^\alpha+\vartheta^2\right)^\mu(\omega)_\ell\ell!}E_{\varsigma,\varsigma\ell+1}\left(-d^\varsigma t^\varsigma\right). \tag{45}$$

Corollary 6. *If* $d,\varsigma>0;\alpha,\beta,\mu,\nu,\vartheta,a\in\mathfrak{R}^+,|t|\le 1,$ *then equation*

$$\mathfrak{I}(t)-\mathfrak{I}_0 S_{(\mu,\nu)}^{(\alpha,\beta)}\left(\vartheta,a;d^\varsigma t^\varsigma\right)=-d^\varsigma{}_0D_t^{-\varsigma}\mathfrak{I}(t), \tag{46}$$

has the solution

$$\mathfrak{I}(t)=\mathfrak{I}_0\sum_{\ell=1}^{\infty}\frac{2a_\ell^\beta(\nu)_\ell\Gamma(\varsigma\ell+1)(dt)^\ell}{\left(a_\ell^\alpha+\vartheta^2\right)^\mu\ell!}E_{\varsigma,\varsigma\ell+1}\left(-d^\varsigma t^\varsigma\right). \tag{47}$$

4. Examples

Details of the Mathieu-type series and their implementations can be contained in the monographs by different authors [21, 31]. The integral transform, named the Sumudu Transform, promotes the method of fathoming differential and integral equations inside time. It turns out that the Sumudu transform has exceptional and useful properties and that it is important in knowing the issues of research and regulation of the management of active situations. Here, we use the concept of the Sumudu transform given by Watugala [32] as follows:

$$\mathcal{G}(\omega)=\mathfrak{S}[f(z);\omega]=\int_0^\infty e^{-z}f(\omega z)\mathrm{d}z;\quad \omega\in\left(-\varepsilon_1,\varepsilon_2\right), \tag{48}$$

where the exponentially bound function class in the $\mathbb{T}$ is as follows:

$$\mathbb{T}=\left\{f(z)\,|\,\exists M,\varepsilon_1,\varepsilon_2>0,|f(z)|<Me^{\left(|z|/\varepsilon_j\right)},t\in(-1)^j\times[0,\infty]\right\}. \tag{49}$$

In addition, the Sumudu transform given in (48) can be calculated directly from the Fourier integral. The Sumudu transformation tends to be the theoretical dual transformation of Laplace. It is interesting to equate the Sumudu transform (48) with the well-known Laplace transform (see, for example, [33]):

$$\mathcal{F}(p)=\mathfrak{L}[f(z)]=\int_0^\infty e^{-pz}f(z)\mathrm{d}z,\quad \mathfrak{R}(p)>0. \tag{50}$$

Equation (15) can be described in the following form by using the Sumudu transformation theorem [34–36]:

$$\mathfrak{S}\{{}_0D_z^{-\varsigma}f(z)\}=\mathfrak{S}\left\{\frac{z^{\varsigma-1}}{\Gamma(\varsigma)}\right\}\cdot\mathfrak{S}\{f(z)\}=u^\varsigma\mathcal{G}(u). \tag{51}$$

It is simple to see that the function $f(z)=z^\delta$ by using the Sumudu transform is given as

$$\mathfrak{S}[f(z)]=\int_0^\infty e^{-z}(yz)^\delta\mathrm{d}z=u^\delta\Gamma(1+\delta),\quad (\mathfrak{R}(\delta)>-1). \tag{52}$$

The interested readers should search [37–41] for more subtle elements almost transforming the Sumudu and its properties as opposed to the Laplace transform.

Because of the significance of the abovementioned observation, in this section, we evaluate the solutions of generalized fractional kinetic equations by applying the Sumudu transform using the same analytical method as in Theorems 1, 2, and 3, presented in examples 1, 2, and 3.

