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SOLUTION OF SUBSONIC AXIALLY-SYMMETRIC STREAM FIELDS

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In this paper we follow the results from [1] where we have formulated the boundary value model problem describing three-dimensional, axially-symmetric, irrotational, subsonic stream fields of ideal compressible fluid. Here we shall study solvability of this problem. We shall not deal with "classical" (i.e. smooth) solutions, but we shall devote ourselves to the problem of existence and uniqueness of the so called "generalized" (weak) solutions, making use of functional analytic methods.

1. FUNDAMENTAL ASSUMPTIONS, NOTATION AND FORMULATION OF THE PROBLEM

First, let us summarize the fundamental assumptions, notation and results from [1], where we have shown that the problem of irrotational, subsonic, adiabatic, axially-symmetric channel flow can be transformed to a nonlinear elliptic boundary value problem in the domain $P \subset E_2$ (E_2 denotes the plane).

P is a bounded domain with the Lipschitz boundary ∂P which consists of four arcs $L_1, L_2, \Gamma^1, \Gamma^2$. Γ^1 and Γ^2 are segments parallel to one of the coordinate axis z or r. The closure of P in the Euclidian topology of E_2 will be denoted by $\bar{P}$. Let $r > 0$ for every $(z, r) \in \bar{P}$. If we put

$$
\begin{aligned}
(1.1) \qquad R_1 &= \min \{r;\ \exists z \in E_1((z, r) \in \bar{P})\}\,,\\
R_2 &= \max \{r;\ \exists z \in E_1((z, r) \in \bar{P})\}\,,\\
Z_1 &= \min \{z;\ \exists r \in E_1((z, r) \in \bar{P})\}\,,\\
Z_2 &= \max \{z;\ \exists r \in E_1((z, r) \in \bar{P})\}
\end{aligned}
$$

(E_1 is the set of all real numbers), then $R_1 > 0$ and $\bar{P} \subset D = \langle Z_1, Z_2 \rangle \times \langle R_1, R_2 \rangle$. Let us assume that the arcs Γ^1 and Γ^2 lie on the boundary of the rectangle D.

In paper [1], we have studied the dependence of the fluid density on the gradient of a stream function and introduced the function $\beta(r, \zeta)$ with the following properties:

a) The domain of β is

$$\mathscr{D}(\beta) = \langle R_1, R_2 \rangle \times \langle 0, +\infty) .$$

b) There exist constants $\hat{\zeta}$, C_1, $C_2 > 0$ such that

$$0 < \frac{1}{R_2} \leqq \beta(r, \zeta) \leqq C_1 , \tag{1.2}$$

$$0 \leqq \frac{\partial \beta}{\partial \zeta}(r, \zeta) \leqq C_2 \quad \text{for every} \quad (r, \zeta) \in \mathscr{D}(\beta) ,$$

$$\frac{\partial \beta}{\partial \zeta}(r, \zeta) = 0 \quad \text{if} \quad r \in \langle R_1, R_2 \rangle \quad \text{and} \quad \zeta \in \langle \hat{\zeta}, +\infty) ,$$

c) $\beta \in C^1(\mathscr{D}(\beta))$ (i.e. β has continuous partial derivatives of the first order on its domain).

The "classical" problem (A) which describes subsonic irrotational compressible channel flow (formulated in [1]) consists in finding a function ψ which satisfies the equation

$$\frac{\partial}{\partial z}\left(\beta(r, (\nabla\psi)^2) \frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\beta(r, (\nabla\psi)^2) \frac{\partial \psi}{\partial r}\right) = 0 \tag{1.3}$$

in the domain P, and the boundary value conditions on ∂P:

$$\left.\frac{\partial \psi}{\partial n}\right| \Gamma^i = 0 , \quad i = 1, 2 , \tag{1.4}$$

$$\psi \mid L_1 = 0 , \quad \psi \mid L_2 = Q . \tag{1.5}$$

Here, $\nabla\psi = \operatorname{grad} \psi = (\partial\psi/\partial z, \partial\psi/\partial r)$, $(\nabla\psi)^2 = (\partial\psi/\partial z)^2 + (\partial\psi/\partial r)^2$, $\partial/\partial n$ is the derivative in the direction of the outer normal to Γ^i with respect to P, Q is a given constant (see [1]).

We do not specify the smoothness of ψ because, as we have already emphasized, we are concerned with the study of "weak" solutions. We shall not discuss the question in which measure the "weak" solutions give a true picture of real stream fields, but we know from experience (gained especially in the theory of elasticity) that problems formulated in a "weak" sense describe actual problems often better than their "classical" formulations.

