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SOLUTION OF THE FIRST PROBLEM OF PLANE ELASTICITY FOR MULTIPLY CONNECTED REGIONS BY THE METHOD OF LEAST SQUARES ON THE BOUNDARY (Part I)

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INTRODUCTION

For simply connected regions, the so-called first problem of plane elasticity is equivalent, roughly speaking (for details see p. 354), to the first biharmonic problem

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{0.1}$$

$$u = g_0(s)\,, \quad \frac{\partial u}{\partial \nu} = g_1(s) \quad \text{on} \quad \Gamma\,. \tag{0.2}$$

Here G is the considered region with its boundary Γ, ν the unit outward normal. To the solution of this problem, a scale of methods has been developed, each of them having its specific advantages and drawbacks. For example: The method of finite differences is very simple, but the approximation of components of the stress-tensor – which are the second order derivatives of the function u, see (1.5), p. 353 – by the corresponding second difference quotients may be very inaccurate. Variational methods – including the finite element method – are very often applied. However, they are applicable provided the solution $u(x, y)$ belongs to the "energetic" space $W_2^{(2)}(G)$. The Muschelišvilli method based on the theory of functions of a complex variable (cf. [4]) has its main drawback in the requirement of a sufficiently smooth boundary (permitting no corners, for example).

In the paper [1] by K. Rektorys and V. Zahradník, an approximate method, the so-called method of least squares on the boundary, is developed, requiring only the boundary Γ to be Lipschitzian and $g_1 \in L_2(\Gamma)$, $g_0 \in W_2^{(1)}(\Gamma)$. (Practically, this means a sufficiently general boundary and a very general loading, if the problem is interpreted as a stress-and-strain problem; for the notation see p. 354.) This method is closely related to that presented in [4], Sec. 3.15 (where a sufficiently smooth boundary is assumed). Its idea is very simple: Let

$$(0.3) \qquad z_1(x, y),\ z_2(x, y),\ \ldots$$

be the sequence of basic biharmonic polynomials[1]) (for details see [1]; for every positive integer $n \geqq 2$ there are precisely $4n - 2$ of these polynomials of order $\leqq n$) and let us look for an approximate solution in the form

$$(0.4) \qquad U_n(x, y) = \sum_{i=1}^{4n-2} a_{ni}\, z_i(x, y)\,,$$

where the coefficients a_{ni} are determined in such a way that

$$(0.5) \qquad \int_\Gamma (U_n - g_0)^2\, \mathrm{d}s + \int_\Gamma \left(\frac{\partial U_n}{\partial s} - \frac{\mathrm{d}g_0}{\mathrm{d}s}\right)^2 \mathrm{d}s + \int_\Gamma \left(\frac{\partial U_n}{\partial \nu} - g_1\right)^2 \mathrm{d}s = \min.$$

is satisfied on the linear set of functions of the form

$$(0.6) \qquad V_n(x, y) = \sum_{i=1}^{4n-2} b_{ni}\, z_i(x, y)\,.$$

(Thus, the approximate solution (0.4) fulfils the given biharmonic equation exactly, while the boundary conditions are fulfilled approximately in the sense of (0.5).)

The condition (0.5) leads to the solution of a system of $4n - 2$ linear equations for $4n - 2$ unknowns a_{ni} $(i = 1, \ldots, 4n - 2)$. In [1], this system is shown to be uniquely solvable, and the convergence of the sequence $\{U_n(x, y)\}$ to the so-called very weak solution of the problem (0.1), (0.2) is proved, provided that G is a bounded simply connected region with a Lipschitzian boundary Γ and that $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$. A numerical example is also given in [1].

In the present paper, the above method is extended to the case of multiply connected regions. This case presents two difficulties:

(i) The first problem of plane elasticity is equivalent to a biharmonic problem only if the latter is properly modified. This fact is to be seen from Ex. 1.1, p. 357.

(ii) In [1], when proving the convergence of the above mentioned method of least squares on the boundary, an approximation of holomorphic functions by polynomials has been used. In the case of multiply connected regions, holomorphic functions cannot be approximated in general by polynomials only. Therefore, the functions g_0, g_1 cannot be approximated (in the sense of (0.5)) only by functions of the type (0.4); also some other simple biharmonic functions should be used (Chap. 3).

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[1]) Thus fulfilling the equation (0.1).

In spite of this, the numerical treatment of the method considered remains also in the case of multiply connected regions relatively simple (cf. Exs. 3.1, 3.2). However, theoretical questions, especially those concerning the convergence of the method, are rather complicated. For this reason, the proof of convergence is postponed to Chap. 4, in order that the reader who is not a professional mathematician be able to follow at least the text of the first three chapters, including numerical examples. This is also the reason why the paper is divided into two parts, Part I containing the first three chapters, Part II the chapters 4 and 5 (the "purely mathematical" part of the paper).

In Chap. 1, the connection between the first problem of plane elasticity and the first biharmonic problem is discussed, first for simply connected regions, where the situation is simpler, then for multiply connected regions, where Ex. 1.1 demonstrates the characteristic difficulties. Using some properties of the so-called complex stress-funstions, formulation of the problem in the real form is given in Chap. 2 and basic results on the solution are derived. In Chap. 3 (p. 374), the method of least squares on the boundary is presented and numerical examples are given. Chap. 4 is devoted to the proof of convergence of the method, Chap. 5 contains proofs of some theorems and of some auxiliary lemmas which were postponed to this chapter in order to make the ideas of the proofs of the main theorems of Chaps. 2 and 4 as clear as possible.

