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Solution of Seventh Order Boundary Value Problem by Differential Transformation Method

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Abstract: Differential transformation method is used to find the solution of the seventh order boundary value problem. The approximate solution of the problem is calculated in the form of a rapid convergent series. Two numerical examples have been considered to illustrate the efficiency and implementation of the method and the results are compared with the method developed in [15].

Keywords: Differential transformation method . seventh order boundary value problems . linear and nonlinear problems . series solution

INTRODUCTION

The boundary value problems play an important role in many fields. The seventh order boundary value problems generally arise in modelling induction motors with two rotor circuits. The induction motor behavior is represented by a fifth order differential equation model. This model contains two stator state variables, two rotor state variables and one shaft speed. Normally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage or rotor distributed parameters. To avoid the computational burden of additional state variables when additional rotor circuits are required, model is often limited to the fifth order and rotor impedance is algebraically altered as function of rotor speed. This is done under the assumption that the frequency of rotor currents depends on rotor speed. This approach is efficient for the steady state response with sinusoidal voltage, but it does not hold up during the transient conditions, when rotor frequency is not a single value. So, the behavior of such models show up in the seventh order boundary value problems [7].

Presently, the literature on the numerical solutions of seventh order boundary value problems and associated eigen value problems is rare available. Siddiqi and Ghazala [8-14] presented the solutions of fifth, sixth, eighth, tenth and twelfth order boundary value problems using polynomial and non-polynomial spline techniques. Siddiqi and Twizell [16-18] presented the solution of eighth, tenth and twelfth order boundary value problems using eighth, tenth and twelfth degree splines respectively.

Zhou [19] introduced the differential transformation method to solve linear and nonlinear initial value problems in electric circuit analysis. Differential transformation method has been used to solve integral and integro-differential systems, differential algebraic equation, partial differential equations, ordinary differential equation, system of ordinary differential equations, Lane-Emden type equations arising in astrophysics, Helmholtz equation [1, 2, 3, 5, 6, 20, 21, 22]. Differential transformation method has been also applied to study the free vibration analysis of rotating ron-prismatic beams, unsteady rolling motion of spheres equation in inclined tubes [23, 24]. This method constructs approximate solution in the form of a polynomial and finally gives a series solution. It is different from the traditional high order Taylor's series, as, it requires a long time in calculation and needs computation of the necessary derivatives of the data functions.

In this paper, the differential transformation method is applied to solve the seventh order boundary value problems. The given problem can be transformed into a recurrence relation, using differential transformation operations, which leads to a series solution. Consider the following seventh order boundary value problem

$$\left.\begin{aligned} &u^{(7)}(x) = f(x,u(x)), a \le x \le b,\\ &u^{(i)}(a) = A_i, i = 0,1,2,3,\\ &u^{(j)}(b) = B_j, i = 0,1,2, \end{aligned}\right\} \quad (1.1)$$

where A_i, i = 0,1,2,3 and B_j, j = 0,1,2 are finite real constants, also $f(x,u(x))$ is a continuous function on [a,b].

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DIFFERENTIAL TRANSFORMATION METHOD [4]

The differential transformation of the kth derivative of a function $f(x)$ is defined by

$$F(k)=\frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{2.1}$$

and the inverse differential transformation of $F(k)$ is defined by

$$f(x)=\sum_{k=0}^{\infty}F(k)(x-x_0)^k \tag{2.2}$$

In real applications, the function $f(x)$ can be expressed as a finite series and Eq. (2.2) can be written as

$$f(x)=\sum_{k=0}^{n}F(k)(x-x_0)^k \tag{2.3}$$

Substituting Eq. (2.1) into Eq. (2.2), gives

$$f(x)=\sum_{k=0}^{\infty}(x-x_0)^k\frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{2.4}$$

which is the Taylor's series for $f(x)$ at $x = x_0$. From Eq. (2.1) and Eq. (2.2) following theorems can be deduced

Theorem 1: If

$$f(x)=g(x)\pm h(x)\text{, then } F(k)=G(k)\pm H(k)$$

Theorem 2: If

$$f(x)=cg(x)\text{, then } F(k)=cG(k)$$

where c is a constant.

Theorem 3: If $f(x)=\dfrac{d^m g(x)}{dx^m}$, then

$$F(k)=\frac{(k+m)!}{k!}G(k+m)$$

Theorem 4 If $f(x)=g(x)h(x)$, then

$$F(k)=\sum_{k_1=0}^{k}G(k_1)H(k-k_1)$$

Theorem 5: If $f(x)=e^{\lambda x}$, then

$$F(k)=\frac{\lambda^k}{k!}$$

Theorem 6: If $f(x)=x^n$, then

$$F(k)=\delta(k-n)=\begin{cases}1, k=n\\ 0, k\neq n\end{cases}$$

Theorem 7: If $f(x)=g_1(x)g_2(x)\cdots g_n(x)$ then

$$F(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_1}^{k_2}G_1(k_1)G_2(k_2-k_1)\cdots G_n(k-k_{n-1})$$

To implement the method, two numerical examples are considered in the following section.