For the solution of the "generalized" problem we shall use the monotone operators method.

2. GENERALIZED FORMULATION OF THE PROBLEM

First, let us briefly recall several important concepts from functional analysis. More detailed information can be found e.g. in [2] or [3].

By integral we mean, in the whole paper, the Lebesque integral.

Let $\Omega \subset E_2$ be a bounded domain with a Lipschitz boundary. Let $\mathscr{E}(\bar{\Omega})$ denote the linear space of all (real) functions infinitely differentiable in $\bar{\Omega}$.

Further, let

$$\mathscr{D}(\Omega) = \{u \in \mathscr{E}(\bar{\Omega});\ \operatorname{supp} u \subset \Omega\}.$$

supp u is defined as the closure of the set

$$\{x \in E_2;\ u(x) \neq 0\}$$

in the Euclidian topology of E_2. Hence $\mathscr{D}(\Omega)$ is the space of all infinitely differentiable functions with compact support in Ω.

We define a scalar product on $\mathscr{E}(\bar{\Omega})$

$$(u, v) = \iint \left(uv + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} \right) \mathrm{d}z\, \mathrm{d}r\,, \tag{2.1}$$

which induces a norm on $\mathscr{E}(\bar{\Omega})$:

$$\|u\| = (u, u)^{1/2}\,. \tag{2.1'}$$

Let $W_2^1(\Omega)$ and $\mathring{W}_2^1(\Omega)$ be the well-known Sobolev spaces, which we get by the completion of $\mathscr{E}(\bar{\Omega})$ and $\mathscr{D}(\Omega)$, respectively in norm (2.1'). It is known that $W_2^1(\Omega)$ is the space of all (equivalence classes of) $u \in L_2(\Omega)$ such that the derivatives in the sense of distributions $\partial u/\partial z$, $\partial u/\partial r \in L_2(\Omega)$. $W_2^1(\Omega)$ and $\mathring{W}_2^1(\Omega)$ are Hilbert spaces.

Now, let us pass to our problem. The set $P \subset E_2$ is a bounded domain with a Lipschitz boundary. Let us define the set

$$\mathfrak{A} = \{u \in \mathscr{E}(\bar{P});\ \operatorname{supp} u \cap (L_1 \cup L_2) = \emptyset\}.$$

$\mathfrak{A}$ is a linear subset of $\mathscr{E}(\bar{P})$. The closure of $\mathfrak{A}$ in the space $W_2^1(P)$ will be denoted by $\tilde{W}_2^1(P)$. It is evident that $\tilde{W}_2^1(P)$ is a Hilbert space. Let $u_0 \in W_2^1(P)$,

$$\begin{aligned} u_0 \mid L_1 &= 0\,, \\ u_0 \mid L_2 &= Q\,, \end{aligned} \tag{2.2}$$

where Q is the constant from the condition (1.5). (In practical cases it is possible to construct the function u_0.)

Problem (B) The "generalized" problem to (A) is to find $u \in W_2^1(P)$ which satisfies the conditions

1) $u - u_0 \in \tilde{W}_2^1(P)$,

2) for every $v \in \tilde{W}_2^1(P)$,

$$\iint_P \beta(r, (\nabla u)^2) \left(\frac{\partial u}{\partial z} \frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \frac{\partial v}{\partial r}\right) \mathrm{d}z \, \mathrm{d}r = 0 \,.^{*)}$$

The function β has been defined in [1] and has the properties (1.2). The function $u \in W_2^1(P)$ which satisfies 1) and 2) will be called a "weak" solution of Problem (A).

Since β is continuous and bounded and u, v with their first partial derivatives are elements of $L_2(P)$, the finite integral in 2) exists.

Remark 1. Let us outline the relation between the "generalized" and the "classical" formulations. If u is a "sufficiently" smooth solution of Problem (B), then $\beta(r, (\nabla u)^2) \in C^1(\bar{P})$ (see the property c) of β in (1.2)). By virtue of the Green theorem we get from the condition 2) of Problem (B) the equality

$$- \iint_P \left\{\frac{\partial}{\partial z}\left(\beta(r, (\nabla u)^2) \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial r}\left(\beta(r, (\nabla u)^2) \frac{\partial u}{\partial r}\right)\right\} v \, \mathrm{d}z \, \mathrm{d}r +$$

$$+ \int_{\partial P} \beta(r, (\nabla u)^2) \frac{\partial u}{\partial n} v \, \mathrm{d}S = 0$$

for every $v \in \mathfrak{A}$. ($\int_{\partial P} \ldots \mathrm{d}S$ denotes the line integral along the boundary ∂P).