Remark 0.1. As said above, the structure of the paper is such that a "consumer" of mathematics be able to read the first three chapters. A reader who is not interested in the application of the method in the theory of elasticity, can start just with the Formulation of the Problem on p. 367. In that formulation, $g_{i0} \in W_2^{(1)}(\Gamma_i)$, $g_{i1} \in L_2(\Gamma_i)$, $i = 0, 1, \ldots, k$ are given functions for him, regardless of whether they have something common with a "loading" on the boundary or not. In Def. 2.1, he should understand under an Airy function such a biharmonic function for which the function (2.9) is single-valued in G. Note that every function $u(x, y)$ biharmonic in G produces three functions (2.4) to which there correspond, according to Lemma 2.2, p. 361, functions $\varphi(z)$, $\psi(z)$ $(=\chi'(z))$ appearing in (2.9). Whether the function (2.9) is single-valued or not, does not depend on the choice of the points z_i, $i = 1, \ldots, k$ from Lemma 2.2. (See the footnote 9 on p. 363.) Of course, it may happen that such a reader will not understand why the problem is formulated precisely as given on p. 361, because the first part of Chap. 2 – which he will omit – is devoted just to the motivation of this formulation.

The "mathematical" reader will then be interested in the basic singular biharmonic functions $r_{ij}(x, y)$ (p. 367) and in the existence theorem 2.1; in Chap. 3 in the algorithm of the method and in Theorems 3.1 and 3.2. The essence of the paper lies in Chap. 4 (proof of the convergence theorem 3.2). The auxiliary mathematical tools are collected in Chap. 5. It is possible that a reader will prefer to start with this last chapter. In this case, he should stop at the assertion (5.35), continue by the proof of Lemma 2.4 and by the text of Chap. 2 up to Theorem 2.1, and then return to Chap. 5.

CHAPTER 1. THE FIRST PROBLEM OF PLANE ELASTICITY AND THE FIRST BIHARMONIC PROBLEM

Convention 1.1. *In this paper, under a region G we shall always understand a bounded region in E_2* (multiply connected, in general) *with the so-called Lipschitzian boundary Γ.*[1])

As usual, $\bar{G} = G \cup \Gamma$.

The classical formulation of the first problem of plane elasticity is the following:

To find sufficiently smooth functions σ_x, σ_y, τ_{xy} (the so-called *components of the stress-tensor*) *which fulfil in G the equations of static equilibrium*

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0\,, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1.1}$$

and the equation of compatibility

$$\Delta(\sigma_x + \sigma_y) = 0 \tag{1.2}$$

(Δ being the Laplace operator), *and on Γ the boundary conditions*

$$\begin{aligned} \nu_x \sigma_x + \nu_y \tau_{xy} &= X(s)\,, \\ \nu_x \tau_{xy} + \nu_y \sigma_y &= Y(s)\,, \end{aligned} \tag{1.3}$$

where $\nu_x(s)$, $\nu_y(s)$ are components of the unit outward normal and $X(s)$, $Y(s)$ are given functions on Γ (components of the loading acting on the boundary of the body G from its exterior). Moreover, if G is multiply connected, it is required that the displacement corresponding to the components σ_x, σ_y, τ_{xy} of the stress-tensor according to the Hooke law be a single-valued function.[2])

If G is a *simply connected region*, this problem can be easily reduced to a biharmonic problem. Actually, we have (see [4], Sec. 2.2, 2.3)

Lemma 1.1. *Let G be a simply connected region. Let the functions σ_x, σ_y, τ_{xy} be twice continuously differentiable in G and let they satisfy* (1.1) *and* (1.2). *Then there exists a function $u(x, y)$ biharmonic in G* (thus satisfying

$$\Delta^2 u = 0 \quad \text{in} \quad G)\,, \tag{1.4}$$

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[1]) The concept of the Lipschitzian boundary is treated in detail in [3], Chap. 28, or in [2]. It represents a slight generalization of the concept of a "piecewise smooth" boundary.

[2]) In case of a simply connected region, this requirement is automatically fulfilled. See also Chap. 2, Lemma 2.1 and Eq. (2.9).

the so-called Airy function, the derivatives of which are the functions σ_x, σ_y, τ_{xy}:

$$\sigma_x = \frac{\partial^2 u}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 u}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 u}{\partial x\, \partial y}. \tag{1.5}$$

This function is uniquely determined by the functions σ_x, σ_y, τ_{xy} *up to an expression of the form*

$$ax + by + c. \tag{1.6}$$

Conversely, if $u(x, y)$ *is a biharmonic function in* G, *then the functions* (1.5) *are sufficiently smooth in* G (they are even infinitely differentiable) *and satisfy the conditions* (1.1), (1.2).

Thus every function biharmonic in a simply connected region G characterizes – through the functions (1.5) – a state of stress in G.

The problem (1.1)–(1.3) being given, it remains to convert the boundary conditions (1.3) for the functions σ_x, σ_y, τ_{xy} into boundary conditions for the biharmonic function $u(x, y)$. This can be carried out in the following way (see e.g. [4], Sec. 2.7):

First, let the loading of Γ be sufficiently smooth. Let l be the length of Γ, s the parameter of arc on Γ $(0 \leqq s < l)$ with $s = 0$ at a chosen point $A \in \Gamma$. Let s be increasing if we run along Γ in the positive sense of its orientation (thus leaving G to the left-hand side). If we put $\partial u/\partial x$ and $\partial u/\partial y$ equal to zero at the point A, then we have on Γ

$$\frac{\partial u}{\partial x}(s) = -\int_0^s Y(t)\, \mathrm{d}t, \quad \frac{\partial u}{\partial y}(s) = \int_0^s X(t)\, \mathrm{d}t. \text{[3]} \tag{1.7}$$

Now, $\partial u/\partial x$, $\partial u/\partial y$ being known on Γ, we compute

$$\frac{\partial u}{\partial s} = -\frac{\partial u}{\partial x} \nu_y + \frac{\partial u}{\partial y} \nu_x, \quad \frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x} \nu_x + \frac{\partial u}{\partial y} \nu_y \tag{1.8}$$

and, putting $u = 0$ at the point A,

$$u(s) = \int_0^s \frac{\partial u}{\partial s}(t)\, \mathrm{d}t. \tag{1.9}$$

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[3]) Conversely, if the functions $\partial u/\partial x$, $\partial u/\partial y$ are given on Γ and are sufficiently smooth, then obviously

$$X(s) = \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial u}{\partial y}, \quad Y(s) = -\frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial u}{\partial x}.$$

