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Solution of Fractional Ordinary Differential Equations by Natural Transform

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Abstract . The fractional calculus for the Natural transform is introduced and some non homogenous fractional ordinary differential equations are solved using Natural transform.

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1. Introduction

The Natural transform, initially was defined by Khan and Khan [5] as N - transform, who studied their properties and applications. Later , Belgacem et al. [3, 10] defined its inverse and studied some additional fundamental properties of this integral transform and named it the Natural transform. Applications of Natural transform in the solution of differential and integral equations and for the distribution and Boehmians spaces can be found in [1, 4, 5, 6,7, 8, 9, 10].

In this paper, we give definitions of fractional calculus and state properties of Natural transform. Further, using derivative of Natural transform of fractional order, solution of fractional ordinary differential equations are derived.

2. Fractional Calculus and Natural Transform

The theory of fractional calculus plays an important role in many fields of pure and applied mathematics. Fractional integrals and derivatives, in association with different integral transforms, are used to solve different types of differential and integral equations. A derivative of fractional order, in the Abel - Riemann sense [4] , is defined by

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[1] The corresponding author

$$D^{\alpha}[f(t)] = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha \le m \\ \frac{d^m}{dt^m} f(t) \;, & \alpha = m \end{cases} \tag{1}$$

where $m \in Z^+$ and $\alpha \in R^+$ and the integral operator is defined by implementing an integral of fractional order in Abel - Riemann sense as

$$D^{-\alpha}[f(t)] = J^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad , \quad t > 0, \alpha > 0 \tag{2}$$

Podlubny [9] introduced the fundamental properties of fractional integration and differentiation, respectively, described as

$$J^{\alpha}[t^n] = \frac{\Gamma(1+n)}{\Gamma(1+n+\alpha)} t^{n+\alpha} \tag{3}$$

$$D^{\alpha}[t^n] = \frac{\Gamma(1+n)}{\Gamma(1+n-\alpha)} t^{n-\alpha} \quad . \tag{4}$$

The Caputo derivative [9] is defined by

$${}^{C}D^{\alpha}[f(t)] = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t) \;, & \alpha = m \end{cases} \tag{5}$$

and one of the property is

$$J^{\alpha}[{}^{C}D^{\alpha}[f(t)]] = f(t) - \sum_{k=0}^{\infty} f^{(k)}(0) \frac{t^k}{k!} \tag{6}$$

The Natural transformation $R(s,u)$ of the function $f(t)$ for all $t \ge 0,$ is given by [3, 10]

$$N^{+}[f(t)] = R(s,u) = \frac{1}{u} \int_0^{\infty} e^{-st} f(ut) dt \quad , \quad s > 0, u > 0 \tag{7}$$

i.e.

$$R(s,u) = \frac{1}{u} \int_0^{\infty} e^{-\frac{st}{u}} f(t) dt \; , \tag{8}$$

where t,u are time variables and s is the frequency variable, provided the function $f(t)$ is defined in the set A by

$$A = \{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{t/\tau_j}; t \in (-1)^j \times [0,\infty) \},$$

where M is a constant of finite number, τ_1 and τ_2 may be finite or infinite.
The discrete form of Natural transform is given by [3]

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$$N^{+}[f(t)] = R(s,u) = \sum_{n=0}^{\infty} \frac{n! a_n u^n}{s^{n+1}} \tag{9}$$

The inverse Natural transformation is defined by [3] and [10]

$$N^{-1}[R(s,u)] = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{st}{u}} R(s,u) ds \ . \tag{10}$$

The duality relation between Natural-Laplace and Natural-Sumudu transform is given by

$$R(s,u) = \frac{1}{u} F\left(\frac{s}{u}\right) \quad ; \quad R(s,u) = \frac{1}{s} G\left(\frac{u}{s}\right) \tag{11}$$

where $R(s,u)$ denote the Natural transform, $F(s)$ is the Laplace transform and $G(u)$ is the Sumudu transform. The Sumudu transform, its properties and applications can be seen in [2,4]. Properties of Natural transform, some mentioned below, can be seen in [3,5,6,10] .

1. *Natural transform of derivative* : The *derivative* of $f(t)$ with respect to t and the n *th order derivative* of the same with respect to t are, respectively, defined by

$$N^{+}[f'(t)] = R_1(s,u) = \frac{s}{u} R(s,u) - \frac{f(0)}{u} \tag{12}$$

$$N^{+}[f''(t)] = R_2(s,u) = \frac{s^2 R(s,u) - sf(0)}{u^2} - \frac{f'(0)}{u} \ , \tag{13}$$

and

$$N[f^{(n)}(t)] = R_n(s,u) = \frac{s^n}{u^n} R(s,u) - \sum_{k=0}^{n} \frac{s^{n-(k+1)}}{u^{n-k}} f^{(k)}(0) \ . \tag{14}$$