In view of the properties of the set $\mathfrak{A}$ and with the help of standard considerations about integrals of continuous functions, we get

$$\frac{\partial}{\partial z}\left(\beta(r, (\nabla u)^2) \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial r}\left(\beta(r, (\nabla u)^2) \frac{\partial u}{\partial r}\right) = 0 \quad \text{in} \quad P \,,$$

$$\left.\frac{\partial u}{\partial n}\right| \Gamma^i = 0 \,, \quad i = 1, 2 \,.$$

Further, condition 1) of Problem (B) yields

$$u \mid L_1 = 0 \,, \quad u \mid L_2 = Q \,.$$

It means that a "sufficiently" smooth solution of Problem (B) is a "classical "solution of Problem (A).

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*) This can be written more simply:

$$\iint_P \beta(r, (\nabla u)^2) \, (\nabla u \,.\, \nabla v) \, \mathrm{d}z \, \mathrm{d}r = 0 \,,$$

where $\nabla u \,.\, \nabla v$ denotes the scalar product of vectors ∇u and ∇v.

In this paragraphs, we introduce several concepts and theorems which will be necessary in the following considerations. A complete theory of monotone operators can be found in Vajnberg's monograph [4].

If we say Banach space, we always mean a real Banach space. If $\mathscr{V}$ is a Banach space, then $\mathscr{V}^*$ denotes its dual, i.e., the Banach space of all real linear continuous functionals defined on $\mathscr{V}$, with the norm

$$\|f\| = \sup_{\substack{v \in \mathscr{V} \\ \|v\| = 1}} |\langle f, v \rangle|$$

for every $f \in \mathscr{V}^*$. The symbol $\langle f, v \rangle$ denotes the value of the functional f at an element $v \in \mathscr{V}$. Under the symbol

$$F : X \to Y$$

we understand the map F of the set X into the set Y with the domain $\mathscr{D}(F) = X$

Definition 1. Let $\mathscr{V}$ be a Banach space. An operator $T : \mathscr{V} \to \mathscr{V}^*$ is *monotone*, if

$$\langle T(u_1) - T(u_2), u_1 - u_2 \rangle \geqq 0 \tag{3.1}$$

for every $u_1, u_2 \in \mathscr{V}$. If the equality holds in (3.1) if and only if $u_1 = u_2$, we say that T is *strictly monotone.*

Definition 2. Let $\mathscr{V}$ be a Banach space. The operator $T : \mathscr{V} \to \mathscr{V}^*$ is *hemicontinuous*, if for arbitrary $u, h \in \mathscr{V}$

$$t \to 0 \Rightarrow T(u + th) \rightharpoonup T(u) .$$

Here $\rightharpoonup$ denotes the convergence in a weak topology of $\mathscr{V}^*$ (i.e. $\lim_{t \to 0} \langle T(u + th), v \rangle = \langle T(u), v \rangle$ for every $v \in \mathscr{V}$).

T is *coercive*, if

$$\lim_{\substack{\|v\| \to +\infty \\ v \in \mathscr{V}}} \frac{\langle T(v), v \rangle}{\|v\|} \to +\infty .$$

The fundamental assertion of the monotone operators theory is the following Browder-Minty theorem.

Theorem 1. *Let $\mathscr{V}$ be a reflexive Banach space, let $T : \mathscr{V} \to \mathscr{V}^*$ be monotone, coercive and hemicontinuous. Then T maps $\mathscr{V}$ onto $\mathscr{V}^*$ and hence the equation*

$$T(u) = f$$

has a solution for every $f \in \mathscr{V}^*$.

For the proof of Theorem 1 see e.g. [4].

Remark 2. If the operator T from Theorem 1 is strictly monotone, then the equation $T(u) = f$ has exactly one solution for every $f \in \mathscr{V}^*$.

Now, we shall introduce a simple concept which has not been used so far. Nevertheless, as we shall see, it may be useful in our considerations.

Definition 3. Let $\mathscr{V}$ be a Banach space, $T : \mathscr{V} \to \mathscr{V}^*$ an operator, $u \in \mathscr{V}$. For every $v, w \in \mathscr{V}$ let there exist finite

$$\tilde{V}T(u)(v, w) = \frac{\mathrm{d}}{\mathrm{d}t} \langle T(u + tw), v \rangle \big|_{t=0},$$

let $\tilde{V}T(u)(v, .)$ be a continuous linear functional defined on the space $\mathscr{V}$ for every $v \in \mathscr{V}$, and thus

$$\tilde{V}T(u) : \mathscr{V} \to \mathscr{V}^* .$$

Then we call $\tilde{V}T(u)$ *the weak Gâteaux differential* of the operator T at the point u.