Denoting $u(s) = g_0(s)$, $(\partial u/\partial v)(s) = g_1(s)$ and taking (1.4) into account, the problem (1.1)–(1.3) is converted in this way into the biharmonic problem

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{1.10}$$

$$u = g_0\,, \quad \frac{\partial u}{\partial v} = g_1 \quad \text{on} \quad \Gamma\,. \tag{1.11}$$

Let us note that the formulae (1.7) permit to take certain singularities of the loading into account. If, for example, at a certain point $B \in \Gamma$ a single load (an isolated force) with components F_x, F_y is acting, then the integrals in (1.7) are to be replaced by the corresponding Stieltjes integrals. At the point B, the function $\partial u/\partial x$ or $\partial u/\partial y$ then has a jump $-F_y$ or F_x, respectively.

In the following text, we shall assume that $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$ *only.* We shall briefly say that *the functions* g_0, g_1 *belong to the space* $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$ *and we shall write* $(g_0, g_1) \in W_2^{(1)}(\Gamma) \times L_2(\Gamma)$. Let us recall that $g_1 \in L_2(\Gamma)$ means that the function $g_1(s)$ is square integrable in the interval $(0, l)$, while $g_0 \in W_2^{(1)}(\Gamma)$ means that $g_0 \in L_2(\Gamma)$ and $\mathrm{d}g_0/\mathrm{d}s \in L_2(\Gamma)$. The spaces $L_2(\Gamma)$ and $W_2^{(1)}(\Gamma)$ are Hilbert spaces with the norms given by

$$\|f\|^2_{L_2(\Gamma)} = \int_0^l f^2(s)\,\mathrm{d}s \quad \text{and} \quad \|f\|^2_{W_2^{(1)}(\Gamma)} = \int_0^l f^2(s)\,\mathrm{d}s + \int_0^l f'^2(s)\,\mathrm{d}s\,, \tag{1.12}$$

respectively.[4])

The assumption $(g_0, g_1) \in W_2^{(1)}(\Gamma) \times L_2(\Gamma)$ is sufficiently general to include a wide class of loadings appearing in applications of the plane elasticity.[5]) In particular, if the loading contains a finite number of single loads and is piecewise continuous elsewhere on the boundary Γ, then we have $(g_0, g_1) \in W_2^{(1)}(\Gamma) \times L_2(\Gamma)$. But the loading can be considerably more general to get this result.

Cf. also the paper [5] by I. Hlaváček and J. Naumann, where the conditions $g_0 \in W_2^{(3/2)}(\Gamma)$, $g_1 \in W_2^{(1/2)}(\Gamma)$ are discussed, which make it possible to work in the space $W_2^{(2)}(G)$.

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[4]) The spaces $L_2(\Gamma)$ and $W_2^{(k)}(\Gamma)$ are defined in [3], Chaps 28 and 30. In the present case, where Γ is the Lipschitzian boundary of a region in E_2 and $k = 1$, it is possible to introduce the norms by (1.12).

In this notation, the condition (0.5), p. 350 can be written in the form

$$\|u - g_0\|^2_{W_2^{(1)}(\Gamma)} + \left\|\frac{\partial u}{\partial v} - g_1\right\|^2_{L_2(\Gamma)} = \min\,. \tag{1.13}$$

[5]) Naturally, it is interesting also from the purely mathematical point of view.

In what follows we always assume that the point A with $s = 0$ is chosen in such a way that the loading has no singularity at that point – more precisely, that it is continuous at the point A. Then it is easy to show that *the functions $\partial u/\partial x$, $\partial u/\partial y$ are continuous at this point, i.e.*

$$\lim_{s \to l-} \frac{\partial u}{\partial x}(s) = \frac{\partial u}{\partial x}(0), \quad \lim_{s \to l-} \frac{\partial u}{\partial y}(s) = \frac{\partial u}{\partial y}(0), \tag{1.14}$$

if and only if the loading satisfies the condition of static equilibrium in forces, i.e. if and only if

$$\int_0^l X(s)\,\mathrm{d}s = 0, \quad \int_0^l Y(s)\,\mathrm{d}s = 0.$$

The function $u(s)$ is then continuous at that point if and only if the loading satisfies the condition of equilibrium in moments,

$$\int_0^l [x\,Y(s) - y\,X(s)]\,\mathrm{d}s = 0.$$

Let us return to the problem (1.10), (1.11). Let there exist such a function $w \in W_2^{(2)}(G)$ that we have

$$w = g_0, \quad \frac{\partial w}{\partial v} = g_1 \quad \text{on} \quad \Gamma \tag{1.15}$$

(in the sense of traces). This case occurs, for example, if the loading as well as the boundary Γ are sufficiently smooth; then it is possible to apply e.g. Theorem 2.5.8 from [2]. See also the paper [5]. As is well known, in this case there exists precisely one weak solution $U(x, y)$ of the problem (1.10), (1.11). The equation (1.10) having constant coefficients, this solution has derivatives of all orders in G [6]) and satisfies (1.10) in the classical sense. The functions (1.5) then satisfy the conditions (1.1), (1.2) and describe a certain state of stress in G. We shall briefly say that the function U *produces* this state of stress in G. The boundary conditions (1.3) are fulfilled by virtue of the boundary conditions (1.11) (in a generalized sense, in general). We shall briefly say that $U(x, y)$ *is the weak Airy function corresponding to the given loading.*

If no function $w \in W_2^{(2)}(G)$ exists satisfying (1.15), then the problem (1.10), (1.11) has no weak solution. But we have $(g_0, g_1) \in W_2^{(1)}(\Gamma) \times L_2(\Gamma)$. In this case, the functions g_0, g_1 can be approximated in $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$ by a sequence of functions g_{0n}, g_{1n} which are traces of functions $w_n \in W_2^{(2)}(G)$ in the sense of (1.15), and the sequence $\{U_n(x, y)\}$ of the corresponding weak solutions of (1.10), (1.11) with g_0, g_1 replaced by g_{0n}, g_{1n} converges in the space $L_2(G)$ to the so-called *very weak solution* $U(x, y)$ of the problem (1.10), (1.11). (See [2], Th. 5.4.2, p. 274; the function

[6]) The reader who is not familiar with this result, see Lemma 5.3.