Example 1. If $\gamma,d,\varsigma,\delta>0;\quad d\ne\gamma;\quad a,\vartheta,\alpha,\beta,\nu,\mu,\tau,\omega\in\mathfrak{R}^+,\mathfrak{R}(p)>0,\mathfrak{R}(q)\ge 0,|\gamma t|\le 1$, then the solution of the equation

$$\mathfrak{I}(t)-\mathfrak{I}_0t^{\delta-1}S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}\left(\vartheta,a;p,q;\gamma t^\varsigma\right)=-d^\varsigma{}_0D_t^{-\varsigma}\mathfrak{I}(t), \tag{53}$$

holds the formula

$$\mathfrak{I}(t)=\mathfrak{I}_0t^{\delta-2}\sum_{\ell=1}^{\infty}\frac{2a_\ell^\beta(\nu)_\ell\mathcal{B}_{p,q}(\tau+\ell,\omega-\tau)\left(\gamma t^\varsigma\right)^\ell}{\left(a_\ell^\alpha+\vartheta^2\right)^\mu\mathcal{B}(\tau,\omega-\tau)}E_{\varsigma,\varsigma\ell+\delta-1}\left(-d^\varsigma t^\varsigma\right). \tag{54}$$

Example 2. If $d,\eta,\varsigma>0;\quad \vartheta,\alpha,\beta,\mu,\nu,\tau,\omega,a\in\mathfrak{R}^+,\mathfrak{R}(p)>0,\mathfrak{R}(q)\ge 0,|\eta t|\le 1$, then equation

$$\mathfrak{I}(t)-\mathfrak{I}_0t^{\delta-1}S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}\left(\vartheta,a;p,q;\eta t^\varsigma\right)=-\left[\sum_{k=1}^{n}\binom{n}{k}d^{-\varsigma k}{}_0D_t^{-\varsigma k}\right]\mathfrak{I}(t), \tag{55}$$

has the solution

$$\mathfrak{I}(t)=\mathfrak{I}_0t^{\delta-2}\sum_{\ell=1}^{\infty}\frac{2a_\ell^\beta(\nu)_\ell\mathcal{B}_{p,q}(\tau+\ell,\omega-\tau)\Gamma(\ell\varsigma+1)\left(\eta t^\varsigma\right)^\ell}{\left(a_\ell^\alpha+\vartheta^2\right)^\mu\mathcal{B}(\tau,\omega-\tau)\ell!}\cdot\mathrm{E}_{\varsigma,\ell\varsigma+\delta-1}^n\left(-d^\varsigma t^\varsigma\right). \tag{56}$$

Example 3. If $\varsigma,d>0,\quad a,\vartheta,\alpha,\beta,\nu,\mu,\omega,\tau\in\mathfrak{R}^+,\mathfrak{R}(p)>0,\mathfrak{R}(q)\ge 0,|t|\le 1$, then the solution of the equation

$$\mathfrak{I}(t)-\mathfrak{I}_0t^{\delta-1}S_{(\mu,\nu,\tau,\omega)}^{(\alpha,\beta)}\left(\vartheta,a;p,q;d^\varsigma t^\varsigma\right)=-d^\varsigma{}_0D_t^{-\varsigma}\mathfrak{I}(t), \tag{57}$$

holds the formula

$$\mathfrak{I}(t)=\mathfrak{I}_0t^{\delta-2}\sum_{\ell=1}^{\infty}\frac{2a_\ell^\beta(\nu)_\ell\mathcal{B}_{p,q}(\tau+\ell,\omega-\tau)\Gamma(\varsigma\ell+1)(dt)^\ell}{\left(a_\ell^\alpha+\vartheta^2\right)^\mu\mathcal{B}(\tau,\omega-\tau)\ell!}\cdot E_{\varsigma,\varsigma\ell+\delta-1}\left(-d^\varsigma t^\varsigma\right). \tag{58}$$

5. Concluding Remarks

It is not troublesome to get a few who encourage closely fractional kinetic equations and their solutions as those displayed here by Theorem 1, 2, and 3 and its Corollaries. It is popular to support that a variety of other special cases of our results can also be obtained as shown in Section 4, if we take $p = q = 0$, $\tau = \omega$, and $\nu = z = 1$, and we can obtain nine different findings. We leave those to the interested reader as an exercise. Moreover, if we set $p = q = 0$ and $\tau = \omega$ in our main results, then we arrive at [4]. In this article, we considered the traditional kinetic equation as a recent fractional generalization and proposed their solutions. Besides, in view of near connections of the (p, q)-Mathieu-type series and (p, q)-Mittag-Leffler with other special functions, it does not seem difficult to construct different known and unused fractional kinetic equations. Hence, the examined which comes about in this paper would, at once, grant numerous outcomes about including assorted special functions happening within the issues of astronomy, scientific mathematical science, and engineering.

Data Availability

The data can be obtained from the corresponding author on reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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