We say that T has the weak Gâteaux differential on $\mathscr{V}$, if $\tilde{V}T(u) : \mathscr{V} \to \mathscr{V}^*$ exists at every $u \in \mathscr{V}$.

Remark 3. If $\mathscr{V}$ is a Banach space, $u \in \mathscr{V}$, $T : \mathscr{V} \to \mathscr{V}^*$ is an operator and if the Gâteaux differential $VT(u)$ of the operator T at the point u exists $(VT(u) : \mathscr{V} \to \mathscr{V}^*)$, then the weak Gâteaux differential $\tilde{V}T(u)$ exists and

$$\tilde{V}T(u) = VT(u) .$$

Conversely, the existence of the weak Gâteaux differential does not imply the existence of the Gâteaux differential in general.

Remark 4. Let $\mathscr{V}$ be a Banach space, $T : \mathscr{V} \to \mathscr{V}^*$, $u, v, h \in \mathscr{V}$ and let T have the weak Gâteaux differential at the point u. It follows from Definition 3 that the function $t \to \langle T(u + th), v \rangle$ has finite derivative at the point $t_0 = 0$ and thus it is continuous at $t_0 = 0$.
It means that

$$\lim_{t \to 0} \langle T(u + th), v \rangle = \langle T(u), v \rangle$$

for every $v, h \in \mathscr{V}$.

Hence, if T has the weak Gâteaux differential on $\mathscr{V}$, then T is hemicontinuous.

The following lemma is a mean value theorem for operators which have the weak Gâteaux differential. We shall use it in the proof of Lemma 2 where we give a criterion of monotonicity.

Lemma 1. *Let $\mathscr{V}$ be a Banach space, $T : \mathscr{V} \to \mathscr{V}^*$ an operator which has the weak Gâteaux differential on $\mathscr{V}$, $u, v, h \in \mathscr{V}$. Then there exists $\tau_v \in (0, 1)$ such that*

$$\langle T(u + h) - T(u), v \rangle = \tilde{V}T(u + \tau_v h)(v, h) .$$

Proof. For $t \in \langle 0, 1 \rangle$, let us put

$$\varphi_v(t) = \langle T(u + th) - T(u), v \rangle .$$

From the assumption that T has the weak Gâteaux differential on $\mathscr{V}$ we can easily deduce that φ_v is continuous on the interval $\langle 0, 1 \rangle$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_v(t) = \tilde{V}T(u + th)(v, h)$$

for $t \in (0, 1)$. By the Lagrange theorem there exists $\tau_v \in (0, 1)$ such that

$$\langle T(u + h) - T(u), v \rangle = \varphi_v(1) - \varphi_v(0) = \frac{\mathrm{d}}{\mathrm{d}t} \varphi_v(\tau_v) = \tilde{V}T(u + \tau_v h)(v, h) .$$

Lemma 2. *Let $\mathscr{V}$ be a Banach space, $T : \mathscr{V} \to \mathscr{V}^*$ an operator which has the weak Gâteaux differential on $\mathscr{V}$. If for arbitrary $u, v \in \mathscr{V}$*

$$\tilde{V}T(u)(v, v) \geqq 0 ,$$

then T is monotone.

Moreover, if $\tilde{V}T(u)(v, v) = 0$ if and only if $v = 0$, then T is strictly monotone.

Proof. Let $u_1, u_2 \in \mathscr{V}$. By Lemma 1, there exists $\tau \in (0, 1)$ such that

$$\langle T(u_1) - T(u_2), u_1 - u_2 \rangle = \tilde{V}T(u_2 + \tau(u_1 - u_2))(u_1 - u_2, u_1 - u_2) .$$

From this equality, we get immediately the assertion of Lemma 2.

4. APPLICATION OF THE MONOTONE OPERATOR METHOD TO THE SOLUTION OF PROBLEM (B)

In order to be able to apply the monotone operator method to the study of Problem (B), we shall reformulate it in the following way.

Problem (C). Let the operator

$$T_{u_0} : \tilde{W}_2^1(P) \to (\tilde{W}_2^1(P))^* = \tilde{W}_2^1(P)$$

be defined by the equality

(4.1)
$$\langle T_{u_0}(u), v \rangle =$$
$$= \iint_P \beta(r, (\nabla(u_0 + u))^2) \left(\frac{\partial(u_0 + u)}{\partial z} \frac{\partial v}{\partial z} + \frac{\partial(u_0 + u)}{\partial r} \frac{\partial v}{\partial r} \right) \mathrm{d}z \, \mathrm{d}r =$$
$$= \iint_P \beta(r, (\nabla(u_0 + u))^2) (\nabla(u_0 + u) \, . \, \nabla v) \, \mathrm{d}z \, \mathrm{d}r$$

$(u, v \in \tilde{W}_2^1(P))$. It is easy to see that the integral in (4.1) is convergent and T_{u_0} is a map of $\tilde{W}_2^1(P)$ into its dual.