$U(x, y)$ is uniquely determined by the functions g_0, g_1.) Also this solution has derivatives of all orders in G and the functions (1.5) give the state of stress in G, corresponding, in this very weak sense, to the given boundary conditions (1.3). In this way we come to the concept of the *very weak Airy function corresponding to the given loading*. The weak or very weak solution $U(x, y)$ can then be sought approximately by the method of least squares on the boundary, as described and discussed in [1].

Let us turn to the case of a *multiply connected* region. (Convention 1.1 (p. 352) concerning the boundedness of the region G and the Lipschitzian boundary Γ remains always valid.) Let

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_k , \tag{1.16}$$

where Γ_0 is the outer boundary curve and $\Gamma_1, \ldots, \Gamma_k$ are inner boundary curves oriented as shown in Fig. 1. On each of the curves Γ_i ($i = 0, 1, \ldots, k$) let the parameter of arc s be chosen, $0 \leqq s < l_i$, where l_i is the length of the curve Γ_i and $s = 0$ is chosen at such a point $A_i \in \Gamma_i$ where the loading on the boundary, given by the components X_i, Y_i in this case, has no singularity. Let us construct the functions $g_{i0}(s)$, $g_{i1}(s)$ on Γ_i ($i = 0, 1, \ldots, k$) in a quite similar way as in (1.7)–(1.9).

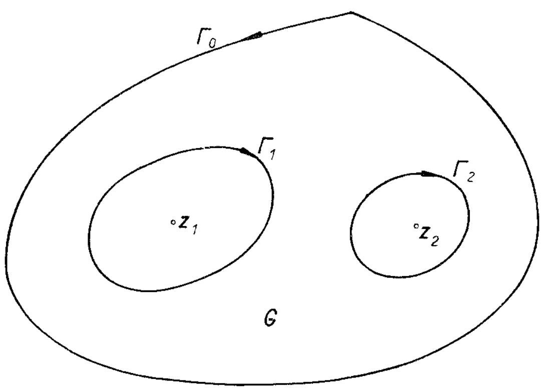

Fig. 1.

Convention 1.2. *In what follows, we shall assume that on every curve Γ_i ($i = 0, 1, \ldots, k$), the loading fulfils the condition of static equilibrium both in forces and moments.*[7]) *The functions g_{i0}, g_{i1} ($i = 0, 1, \ldots, k$) will always be assumed to belong to the space $W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$.*

[7]) From the mathematical point of view, this requirement ensures the continuity of the functions $u(s)$, $(\partial u/\partial x)(s)$, $(\partial u/\partial y)(s)$ at the above mentioned points $A_i \in \Gamma_i$ (in the sense of (1.14)) which makes it possible to work only with *single-valued* functions $u(x, y)$, $\frac{\partial u}{\partial x}(x, y)$, $\frac{\partial u}{\partial y}(x,y)$ on G.

Problems in which this requirement is not fulfilled can be easily reduced to the problem considered by using a proper particular solution. It is clear from Example 3.3, p. 390 how such problems should be treated.

Similarly as in (1.10), (1.11) let us solve the biharmonic problem

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{1.17}$$

$$u = g_{i0}\,, \quad \frac{\partial u}{\partial \nu} = g_{i1} \quad \text{on} \quad \Gamma_i\,, \quad i = 0, 1, \ldots, k\,. \tag{1.18}$$

If the functions g_{i0}, g_{i1} are traces of a function $w \in W_2^{(2)}(G)$ in the sense of (1,15), then there exists precisely one weak solution $u(x, y)$ of the problem (1.17), (1,18). In the opposite case, the functions g_{i0}, g_{i1} belonging to the space $W_2^{(1)}(\Gamma_i) \times \times L_2(\Gamma_i)$ $(i = 0, 1, \ldots, k)$, there exists precisely one very weak solution $u(x, y)$ of this problem.[8] In both cases, the function $u(x, y)$ is a classical solution of the equation (1.17) in G and the functions (1.5) satisfy in G conditions (1.1) and (1.2). *But in contrast to the case of a simply connected region, these functions need not describe a state of stress in G, corresponding* (through the functions g_{i0}, g_{i1}) *to the given loading on the boundary*, as is clear from the following example:

Example 1.1. Let G be a ring with its center at the origin, with the outer circle Γ_0 of radius 2 and the inner circle Γ_1 of radius 1. Let the following biharmonic problem be given:

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{1.19}$$

$$u = 0\,, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0\,, \tag{1.20}$$

$$u = 1\,, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1\,. \tag{1.21}$$

The (unique) solution of this problem is[9]

$$u = \frac{1}{9 - 16 \ln^2 2} \Big[-(3 + 8 \ln 2)(x^2 + y^2) + 3(x^2 + y^2) \ln (x^2 + y^2) + + 8 \ln 2 \,.\, \ln (x^2 + y^2) + (12 + 8 \ln 2 - 16 \ln^2 2) \Big]. \tag{1.22}$$

---

[8]) Its construction is quite similar to that of the problem (1.10), (1.11).

[9]) One transforms the equation (1.19) into polar coordinates r, ω; taking into account that the solution does not depend on ω in our case, one gets an ordinary differential equation with the general integral (cf. [3], Chap. 26)

$$u = C_1 r^2 + C_2 r^2 \ln r + C_3 \ln r + C_4\,;$$

then it is sufficient to apply conditions (1.20), (1.21) and to write $\ln r = \frac{1}{2} \ln r^2 = \frac{1}{2} \ln (x^2 + y^2)$.