2. *Convolution Theorem* : If $F(s,u)$ and $G(s,u)$ are Natural transforms of the functions $f(t)$ and $g(t)$, respectively, defined in set A then the *convolution* is given by

$$N^{+}[(f * g)(t)] = u\, F(s,u) G(s,u) \ . \tag{15}$$

3. When $f(t) = \delta(t)$ (the Dirac delta function), the Natural transform becomes

$$N^{+}[\delta(t)] = R(s,u) = \frac{1}{u}, \tag{16}$$

and when $f(t) = \frac{t^{n-1}}{\Gamma(n)}$ or $\frac{t^{n-1}}{(n-1)!}$; $n > 0$, the Natural transform is

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$$N^{+}[\frac{t^{n-1}}{\Gamma(n)}] = \frac{u^{n-1}}{s^{n}}. \tag{17}$$

4. *Multiple Shift* : When the function $f(t)$ in set A is multiplied with shift functions t and t^{n} , the Natural transforms of these are given by

$$N^{+}[tf(t)] = \frac{u}{s}\frac{d}{du}uR(s,u), \tag{18}$$

and

$$N^{+}[t^{n} f(t)] = \frac{u^{n}}{s^{n}}\frac{d^{n}}{du^{n}}u^{n}R(s,u). \tag{19}$$

On distribution spaces, the Natural transform is defined by $f(t) \in \mathcal{D}'$

$$N^{+}[f(t)] = R(s,u) = \left\langle f(t), \frac{1}{u}e^{-\frac{st}{u}} \right\rangle . \tag{20}$$

One may refer to [6, 7, 8] for details.

The fractional integral for the function $f(t)$, as in (2) , can also be written as

$$D^{-\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(\tau) \tag{21}$$

Proposition 1: If $F(s,u)$ is the Natural transform of the function $f(t)$, then the Natural transform of fractional integral of order α is defined by

$$N^{+}\left[D^{-\alpha}[f(t)]\right] = \frac{u^{\alpha}}{s^{\alpha}}F(s,u) \tag{22}$$

Proof : Applying Natural transform in the equation (21) and invoking properties given by (15) and (17), we have

$$N^{+}[D^{-\alpha}[f(t)]] = \frac{1}{\Gamma(\alpha)}N^{+}[t^{\alpha-1}] * N^{+}[f(\tau)]$$

$$= u\frac{u^{\alpha-1}}{s^{\alpha}}F(s,u) = \frac{u^{\alpha}}{s^{\alpha}}F(s,u).$$

Proposition 2 : If $F(s,u)$ is the Natural transform of the function $f(t)$, then the Natural transform of fractional derivative of order α is defined as

$$N^{+}[f^{(\alpha)}(t)] = \frac{s^{\alpha}}{u^{\alpha}}R(s,u) - \sum_{k=0}^{n}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0) . \tag{23}$$

Proof : We consider the derivative of Laplace transform as

$$L[D^{\alpha} f(t)] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{k} [D^{\alpha-k-1}(f(t))]_{t=0} \ .$$

Further, considering the duality relation of Laplace and Natural transform and property of derivative of Natural transform and the fractional derivative for Natural transform is derived ((proof can be seen in [3, 5]).

3. Applications of Natural Transform to Non - homogenous Fractional Ordinary Differential Equations

We intend to use the Natural transform to obtain solutions of certain fractional differential equations [4]

Example 1 : To obtain the solution of non-homogenous fractional ordinary differential equation

$$D^{\alpha}[U(t)] = -U(t) + \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + t^{2} - t, \ \ t > 0, 0 < \alpha \le 1, \qquad (24)$$

subject to the initial condition $U(0) = 0$.

Solution : Invoking the definition of the Natural transform and, simultaneously, using (12), (13) and (23) together with values from [3,10], we express (24) as

$$N^{+}[D^{\alpha}[U(t)]] + N^{+}[U(t)] = N^{+}\left[\frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}\right] - N^{+}\left[\frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}\right] + N^{+}[t^{2}] - N^{+}[t],$$

i.e.

$$\frac{s^{\alpha}}{u^{\alpha}} R(s,u) - \sum_{k=0}^{n} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} f^{(k)}(0) + R(s,u) = \frac{2}{\Gamma(3-\alpha)} N^{+}\left[t^{2-\alpha}\right] - \frac{1}{\Gamma(2-\alpha)} N^{+}\left[t^{1-\alpha}\right] + N^{+}[t^{2}] - N^{+}[t],$$

i.e.

$$\frac{s^{\alpha}}{u^{\alpha}} R(s,u) + R(s,u) - D^{\alpha-1}[U(t)]_{t=0} = \frac{2}{\Gamma(3-\alpha)} \Gamma(3-\alpha) \frac{u^{2-\alpha}}{s^{3-\alpha}} - \frac{1}{\Gamma(2-\alpha)} \Gamma(2-\alpha) \frac{u^{1-\alpha}}{s^{2-\alpha}} + \Gamma(3) \frac{u^{2}}{s^{3}} - \frac{u}{s^{2}}$$

i.e.