Problem (C) is to find $\tilde{u} \in \tilde{W}_2^1(P)$ which is a solution of the equation

$$T_{u_0}(\tilde{u}) = \Theta \,. \tag{4.2}$$

Θ denotes here the zero element of the space $(\tilde{W}_2^1(P))^*$. Hence we have to find $\tilde{u} \in \tilde{W}_2^1(P)$ such that

$$\langle T_{u_0}(\tilde{u}), v \rangle = 0$$

for every $v \in \tilde{W}_2^1(P)$.

Remark 5. If $\tilde{u}$ is a solution of Eq. (4.2), then the function $\tilde{w} = \tilde{u} + u_0$ is a solution of Problem (B). Conversely, every solution $\tilde{w}$ of Problem (B) gives a solution of Problem (C). It is sufficient to put $\tilde{u} = \tilde{w} - u_0$.

The following lemma will be important in the proof of Theorem 2 on the unique solution of Problem (C).

Lemma 3. *If we put*

$$\|u\|^{\tilde{W}} = \left(\iint_P (\nabla u)^2 \, dz \, dr\right)^{1/2} \tag{4.3}$$

for every $u \in \tilde{W}_2^1(P)$, then $\|\ldots\|^{\tilde{W}}$ is a norm defined on $\tilde{W}_2^1(P)$ and the norms (2.1′) and (4.3) are equivalent.

Proof. We have to prove that (4.3) is a norm on $\tilde{W}_2^1(P)$ and there exist constants $K_1, K_2 > 0$ such that

$$K_1\|u\| \leqq \|u\|^{\tilde{W}} \leqq K_2\|u\| \tag{4.4}$$

for every $u \in \tilde{W}_2^1(P)$.

It is evident that if $\alpha \in E_1$ and $u \in \tilde{W}_2^1(P)$, then

$$\|u\|^{\tilde{W}} \geqq 0 \,,$$
$$\|\alpha u\|^{\tilde{W}} = |\alpha| \, \|u\|^{\tilde{W}} \,.$$

In view of the Minkowski inequality,

$$\|u + v\|^{\tilde{W}} \leqq \|u\|^{\tilde{W}} + \|v\|^{\tilde{W}}$$

for $u, v \in \tilde{W}_2^1(P)$. If (4.4) is valid, then $\|u\|^{\tilde{W}} = 0$ if and only if $u = 0$. The right-hand inequality in (4.4) is satisfied if we put e.g. $K_2 = 1$.

Now it remains to prove the existence of $K_1 > 0$ and the validity of the left-hand inequality in (4.4). It is sufficient to prove the existence of $K > 0$ such that

$$\iint_P u^2 \, dz dr \leqq K \iint_P \left(\frac{\partial u}{\partial r}\right)^2 dz dr$$

for every $u \in \mathfrak{A}$ (see Paragraph 2). Following the notation from Paragraph 1, we have

$$P \subset D = \langle Z_1, Z_2 \rangle \times \langle R_1, R_2 \rangle .$$

Let $u \in \mathfrak{A}$. If we put $u = 0$ in $D - P$, we get the extension of u to the rectangle D. With respect to the geometry of P (see Paragraph 1) and the properties of the set $\mathfrak{A}$, the extended function u is infinitely differentiable on D. For every $(z, r) \in D$ we have

$$u(z, r) = \int_{R_1}^{r} \frac{\partial u}{\partial r}(z, \tau)\, d\tau .$$

If we square this equality and use the Cauchy inequality, we get

$$(u(z, r))^2 = \left(\int_{R_1}^{r} 1 \cdot \frac{\partial u}{\partial r}(z, \tau)\, d\tau\right)^2 \leqq \int_{R_1}^{r} d\tau \int_{R_1}^{r} \left(\frac{\partial u}{\partial r}(z, \tau)\right)^2 d\tau \leqq$$
$$\leqq (R_2 - R_1) \int_{R_1}^{R_2} \left(\frac{\partial u}{\partial r}(z, \tau)\right)^2 d\tau .$$