The equations (1.5) then formally yield

$$(1.23)\quad \sigma_x = \frac{\partial^2 u}{\partial y^2} = \frac{2}{9-16\ln^2 2}\left\{-(3+8\ln 2) + 3\left[\frac{2y^2}{x^2+y^2} + \ln(x^2 + y^2) + 1\right] + 8\ln 2 \,.\, \frac{x^2-y^2}{(x^2+y^2)^2}\right\},$$

$$\sigma_y = \frac{\partial^2 u}{\partial x^2} = \frac{2}{9-16\ln^2 2}\left\{-(3+8\ln 2) + 3\left[\frac{2x^2}{x^2+y^2} + \ln(x^2+y^2) + 1\right] + 8\ln 2 \,.\, \frac{y^2-x^2}{(x^2+y^2)^2}\right\},$$

$$\tau_{xy} = -\frac{\partial^2 u}{\partial x\, \partial y} = -\frac{4xy}{9-16\ln^2 2}\left[\frac{3}{x^2+y^2} - \frac{8\ln 2}{(x^2+y^2)^2}\right].$$

We have received a paradoxical result: The function u "produces" (at least formally) an evidently nonzero stress-tensor $(\sigma_x, \sigma_y, \tau_{xy})$ in G, while in accordance with the footnote 3, p. 353 and according to (1.20), (1.21) the loading on the boundary is equal to zero.[10])

This "perpetuum mobile" cannot correspond to the reality, of course. In fact, the functions (1.23) satisfy the conditions of static equilibrium (1.1) and the equation of compatibility (1.2); however, as will be shown in the next chapter, no single-valued displacement corresponds to these functions. To be able to understand well the whole problem (and also to avoid the concept of a multi-valued real function), let us remind at the beginning of the next chapter the connection between the "real" and the "complex" theory of plane elasticity. This will make it easier to formulate the problem for multiply connected regions in a proper real form and then to give the basic results concerning its solution.

CHAPTER 2. FORMULATION OF THE PROBLEM FOR MULTIPLY CONNECTED REGIONS. EXISTENCE THEOREM

In Chap. 1, we have discussed the relation between the first problem of plane elasticity and the first biharmonic problem. In the first part of the present chapter, we give a short survey concerning the connection between the components σ_x, σ_y, τ_{xy} of the stress-tensor, the corresponding biharmonic function and the so-called complex

[10]) Let us note that the first conditions in (1.20), (1.21) imply $\partial u/\partial s \equiv 0$ on Γ which together with the remaining conditions gives $\partial u/\partial x \equiv 0$, $\partial u/\partial y \equiv 0$ on Γ and according to the footnote 3, p. 353 we have $X(s) \equiv 0$, $Y(s) \equiv 0$.

The result can be obtained, of course, also by direct computation, if we evaluate σ_x, σ_y and τ_{xy} on Γ_0, Γ_1 and use (1.3).

stress-functions.[1]) Then we give a formulation of the problem in the real form and present the basic existence theorem.

a) *Simply connected regions*[2])

In Lemma 1.1, p. 352, the existence of the so-called Airy function corresponding to the stress-tensor with components $\sigma_x, \sigma_y, \tau_{xy}$ was shown for a simply connected region G. The functions $\sigma_x, \sigma_y, \tau_{xy}$ being sufficiently smooth in G and fulfilling the equations of static equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0\,, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{2.1}$$

and the equation of compatibility

$$\Delta(\sigma_x + \sigma_y) = 0\,, \tag{2.2}$$

there exists a biharmonic function $u(x, y)$, uniquely determined by the functions $\sigma_x, \sigma_y, \tau_{xy}$ up to an expression of the form

$$ax + by + c \tag{2.3}$$

(a, b, c arbitrary real constants), and satisfying in G the relations

$$\sigma_x = \frac{\partial^2 u}{\partial y^2}\,, \quad \sigma_y = \frac{\partial^2 u}{\partial x^2}\,, \quad \tau_{xy} = -\frac{\partial^2 u}{\partial x\, \partial y}\,. \tag{2.4}$$

Conversely, if $u(x, y)$ is an arbitrary biharmonic function in G, then the functions (2.4) satisfy equations (2.1) and (2.2).

Using this lemma and relations (1.7)–(1.9), p. 353 we have shown in Chapter 1 how to transform the first problem of plane elasticity into a biharmonic problem.

Now, we give another lemma which enables us to express the components of the stress-tensor $\sigma_x, \sigma_y, \tau_{xy}$ in terms of some holomorphic functions of the complex variable $z = x + \mathrm{i}y$ and which yields a very simple expression for the vector $(d_1(x, y), d_2(x, y))$ of displacement corresponding to this stress-tensor:

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[1]) The reader is referred especially to the book [4], Chap. 2, Sections 2.1–2.10, where he can find the results given below – possibly in a slightly different form. Especially, we use a rather different notation here. For example, we write $\sigma_x, \sigma_y, \tau_{xy}$ instead of X_x, Y_y, X_y for the components of the stress-tensor, d_1, d_2 instead of u, v for the components of the vector of displacement, etc.

[2]) According to Convention 1.1, bounded regions with Lipschitzian boundaries are always considered throughout this paper.

Lemma 2.1. ([4], Sec. 2.4). *In a simply connected region G, every biharmonic function $u(x, y)$ can be expressed in the form*

$$u(x, y) = \mathrm{Re}\,(\bar{z}\,\varphi(z) + \chi(z))\,,^{3}) \tag{2.5}$$

where $\varphi(z)$, $\chi(z)$ are holomorphic functions in G. By the function $u(x, y)$, the functions $\varphi(z)$ and $\chi(z)$ are determined uniquely up to an expression of the form

$$\mathrm{i}C_1 z + C_2 + \mathrm{i}C_3 \quad or \quad -(C_2 - \mathrm{i}C_3)\,z + \mathrm{i}C_4\,, \tag{2.6}$$

respectively, where $C_1, \ldots, C_4$ are real constants.