$$R(s,u)\left(1+\frac{s^\alpha}{u^\alpha}\right)=2\frac{u^{2-\alpha}}{s^{3-\alpha}}-\frac{u^{1-\alpha}}{s^{2-\alpha}}+\Gamma(3)\frac{u^2}{s^3}-\frac{u}{s^2}$$

i.e.
$$R(s,u)\left(u^\alpha+s^\alpha\right)=2\frac{u^2}{s^{3-\alpha}}-\frac{u}{s^{2-\alpha}}+2\frac{u^{\alpha+2}}{s^3}-\frac{u^{\alpha+1}}{s^2}$$

i.e.
$$R(s,u)\left(1+\frac{s^\alpha}{u^\alpha}\right)=2\frac{u^2}{s^3}-\frac{u}{s^2}+2\frac{u^{\alpha+2}}{s^{\alpha+3}}-\frac{u^{\alpha+1}}{s^{\alpha+2}}$$

$$=2\frac{u^2}{s^3}+2\frac{u^\alpha u^2}{s^\alpha s^3}-\frac{u}{s^2}-\frac{u^\alpha u}{s^\alpha s^2}$$

$$=2\frac{u^2}{s^3}\left(1+\frac{s^\alpha}{u^\alpha}\right)-\frac{u}{s^2}\left(1+\frac{s^\alpha}{u^\alpha}\right)$$

i.e.
$$R(s,u)=2\frac{u^2}{s^3}-\frac{u}{s^2} \tag{25}$$

Using inverse Natural transform (10), we obtain the solution of (24) as
$$U(t)=t^2-t. \tag{26}$$

Example 2 : Solve the non - homogenous fractional ordinary differential equation
$$D^{0.5}[U(t)]+U(t)=t^2-\frac{\Gamma(3)}{\Gamma(2.5)}t^{1.5} \quad , \quad t>0, \tag{27}$$
with the initial condition $U(0)=0.$

Solution : We invoke the Natural transform in (27) and employ given conditions and properties using tables of Natural transform from [3,10], we obtain
$$N^+[D^{0.5}[U(t)]]+N^+[U(t)]=N^+\left[t^2\right]+\frac{\Gamma(3)}{\Gamma(2.5)}N^+\left[t^{1.5}\right] \quad ,$$

i.e.
$$\frac{u^{0.5}}{s^{0.5}}R(s,u)-[D^{\alpha-1}U(t)]_{t=0}+R(s,u)=2\frac{u^2}{s^3}+2\frac{u^{1.5}}{s^{2.5}}$$

i.e.
$$R(s,u)\left(1+\frac{u^{0.5}}{s^{0.5}}\right)=2\frac{u^{1.5}u^{0.5}}{s^{2.5}u^{0.5}}+2\frac{u^{1.5}}{s^{2.5}} \quad ,$$

$$=2\frac{u^{1.5}}{s^{2.5}}\left(1+\frac{u^{0.5}}{s^{0.5}}\right)$$

i.e.
$$R(s,u)=2\frac{u^{1.5}}{s^{2.5}} \tag{28}$$

Taking inverse Natural transform (10), in (28) , we obtain the required solution

$$U(t) = t^2 \tag{29}$$

Conclusion : The current paper describes an approach by which it is found that the Natural transform has an extensive affinity with the solutions of differential and integral equations, and more specifically with the fractional differential equations which has been the centre forum of this paper. The solution of fractional ordinary differential equations can be obtained in the form of distributional fractional ordinary differential equations, when distributional Natural transform are invoked.

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References

1. Al-Omari, S. K. Q. On the applications of Natural transform, International Journal of Pure and Applied Mathematics, 85 (4) , 729 -- 744 (2013).
2. Belgacem, F. B. M., Karaballi, A. A. and Kalla, S. L. Analytical investigations of the Sumudu transform and applications to integral production equations, Mathematical Problems in Engineering 2003 (3), 103 --118 (2003).
3. Belgacem, F. B. M. and Silambarasan, R. Theory of Natural Transform, Math. Engg, Sci. Aerospace (MESA), 3 (1),, 99--124 (2012).
4. Bulut, H., Baskonus, H. M. and Belgacem, F. B. M. The analytical solution of some fractional ordinary differential equations by the Sumudu transform method, Abstract and Applied Analysis 2013, 1-6 (2013), Article ID 2013875.
5. Khan, Zafar H. and Khan, Waqar A., N- Transform - Properties and Applications, NUST J. Engg. Sci. 1 (1), 127--133 (2008).
6. Loonker, Deshna and Banerji, P. K. Natural transform for distribution and Boehmian spaces, Math. Engg. Sci. Aerospace 4 (1) , 69 -- 76 (2013).
7. Loonker, Deshna and Banerji, P. K. Natural transform and solution of integral equations for distribution spaces, Amer. J. Math. Sci. (2013) (To Appear).
8. Loonker, Deshna and Banerji, P. K. Applications of Natural transform to differential equations, J. Indian Acad. Math. 35 (1), 151 --158 (2013).
9. Podlubny, I. Fractional Differential Equations, Vol. 198, Mathematics in Science and Engineering, Academic Press, San Diego, USA (1999).
10. Silambarasan, R. and Belgacem, F. B. M. Applications of the Natural transform to Maxwell's equations, Prog. Electromagnetic Research Symposium Proc. Suzhou, China , 899 -- 902 (2011).