Integrating the inequality

$$(u(z, r))^2 \leqq (R_2 - R_1) \int_{R_1}^{R_2} \left(\frac{\partial u}{\partial r}(z, \tau)\right)^2 d\tau$$

with respect to r on the interval $\langle R_1, R_2 \rangle$, we obtain

$$\int_{R_1}^{R_2} (u(z, r))^2\, dr \leqq (R_2 - R_1)^2 \int_{R_1}^{R_2} \left(\frac{\partial u}{\partial r}(z, \tau)\right)^2 d\tau .$$

Integration of this result with respect to z on the interval $\langle Z_1, Z_2 \rangle$ yields

$$\iint_P u^2\, dz\, dr = \iint_D u^2\, dz\, dr \leqq (R_2 - R_1)^2 \iint_D \left(\frac{\partial u}{\partial r}\right)^2 dz\, dr =$$
$$= (R_2 - R_1)^2 \iint_P \left(\frac{\partial u}{\partial r}\right)^2 dz\, dr ,$$

which completes the proof of Lemma 3.

Lemma 4. *The operator T_{u_0} is strictly monotone and hemicontinuous.*

Proof. Let us show that the weak Gâteaux differential of the operator T_{u_0} on the space $\tilde{W}_2^1(P)$ exists. If $u, v, w \in \tilde{W}_2^1(P)$ are given, the function f is defined on the set $\mathscr{D}(f) = P \times (-1, 1)$ by

$$f(z, r, t) = \beta(r, (\nabla(u_0 + u + tw))^2 (z, r)) (\nabla(u_0 + u + tw) \cdot \nabla v)(z, r) .$$

With respect to Definition 3,

$$\tilde{V}T_{u_0}(u)(v, w) = \frac{\mathrm{d}}{\mathrm{d}t} \langle T_{u_0}(u + tw), v \rangle|_{t=0} =$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \left(\iint_P f(z, r, t) \, \mathrm{d}z \, \mathrm{d}r \right) \bigg|_{t=0},$$

if the derivative exists and is finite.

The partial derivative $\partial f / \partial t$ on $\mathscr{D}(f)$ exists and

$$\frac{\partial f}{\partial t}(\cdot, \cdot, t) =$$

$$= 2 \frac{\partial \beta}{\partial \zeta}(\cdot, (\nabla(u_0 + u + tw))^2)(\nabla(u_0 + u + tw) . \nabla w)(\nabla(u_0 + u + tw) . \nabla w) +$$

$$+ \beta(\cdot, (\nabla(u_0 + u + tw))^2)(\nabla w . \nabla v).$$

We shall denote

$$h_1(z, r, t) =$$

$$= 2 \frac{\partial \beta}{\partial \zeta}(r, (\nabla(u_0 + u + tw))^2(z, r))(\nabla(u_0 + u + tw) . \nabla w) \times$$

$$\times (\nabla(u_0 + u + tw) . \nabla v)(z, r),$$

$$h_2(z, r, t) = \beta(r, (\nabla(u_0 + u + tw))^2(z, r))(\nabla w . \nabla v)(z, r)$$

$((z, r, t) \in \mathscr{D}(f))$ and

$$A_t = \{(z, r) \in P; \ (\nabla(u_0 + u + tw))^2(z, r) \leqq \hat{\zeta}\}, \quad B_t = P - A_t$$

$(t \in (-1, 1))$. $\hat{\zeta}$ is the constant from Paragraph 1.

If $t \in (-1, 1)$, then it holds in view of (1.2):

$$(4.5) \qquad 0 \leqq \frac{\partial \beta}{\partial \zeta}(r, (\nabla(u_0 + u + tw))^2(z, r)) \leqq C_2 \quad \text{for} \quad (z, r) \in A_t,$$

$$\frac{\partial \beta}{\partial \zeta}(r, (\nabla(u_0 + u + tw))^2(z, r) = 0 \quad \text{for} \quad (z, r) \in B_t,$$

$$\left| \frac{\partial(u_0 + u + tw)}{\partial z}(z, r) \right| \leqq (\hat{\zeta})^{1/2} \quad \text{for} \quad (z, r) \in A_t,$$

$$\left| \frac{\partial(u_0 + u + tw)}{\partial r}(z, r) \right| \leqq (\hat{\zeta})^{1/2} \quad \text{for} \quad (z, r) \in A_t,$$

$$0 < \frac{1}{R_2} \leqq \beta(r, (\nabla(u_0 + u + tw))^2(z, r)) \leqq C_1 \quad \text{for} \quad (z, r) \in P.$$