On the other hand, if $\varphi(z)$ and $\chi(z)$ are arbitrary functions holomorphic in G, then the function $u(x, y)$ given by (2.5) is biharmonic in G.

Remark 2.1. It follows from Lemmas 1.1 and 2.1 that to every sufficiently smooth functions σ_x, σ_y, τ_{xy} which fulfil (2.1) and (2.2) there correspond holomorphic functions $\varphi(z)$, $\chi(z)$ in G so that we have (2.5) and (2.4). An easy computation (cf. [4], Sec. 2.8) yields the following relations between these functions and the original functions σ_x, σ_y, τ_{xy}:

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\,(\varphi')\,, \tag{2.7}$$

$$\sigma_y - \sigma_x + 2\mathrm{i}\tau_{xy} = 2(\bar{z}\varphi'' + \chi'')\,. \tag{2.8}$$

For the components $d_1(x, y)$, $d_2(x, y)$ of the vector of displacement corresponding to the stress-tensor $(\sigma_x, \sigma_y, \tau_{xy})$ according to the Hooke law one gets the expression ([4], Sec. 2.6 and 2.8; μ and $\varkappa$ are positive constants depending on the material considered)

$$d_1 + \mathrm{i}d_2 = \frac{1}{2\mu}(\varkappa\varphi - z\bar{\varphi}' - \bar{\chi}') \tag{2.9}$$

which will be of particular significance in the following text.[4])

The functions $\varphi(z)$, $\chi(z)$ (connected with the functions σ_x, σ_y, τ_{xy} by the relations (2.7), (2.8)) are called the *stress-functions* (corresponding to the functions σ_x, σ_y, τ_{xy}).[5])

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[3]) By the symbols Re $(f(z))$, Im $(f(z))$ we denote respectively the real and imaginary parts of the function $f(z)$. By $\overline{f(z)}$ we denote the complex conjugate to $f(z)$. In Particular we have Re $(z) = x$, Im $(z) = y$, $\bar{z} = x - \mathrm{i}y$.

[4]) To get this simple expression for the displacement was one of the reasons why we introduced the complex stress functions $\varphi(z)$ and $\chi(z)$. Note that it is possible to avoid this "complex theory" here, but in the case of multiply connected regions this means to meet difficulties — although of formal character — consisting in the necessity of working with multi-valued real functions.

[5]) In the following text, it will be often useful to consider the pair of functions $\varphi(z)$, $\psi(z) = \chi'(z)$ instead of the pair of functions $\varphi(z)$, $\chi(z)$. Also these functions $\varphi(z)$, $\psi(z)$ will be called the stress-functions.

Remark 2.2. It follows from (2.3) and (2.6) that to the functions $\sigma_x, \sigma_y, \tau_{xy}$ there exist a set of the corresponding Airy functions and a set of the corresponding stress-functions. Nevertheless, in every case (i.e. when chosing the constants a, b, c or $C_1, \ldots, C_4$ arbitrarily) (2.4) or (2.7), (2.8) yield precisely the original functions σ_x, σ_y, τ_{xy}. As to the components d_1, d_2 of the vector of displacement, they may differ by certain linear functions, the physical meaning of which is a "small displacement" or a "small rotation" of G as of a rigid body. For details see [44], Sec. 2.6.

b) *Multiply connected regions*

Let us consider a $(k + 1)$ – tuply connected region G with the boundary

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_k$$

as discussed in Chap. 1, p. 356.[6]) Let $\sigma_x, \sigma_y, \tau_{xy}$ be sufficiently smooth functions fulfilling (2.1) and (2.2). In this case it is again possible to construct a biharmonic function $u(x, y)$ such that in G the relations (2.4) hold. But in contrast to the case of a simply connected region, this function need not be a single-valued function. We are not going to introduce here the concept of a multi-valued function and of its derivatives. Instead we give directly a lemma concerning the form of the corresponding complex stress-functions.[7]) These functions will also appear to be multi-valued, but this multi-validity is of a very simple – namely of logarithmic – character:

Lemma 2.2. ([4], Sec. 2.10). *Let G be a bounded $(k + 1)$ – tuply connected region with inner boundary curves $\Gamma_1, \ldots, \Gamma_k$. Let $z_i = x_i + iy_i$ $(i = 1, \ldots, k)$ be arbitrary* (but fixed) *points lying inside Γ_i* (and, consequently, outside G; cf. Fig. 1). *Let $\sigma_x, \sigma_y, \tau_{xy}$ be continuously differentiable functions in G* [8]) *fulfilling* (2.1) *and* (2.2). *Then the stress-functions $\varphi(z)$, $\psi(z)$* (cf. the footnote 5, p. 360) connected with the functions $\sigma_x, \sigma_y, \tau_{xy}$ by the relations (2.7), (2.8) exist and can be written in the form

$$(2.10) \qquad \varphi(z) = z \sum_{i=1}^{k} A_i \ln (z - z_i) + \sum_{i=1}^{k} B_i \ln (z - z_i) + \varphi_0(z) \,,$$

$$(2.11) \qquad \psi(z) = \chi'(z) = \sum_{i=1}^{k} C_i \ln (z - z_i) + \psi_0(z) \,,$$

[6]) $\Gamma_1, \ldots, \Gamma_k$ are inner boundary curves, oriented as shown in Fig. 1, p. 356.

[7]) We could have chosen this way also in the case of simply connected regions, of course. But there was no need to do it there.

[8]) In [4], the proof is carried out under supplementary assumptions on the smoothness of the boundary Γ and of the functions $\sigma_x, \sigma_y, \tau_{xy}$ up to the boundary. Then it is shown that these assumptions are superfluous. The sketch of the proof (from which this fact is also clear) is given in Chap. 5,

where A_i are real constants, B_i, C_i are complex constants and $\varphi_0(z)$, $\psi_0(z)$ are holomorphic functions in G.

Moreover, the constants A_i are independent of the choice of the points z_i inside of Γ_i. So are the constants B_i and C_i in the case that all the A_i are zeros.