NUMERICAL EXAMPLES

Example 3.1: Consider the linear seventh order boundary value problem

$$u^{(7)}(x)=xu(x)+e^x(x^2-2x-6), 0\leq x\leq 1 \tag{3.1}$$

Subject to the boundary conditions

$$\left.\begin{array}{l}u(0)=1, u^{(1)}(0)=0, u^{(2)}(0)=-1, u^{(3)}(0)=-2,\\ u(1)=0, u^{(1)}(1)=-e, u^{(2)}(1)=-2e,\end{array}\right\} \tag{3.2}$$

The exact solution of the problem (3.1) is $u(x)=(1-x)e^x$.

Applying the above theorems, the differential transformation of Eq. (3. 1) is obtained as

$$U(k+7)=\frac{k!}{(k+7)!}\left[\sum_{k_1=0}^{k}\delta(k_1-1)U(k-k_1)+\sum_{k_1=0}^{k}\frac{\delta(k_1-2)}{(k-k_1)!}-2\sum_{k_1=0}^{k}\frac{\delta(k_1-1)}{(k-k_1)!}-\frac{6}{k!}\right] \tag{3.3}$$

Using Eq. (2.1), the boundary conditions (3.2) can be transformed at $x_0 = 0$ as:

$$\left.U(0)=1, U(1)=0, U(2)=\frac{-1}{2!}, U(3)=\frac{-2}{3!}, \sum_{k=0}^{n}U(k)=0, \sum_{k=0}^{n}kU(k)=-e, \sum_{k=0}^{n}k(k-1)U(k)=-2e\right\} \tag{3.4}$$

where n is a sufficiently large integer. Using the recurrence relation (3.3) and the transformed boundary conditions (3.4), the following series solution up to $O(x^{14})$ can be obtained as:

$$u(x)=1-\frac{x^2}{2}-\frac{x^3}{3}+Ax^4+Bx^5+Cx^6-\frac{x^7}{840}-\frac{x^8}{5760}-\frac{x^9}{45360}-\frac{x^{10}}{403200}-\frac{x^{11}}{3991680}+\frac{(\frac{1}{30}+A)x^{12}}{3991680}+\frac{(\frac{1}{60}+B)x^{13}}{8648640}+O(x^{14}) \tag{3.5}$$

where the constants A, B and C can be determined, using Eq. (2.1), as:

$$A=\frac{u^{(4)}(0)}{4!}=U(4),B=\frac{u^{(5)}(0)}{5!}=U(5),C=\frac{u^{(6)}(0)}{6!}=U(6) \tag{3.6}$$

Taking n = 13, an algebraic system of linear equations in terms of A, B and C is obtained. The solution of the system yields

$$A=-0.1250000058927127,B=-0.033333320250896525,C=-0.006944451794759032$$

Finally, the series solution can be written as:

$$u(x)=1-\frac{x^2}{2}-\frac{x^3}{3}-0.125x^4-0.0333333x^5-0.00694445x^6-\frac{x^7}{840}-\frac{x^8}{5760}-\frac{x^9}{45360}-\frac{x^{10}}{403200}$$
$$-\frac{x^{11}}{3991680}-(2.29644\times10^{-8})x^{12}-(1.92708\times10^{-9})x^{13}+O(x^{14}) \tag{3.7}$$

The comparison of the approximate solution of problem (3.1) obtained by the differential transformation method and Adomian decomposition method [15] is given in Table 1, which shows that the present method is quite efficient.

Example 3.2 Consider the following seventh order nonlinear boundary value problem

$$u^{(7)}(x)=-e^{x}u^{2}(x)\,,0\le x\le1 \tag{3.8}$$

subject to the boundary conditions

$$\left.u(0)=1,u^{(1)}(0)=-1,u^{(2)}(0)=1,u^{(3)}(0)=-1,u(1)=e^{-1},u^{(1)}(1)=-e^{-1},u^{(2)}(1)=e^{-1}\right\} \tag{3.9}$$

The exact solution of the problem (3.2) is $u(x)=e^{-x}$.

Applying the above theorems, the differential transformation of Eq. (3. 8) is obtained as

$$U(k+7)=\frac{k!}{(k+7)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{U(k-k_2)U(k_2-k_1)}{k_1!} \tag{3.10}$$

Using Eq. (2.1) the boundary conditions (3.9) can be transformed at $x_0=0$ as:

$$\left.U(0)=1,U(1)=-1,U(2)=\frac{-1}{2!},U(3)=\frac{-1}{3!},\sum_{k=0}^{n}U(k)=e^{-1},\sum_{k=0}^{n}kU(k)=-e^{-1},\sum_{k=0}^{n}k(k-1)U(k)=e^{-1}.\right\} \tag{3.11}$$

where n is a sufficiently large integer. Using the recurrence relation (3.10) and the transformed boundary conditions (3.11), the following series solution up to $O(x^{14})$ can be obtained as:

$$u(x)=1-x+\frac{x^2}{2}-\frac{x^3}{6}+Ax^4+Bx^5+Cx^6-\frac{x^7}{5040}+\frac{x^8}{40320}-\frac{x^9}{362880}+\frac{x^{10}}{3628800}-\frac{(-1+48A)x^{11}}{39916800}-\frac{(1+240B)x^{12}}{479001600}+\frac{(\frac{1}{120}-2C)x^{13}}{8648640}+O(x^{14}) \tag{3.12}$$