Now, if we put

$$g_1 = 2C_2\,\hat{\zeta}\left(\left|\frac{\partial w}{\partial z}\right| + \left|\frac{\partial w}{\partial r}\right|\right)\left(\left|\frac{\partial v}{\partial z}\right| + \left|\frac{\partial v}{\partial r}\right|\right),$$

$$g_2 = C_1\left(\left|\frac{\partial w}{\partial z}\right|\left|\frac{\partial v}{\partial z}\right| + \left|\frac{\partial w}{\partial r}\right|\left|\frac{\partial v}{\partial r}\right|\right),$$

then in view of (4.5) we have

$$\left|\frac{\partial f}{\partial t}(z, r, t)\right| \leqq |h_1(z, r, t)| + |h_2(z, r, t)| \leqq g_1(z, r) + g_2(z, r)$$

for all $(z, r, t) \in \mathscr{D}(f)$. Moreover, the integrals

$$\iint_P g_i(z, r)\,\mathrm{d}z\,\mathrm{d}r\,, \quad i = 1, 2$$

are convergent. Since for every $t \in (-1, 1)$ the integral

$$\iint_P f(z, r, t)\,\mathrm{d}z\,\mathrm{d}r$$

is convergent, the use of the theorem on differentiating under the integral sign is justified and we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\iint_P f(z, r, t)\,\mathrm{d}z\,\mathrm{d}r\right)\bigg|_{t=0} = \iint_P \frac{\partial f}{\partial t}(z, r, 0)\,\mathrm{d}z\,\mathrm{d}r\,.$$

Here the right hand side integral is convergent and thus

$$\tilde{V}T_{u_0}(u)(v, w) = \iint_P \frac{\partial f}{\partial t}(z, r, 0)\,\mathrm{d}z\,\mathrm{d}r$$

for arbitrary $u, v, w \in \tilde{W}_2^1(P)$. Further, we can see that $\tilde{W}T_{u_0}(u)(v, \cdot)$ is a continuous linear functional defined on the space $\tilde{W}_2^1(P)$.

We have proved the existence of the weak Gâteaux differential of T_{u_0} on $\tilde{W}_2^1(P)$. It follows from Remark 4 that T_{u_0} is hemicontinuous.

If we put $v = w$, we get

$$\tilde{V}T_{u_0}(v, v) = \iint_P \left\{2\frac{\partial \beta}{\partial \zeta}(r, (\nabla(u_0 + u))^2)(\nabla(u_0 + u)\,.\,\nabla v)^2 + \right.$$

$$\left. + \beta(r, (\nabla(u_0 + u))^2)(\nabla v)^2\right\}\mathrm{d}z\,\mathrm{d}r \geqq \iint_P \{\beta(r, (\nabla(u_0 + u))^2)(\nabla v)^2\}\,\mathrm{d}z\,\mathrm{d}r\,.$$

In view of (4.5) and Lemma 3 (see (4.4))

$$\tilde{V}T_{u_0}(u)(v, v) \geqq \frac{1}{R_2} (\|v\|^{\tilde{W}})^2 \geqq \frac{K_1^2}{R_2} \|v\|^2$$

for all $u, v \in \tilde{W}_2^1(P)$. This implies that the inequality

$$\tilde{V}T_{u_0}(u)(v, v) \geqq 0$$

is valid for all $u, v \in \tilde{W}_2^1(P)$, and

$$\tilde{V}T_{u_0}(u)(v, v) = 0$$

if and only if $v = 0$. Consequently, in virtue of Lemma 2, the operator T_{u_0} is strictly monotone.

Lemma 5. *The operator T_{u_0} is coercive.*

Proof. If $v \in \tilde{W}_2^1(P)$ then, by (1.2),

$$\langle T_{u_0}(v), v\rangle = \iint_P \{\beta(r, (\nabla(u_0 + u))^2)(\nabla u_0 \,.\, \nabla v + \nabla v \,.\, \nabla v)\} \, dz \, dr \geqq$$

$$\geqq \frac{1}{R_2} \iint_P (\nabla v)^2 \, dz \, dr - C_1 \iint_P |(\nabla u_0 \,.\, \nabla v)| \, dz \, dr \,.$$

By means of the Cauchy inequality we get

$$\iint_P |(\nabla u_0 \,.\, \nabla v)| \, dz \, dr \leqq \|u_0\|^{\tilde{W}} \|v\|^{\tilde{W}} \,.$$

It follows from Lemma 3 that there exist constants $K_1, K_2 > 0$ such that (4.4) is valid for all $u \in \tilde{W}_2^1(P)$ and thus,

$$\langle T_{u_0}(v), v\rangle \geqq \frac{1}{R_2} (\|v\|^{\tilde{W}})^2 - C_1 \|u_0\|^{\tilde{W}} \|v\|^{\tilde{W}} \geqq \frac{K_1^2}{R_2} \|v\|^2 - C_1 K_2^2 \|u_0\| \|v\| \,.$$

Hence we get

$$\lim_{\substack{\|v\| \to +\infty \\ v \in \tilde{W}_2^1(P)}} \frac{\langle T_{u_0}(v), v\rangle}{\|v\|} \geqq \frac{K_1^2}{R_2} \lim_{\substack{\|v\| \to +\infty \\ v \in \tilde{W}_2^1(P)}} \|v\| - C_1 K_2^2 \|u_0\| = +\infty \,,$$

Q.E.D.