Remark 2.3. The functions σ_x, σ_y, τ_{xy} being given and the points z_i being chosen *fixed*, the functions (2.10), (2.11) are uniquely determined up to some linear functions of z (cf. (2.6)). In particular, the coefficients A_i, B_i, C_i are uniquely determined.

Remark 2.4. On the other hand, the functions (2.10), (2.11) being given, the functions σ_x, σ_y, τ_{xy} computed by (2.7), (2.8) fulfil equations (2.1), (2.2). These functions can be obtained also in such a way that we construct the function (2.5) biharmonic in G, and then use (2.4). However, the function (2.5) need not be single-valued, and, as said above, the concept of a multi-valued real function and its derivatives will not be introduced here. *Thus we shall speak of the function* (2.5) (and of relations (2.4)) *only if it is a single-valued function.*

Note that the function (2.5) may be single-valued even when the stress-functions $\varphi(z)$, $\chi(z)$ are multi-valued: For example, if we take in Ex. 1.1 (p. 357)

$$\varphi(z) = \frac{1}{9 - 16 \ln^2 2}\left[6z \ln z - (3 + 8 \ln 2)\, z\right],$$

$$\chi(z) = \frac{1}{9 - 16 \ln^2 2}\left[16 \ln 2 \,.\, \ln z + (12 + 8 \ln 2 - 16 \ln^2 2)\right],$$

we get by (2.5) precisely the function (1.22), and this is a single-valued function. (The reader may easily check also the validity of (2.7), (2.8) with σ_x, σ_y, τ_{xy} given by (1.23).)

Let us show that the corresponding displacement (2.9) is not a single-valued function in this case. To this purpuse it is sufficient to examine only the functions $\tilde{\varphi}(z) = z \ln z$, $\tilde{\chi}(z) \equiv 0$, because the function $(3 + 8 \ln 2)\, z$ as well as the function $\chi'(z)$ appearing in (2.9) are single-valued functions. But we have (writing $\ln z$ in the usual form $\ln r + \mathrm{i}\omega$)

$$\varkappa\tilde{\varphi} - z\overline{\tilde{\varphi}'} - \overline{\tilde{\chi}'} = z[(\varkappa \ln r - \ln r - 1) + \mathrm{i}(\varkappa + 1)\, \omega],$$

and this function is not single-valued in G because of the multi-validity of the function ω.

Thus in the case of multiply connected regions it may happen that a single-valued biharmonic function $u(x, y)$ "produces" (by virtue of (2.4)) the functions σ_x, σ_y, τ_{xy} which fulfil the equations of static equilibrium and of compatibility, while the corresponding components of the vector of displacement are not single-valued functions. Such a biharmonic function thus cannot describe a real stress-and-strain state in G.

Definition 2.1. A (single-valued) biharmonic function to which there correspond single-valued components of the vector of displacement in the just discussed sense is called an *Airy function*.[9]) In the opposite case we call it a *singular biharmonic function*.

An example of an Airy function is every function biharmonic in a simply connected region G. An example of a singular biharmonic function is the function (1.22) from Ex. 1.1.

In Chap. 1 we have shown how to transform the first problem of plane elasticity in a *simply connected* region into a biharmonic problem

$$\Delta^2 u = 0 \quad \text{in} \quad G, \tag{2.12}$$

$$u = g_0(s), \quad \frac{\partial u}{\partial v} = g_1(s) \quad \text{on} \quad \Gamma. \tag{2.13}$$

Here the functions $g_0(s)$, $g_1(s)$ are constructed from the given loading on the boundary by (1.7)–(1.9) (using if necessary the correspoding Stieltjes integrals, etc.). Assuming $(g_0, g_1) \in W_2^{(1)}(\Gamma) \times L_2(\Gamma)$, we have shown the existence (and uniqueness) of the so-called very weak solution $u(x, y)$ (or of a weak solution, as a special case) of the problem (2.12), (2.13). The function $u(x, y)$ is biharmonic in G in the classical sense, the functions $\sigma_x, \sigma_y, \tau_{xy}$ given by (2.4) fulfil equations (2.1), (2.2) of static equilibrium and of compatibility and the corresponding displacement (2.9) is a single-valued function, because $\varphi(z)$ and $\psi(z)$ are holomorphic in G. Thus the functions $\sigma_x, \sigma_y, \tau_{xy}$ characterize actually a state of stress in G. The boundary conditions are fulfilled in a generalized sense.

In the case of a *multiply connected* region, we have proceeded similarly. We have shown the existence and uniqueness of a very weak (or weak) solution $u(x, y)$ of the problem

$$\Delta^2 u = 0 \quad \text{in} \quad G, \tag{2.14}$$

$$u = g_{i0}(s), \quad \frac{\partial u}{\partial v} = g_{i1}(s) \quad \text{on} \quad \Gamma_i, \quad i = 0, 1, \ldots, k, \tag{2.15}$$

provided $(g_{i0}, g_{i1}) \in W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$. The functions g_{i0}, g_{i1} have been constructed similarly as the functions g_0, g_1 by (1.7)–(1.9). The functions $\sigma_x, \sigma_y, \tau_{xy}$ given by

[9]) Thus this function describes a real stress-and-strain state in G.

Note that the concept of the Airy function does not depend on the choice of the points z_i appearing in Lemma 2.2, in spite of the fact that in general the coefficients B_i, C_i of the logarithmic terms in (2.10, (2.11) depend on this choice. In fact, if at least one on the coefficients A_i ($i = 1, \ldots, k$) is different from zero, the function (2.9) cannot be single-valued. This fact follows from an easy computation similar to that carried out in the preceding footnote. (See also (2.43) in the following text, which implies this fact immediately.) If all A_i ($i = 1, \ldots, k$) are equal to zero, then B_i and C_i are independent of the choice of the points z_i.

