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Solution of Potts Model for Phase Transition

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It is interesting to study how the properties of thermodynamic quantities near the transition point depend on the type of interaction in cooperative systems. In spin systems, Onsager's exact solution of the two-dimensional Ising model is well known. Here, the solution of another model with a little different type of interaction is given exactly in the same two-dimensional square lattice. In this model, the energy of interaction between two neighbouring units is proportional to the scalar product of the vectors representing them, and each unit is capable of four configurations, represented by four directions at right angles, as shown

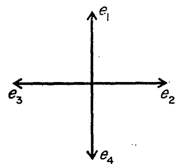

Fig. 1.

in Fig. 1. Originally, using a duality transformation, Potts has found the transition point of this model,[1] together with the other ones which have three directions at 120°, or r configurations with only two different energies of interaction.[1)~4)]

The Hamiltonian of this model is given by

$$\mathcal{H} = -J\sum(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j), \tag{1}$$

where $\boldsymbol{\mu}_i$ and $\boldsymbol{\mu}_j$ assume $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ or $\boldsymbol{e}_4$, respectively.

Now, the transfer matrix of the i-th row, as shown in Fig. 2, is

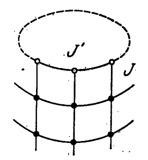

Fig. 2.

$$\widetilde{V}_i=\begin{pmatrix} e^{K} & 1 & 1 & e^{-K} \\ 1 & e^{K} & e^{-K} & 1 \\ 1 & e^{-K} & e^{K} & 1 \\ e^{-K} & 1 & 1 & e^{K} \end{pmatrix}, \quad \text{and} \quad K=J/kT. \tag{2}$$

Notice that the matrix $\widetilde{V}_i$ can be expressed in the following direct product,

$$\widetilde{V}_i=V_i\dot{X}V_i' \tag{3}$$

where

$$V_i=\begin{pmatrix} e^{K/2} & e^{-K/2} \\ e^{-K/2} & e^{K/2} \end{pmatrix}=e^{K/2}+e^{-K/2}C_i$$

$$=2\sinh K\exp\left(\frac{K}{2}C_i\right), \tag{4}$$

$$V_i'=2\sinh K\exp\left(\frac{K}{2}C_i'\right), \tag{5}$$

$$\text{and} \quad K^*=\log\coth\left(\frac{K}{2}\right)=2\tanh^{-1}(e^{-K}). \tag{6}$$

The operator which describes the addition of a new tier is

$$\widetilde{V}_I=\widetilde{V}_1\dot{X}\widetilde{V}_2\cdots\dot{X}\widetilde{V}_n=\prod_{i=1}^{n}\widetilde{V}_i. \tag{7}$$

The state function of the added tier can be represented as follows:

$$\psi=\psi(\cdots;\mu_j,\mu_j';\cdots);\quad \mu_j,\mu_j'=\pm1, \tag{8}$$

and the operators $\{C_j\}$ and $\{C_j'\}$ have the following effect:

$$C_j\psi(\cdots;\mu_j,\mu_j';\cdots)$$
$$=\psi(\cdots;-\mu_j,\mu_j';\cdots),\quad \text{etc.} \tag{9}$$

Moreover, the operator which introduces a similar interaction J between adjacent units in a tier is represented as follows:

$$\widetilde{V}_{II}=\exp\left(\frac{K'}{2}\Sigma S_jS_{j+1}\right)\dot{X}$$
$$\times\exp\left(\frac{K'}{2}\Sigma S_j'S_{j+1}'\right), \tag{10}$$

using the diagonal operators $\{S_j\}$ and $\{S_j'\}$:

$$S_j\psi(\cdots;\mu_j,\mu_j';\cdots)$$
$$=\mu_j\psi(\cdots;\mu_j;\mu_j';\cdots),\quad \text{etc.} \tag{11}$$

Therefore, the addition of one tier of units with interactions J and J' is represented by

$$\widetilde{V}=\widetilde{V}_{II}\widetilde{V}_I. \tag{12}$$

Consequently, we obtain the partition function of the system as follows:

$$Z_{n,m}(J,J')=\mathrm{Tr}\,\widetilde{V}^m=(2\sinh K)^{nm}$$
$$\times\mathrm{Tr}\left[\exp\left(\frac{K^*}{2}\Sigma C_j\right)\exp\left(\frac{K'}{2}\Sigma S_jS_{j+1}\right)\right]^m$$
$$\times\dot{X}\left[\exp\left(\frac{K^*}{2}\Sigma C_j'\right)\exp\left(\frac{K'}{2}\Sigma S_j'S_{j+1}'\right)\right]^m$$
$$=\left[Z^I_{n,m}\left(\frac{J}{2},\frac{J'}{2}\right)\right]^2, \tag{13}$$

where $Z^I_{n,m}$ indicates the solution for the Ising model. The partition function per unit, λ is given as follows, using Onsager's solution,[5] in the limit of an infinitely large crystal:

$$\log\frac{\lambda}{4}=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi\log(\cosh K\cosh K'$$
$$-\sinh K\cos\omega-\sinh K'\cos\omega')d\omega\,d\omega'. \tag{14}$$

Near the transition point, the specific heat is, in the case $J'=J$,

$$C_V/Nk\sim-0.9880\log|T-T_c|;$$
$$(-0.4945\log|T-T_c|$$
$$\text{for the Ising model}). \tag{15}$$

After we have known the final results, we easily find the following one-to-one correspondence between the vector $\boldsymbol{\mu}_i$ and a pair of Ising spins $(\delta_{i1},\delta_{i2})$:

$$\boldsymbol{e}_1\leftrightarrow(1,1),\ \boldsymbol{e}_2\leftrightarrow(1,-1),\ \boldsymbol{e}_3\leftrightarrow(-1,1),$$
$$\boldsymbol{e}_4\leftrightarrow(-1,-1). \tag{16}$$

The following relation holds owing to the above correspondence (16):

$$(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j) = \frac{1}{2}(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}). \qquad (17)$$

This is equivalent to the representation of the matrix $\widetilde{V}_i$ as the direct product (3). Therefore, in general, the following relation can be obtained, irrespective of both dimensions and the range of interaction:

$$\begin{aligned} Z_N\{J_{ij}\} &= \mathrm{Tr}\exp \sum K_{ij}(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j) \\ &= \mathrm{Tr}_{1,2}\exp\left[\frac{1}{2}\sum k_{ij}(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2})\right] \\ &= [Z_N^I\{J_{ij}/2\}]^2. \end{aligned} \qquad (18)$$

This means that the Potts model is equivalent to uncoupled double Ising models with interaction of half strength. The correlation function is given by

$$<(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j)>(J) = <\delta_i \cdot \delta_j>^I\left(\frac{J}{2}\right). \qquad (19)$$

Consequently, the long range order $R(J)$ is given by

$$R^2(J) = \lim_{|i-j|\to\infty} <(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j)>(J) = \left[R^I\left(\frac{J}{2}\right)\right]^2. \qquad (20)$$

These relations seems to be intuitively obvious from the above equivalence. In particular, in the two-dimensional square lattice ($J' = J$), the long range order near the transition point is

$$\begin{aligned} R &= (1 - 1/\sinh^4 K)^{1/8} \\ &\sim 1.0945(1 - T/T_c)^{1/8}, \end{aligned} \qquad (21)$$

and the final expression in the above equation (21) is quite the same as that of the Ising model.[5]

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