Theorem 2. *Problem (C) has exactly one solution.*

Proof. The operator $T_{u_0} : \tilde{W}_2^1(P) \to (\tilde{W}_2^1(P))^*$ is strictly monotone, coercive and hemicontinuous. By Theorem 1 and Remark 2, Eq. (4.2) has exactly one solution $\tilde{u} \in \tilde{W}_2^1(P)$.

Remark 6. The operator T_{u_0} depends on the function u_0. Problem (C), as we have proved, has for the given u_0 the unique solution $\tilde{u}$. It follows from this that Problem (B) has also the unique solution $w = u_0 + \tilde{u}$. Let us show that the solution w of Problem (B) does not depend on the function u_0 satisfying conditions (2.2).

Let $u_1, u_2 \in \tilde{W}_2^1(P)$ satisfy (2.2). Let T_{u_1} and T_{u_2} be operators defined by (4.1), where we write u_1 and u_2, respectively, instead of u_0. If we put

$$q = u_2 - u_1 ,$$

then $q \in \tilde{W}_2^1(P)$. Let $\tilde{u}_1$ be a solution of the equation

$$T_{u_1}(\tilde{u}_1) = \Theta .$$

Then $\tilde{u}_2 = \tilde{u}_1 - q$ is a (unique) solution of the equation

$$T_{u_2}(\tilde{u}_2) = \Theta .$$

The functions $w_i = u_i + \tilde{u}_i$ $(i = 1, 2)$ are the solutions of Problem (B) (where we use u_i instead of u_0). But

$$w_2 = u_2 + \tilde{u}_2 = u_2 + \tilde{u}_1 - q = u_1 + \tilde{u}_1 = w_1 ,$$

which we wanted to prove.

This also guarantees the uniqueness of solution of the "classical" Problem (A), provided the first derivatives of solutions of Problem (A) are elements of the space $L_2(P)$. This assumption is physically well-founded because it is equivalent to the requirement that the total energy of the moving fluid, which fills up the channel, is finite.

Let $u \in W_2^1(P)$ be the solution of Problem (B). We can easily find out that the functions

$$v_z = \beta(r, (\nabla u)^2) \frac{\partial u}{\partial r}, \quad v_r = -\beta(r, (\nabla u)^2) \frac{\partial u}{\partial z}$$

are elements of $L_2(P)$. We shall call the vector $\boldsymbol{V} = (v_z, v_r)$ the "generalized" velocity field of compressible irrotational flow. It follows from Theorem 2 that for the given total flow through the channel, determined by the constant Q, there exists exactly one "generalized" velocity $\boldsymbol{V}$ of compressible irrotational channel flow. By $\boldsymbol{V}$ the fluid density ϱ and the pressure p are uniquely determined, which completes the solution of the channel flow.

In conclusion, let us add that in practice it is required to determine the solution, the existence and uniqueness of which we have proved. The only way possible is to use an appropriate approximate method. In this case, where we have used the monotone operators method, it would be convenient to solve the problem approximately by the finite element method. However, the solution of equations on appropriate finite-dimensional spaces and the estimates of convergence remain an open problem.

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Souhrn

ŘEŠENÍ PODZVUKOVÝCH OSOVĚ SYMETRICKÝCH PROUDOVÝCH POLÍ

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Tímto článkem navazujeme na výsledky z [1], kde jsme formulovali okrajovou modelovou úlohu, která popisuje třírozměrná osově symetrická proudová pole stlačitelné tekutiny. Zde se zabýváme řešitelností této úlohy. Nestudujeme však řešení „klasické", ale věnujeme se otázkám existence a jednoznačnosti „zobecněných" řešení, které vyšetřujeme pomocí metody monotónních operátorů.

Hlavním výsledkem je věta o existenci a jednoznačnosti řešení úlohy (C), odkud plyne, že je-li dán celkový průtok tekutiny kanálem, pak v daném kanálu existuje právě jedno nevířivé proudové pole stlačitelné tekutiny.

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