Table 1: Comparison of numerical results for problem (3.1)

x	Exact solution	Approximate series solution	Absolute Error Present method	Absolute Error by ADM [15]
0.0	1.0000	1.0000	0.0000	0.0000
0.1	0.0994	0.0994	4.6585E-13	4.3972E-10
0.2	0.9771	0.9771	5.7126E-12	4.9251E-10
0.3	0.9449	0.9449	2.1299E-11	7.4067E-10
0.4	0.8950	0.8950	4.6995E-11	6.6537E-10
0.5	0.8243	0.8243	7.4307E-11	3.0059E-11
0.6	0.7288	0.7288	8.9219E-11	4.3591E-10
0.7	0.6041	0.6041	7.9767E-11	3.6735E-10
0.8	0.4451	0.4451	4.6686E-11	7.2753E-10
0.9	0.2459	0.2459	1.0960E-11	7.0036E-10
1.0	0.0000	6.9252E-16	6.9252E-16	2.2191E-10

Table 2: Comparison of numerical results for problem (3.2)

x	Exact solution	Approximate series solution	Absolute Error Present method	Absolute Error by ADM [15]
0.0	1.0000	1.0000	0.0000	0.0000
0.1	0.9048	0.9048	3.0198E-14	1.5676E-9
0.2	0.8187	0.8187	3.6903E-13	1.6418E-9
0.3	0.7408	0.7408	1.3749E-12	4.9680E-9
0.4	0.6703	0.6703	3.0308E-12	1.5514E-9
0.5	0.6065	0.6065	4.7868E-12	1.5274E-9
0.6	0.5488	0.5488	5.7388E-12	2.4958E-9
0.7	0.4965	0.4965	5.1207E-12	1.3993E-8
0.8	0.4493	0.4493	2.9893E-12	2.5593E-9
0.9	0.4065	0.4065	6.9944E-13	5.4089E-9
1.0	0.3678	0.3678	1.1102E-16	1.1034E-9

where the constants A, B and C can be determined, using Eq. (2.1), as:

$$A=\frac{u^{(4)}(0)}{4!}=U(4), B=\frac{u^{(5)}(0)}{5!}=U(5), C=\frac{u^{(6)}(0)}{6!}=U(6) \tag{3.13}$$

Taking n = 13, an algebraic system of linear equations in terms of A, B and C is obtained. The solution of the system yields

$$A=0.04166666704765717\,,\ B=-0.008333334180647071\,,\ C=0.001388889365963449$$

Finally, the series solution can be written as

$$\begin{aligned} u(x) &= 1-x+\frac{x^2}{2}-\frac{x^3}{6}+0.0416667x^4-0.00833333x^5+0.00138889x^6-\frac{x^7}{5040}+\frac{x^8}{40320}-\frac{x^9}{362880} \\ &+\frac{x^{10}}{3628800}-(2.50521\times 10^{-8})x^{11}+(2.08768\times 10^{-9})x^{12}-(1.60591\times 10^{-10})x^{13}+O(x^{14}) \end{aligned} \tag{3.14}$$

The comparison of the approximate solutions of problem (3.2) obtained by the differential transformation method and Adomian decomposition method [15] is given in Table 2, which shows that the present method is quite efficient.

CONCLUSION

In this paper, the differential transformation method has been applied to obtain the numerical solution of linear and nonlinear seventh order boundary

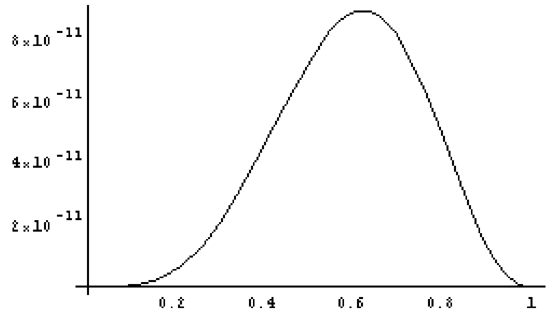

Fig. 1: Absolute error for problem (3.1)

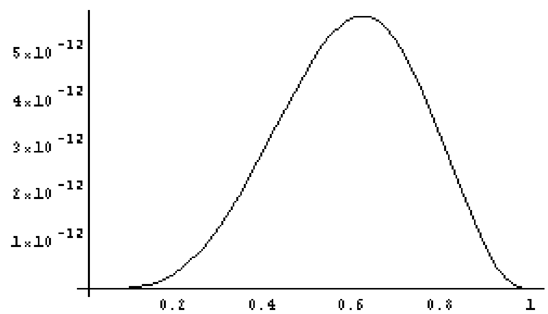

Fig. 1: Absolute error for problem (3.2)

value problems. The present method has been applied in a direct way without using linearization, discretization, or perturbation. Comparison of the numerical results with the existing technique [15] shows that the present method is more accurate.

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