(2.4) satisfy again equations (2.1), (2.2). But in contrast to the previous case, the corresponding displacement need not be a single-valued function, as mentioned before (see especially Example 1.1). Then the functions σ_x, σ_y, τ_{xy} do not describe a real stress-and-strain state in G and cannot be taken as components of a real stress-tensor. At the same time, if a body is in a static and moment equilibrium, it is to be expected that a real stress-tensor (to which a single-valued displacement corresponds) should exist.

What is the cause of such a discrepancy?

Let us consider first the case of a simply connected region. The functions $g_0(s)$, $g_1(s)$ are constructed from the given loading with components $X(s)$, $Y(s)$ by (1.7) to (1.9): We start with the construction of the functions

$$(2.16) \qquad \frac{\partial u}{\partial x} = -\int_0^s Y(t)\,\mathrm{d}t\,, \quad \frac{\partial u}{\partial y} = \int_0^s X(t)\,\mathrm{d}t\,,$$

putting $s = 0$ at a chosen point $A \in \Gamma$, at which we then have $\partial u/\partial x = 0$, $\partial u/\partial y = 0$.[10]) If we had chosen $s = 0$ at another point $B \in \Gamma$, then the new functions – denote them by $\partial \hat{u}/\partial x$, $\partial \hat{u}/\partial y$ – would be equal to zero at that point B, and each of them would differ on Γ from the original functions $\partial u/\partial x$ and $\partial u/\partial y$ by a constant. It is easy to compute from (1.8), (1.9) that, if we put $s = 0$ at a point B, the functions

$$(2.17) \qquad \hat{u} = \hat{g}_0(s)\,, \quad \frac{\partial \hat{u}}{\partial v} = \hat{g}_1(s) \quad \text{on} \quad \Gamma$$

differ from the original functions

$$(2.18) \qquad u = g_0(s)\,, \quad \frac{\partial u}{\partial v} = g_1(s) \quad \text{on} \quad \Gamma$$

by expressions of the form

$$(2.19) \qquad l = ax + by + c \quad \text{and} \quad av_x + bv_y = a\frac{\partial x}{\partial v} + b\frac{\partial y}{\partial v} = \frac{\partial l}{\partial v}\,,$$

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[10]) Thus

$$(\mathrm{i}) \qquad \frac{\partial u}{\partial x}(s_C) = -Y_{AC}\,, \quad \frac{\partial u}{\partial y}(s_C) = X_{AC}\,.$$

Here AC is an arc on Γ (its end-point C having the coordinate s_C) with the same orientation as Γ, and X_{AC} or Y_{AC} is the x- or y-component of the main vector, respectivey (i.e. of the total force. acting on AC). While the form (1.7) is suitable for a "regular" loading, the form (i) is suitable for a "general" loading (containing various singularities, e.g. single loads).

respectively, where a, b, c are real constants depending on the choice of the point B; ν_x, ν_y are components of the unit outward normal.[11]) Now, let $u(x, y)$ be a (weak or very weak) solution of the biharmonic problem (2.14), (2.18). Then obviously the function

$$\hat{u}(x, y) = u(x, y) + ax + by + c$$

is a solution of the problem (2.14), (2.17). By the uniqueness of the weak or very weak solution, it is its only solution. From (2.4) it is clear that the same stress-tensor corresponds to both the functions $u(x, y)$ and $\hat{u}(x, y)$.

Thus in the case of a simply connected region the change of the "starting" point A on Γ does not cause any difficulties: The functions (2.17), (2.18) which correspond to the same loading lead to the same stress-tensor in G.

If the region G is $(k + 1) -$ *tuply connected* $(k > 0)$, then replacing the "starting" points A_i on Γ_i $(i = 0, 1, \ldots, k)$ by new points B_i we get new functions

$$\hat{u} = g_{i0} + l_i\,, \quad \frac{\partial \hat{u}}{\partial \nu} = g_{i1} + \frac{\partial l_i}{\partial \nu} \quad \text{on} \quad \Gamma \tag{2.20}$$

[11]) Conversely, the same loading of Γ corresponds to the functions (2.18) and (2.17) with

$$\hat{g}_0 = g_0 + ax + by + c\,, \quad \hat{g}_1 = g_1 + a\nu_x + b\nu_y\,, \tag{ii}$$

where a, b, c are arbitrary (real) constants. In fact, using the well-known formulae

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s}\nu_y + \frac{\partial u}{\partial \nu}\nu_x\,, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\nu_x + \frac{\partial u}{\partial \nu}\nu_y\,,$$

we get

$$\frac{\partial u}{\partial x} = -\frac{\mathrm{d}g_0}{\mathrm{d}s}\nu_y + g_1\nu_x\,, \quad \frac{\partial u}{\partial y} = \frac{\mathrm{d}g_0}{\mathrm{d}s}\nu_x + g_1\nu_y\,,$$

$$\frac{\partial \hat{u}}{\partial x} = \frac{\partial u}{\partial x} + a(\nu_y^2 + \nu_x^2) = \frac{\partial u}{\partial x} + a\,, \quad \frac{\partial \hat{u}}{\partial y} = \frac{\partial u}{\partial y} + b(\nu_x^2 + \nu_y^2) = \frac{\partial u}{\partial y} + b\,.$$

Consequently, for an arc PQ on Γ with the same orientation as Γ we have

$$\frac{\partial \hat{u}}{\partial x}(Q) - \frac{\partial \hat{u}}{\partial x}(P) = \frac{\partial u}{\partial x}(Q) - \frac{\partial u}{\partial x}(P) \quad \text{and}$$

$$\frac{\partial \hat{u}}{\partial y}(Q) - \frac{\partial \hat{u}}{\partial y}(P) = \frac{\partial u}{\partial y}(Q) - \frac{\partial u}{\partial y}(P)$$

so that (see the preceding footnote) the main vectors acting on PQ are the same. The arc PQ being arbitrary, the same loading of Γ corresponds to the functions (2.17), (2.18) fulfilling (ii).

instead of the original functions

$$u = g_{i0}\,, \quad \frac{\partial u}{\partial \nu} = g_{i1} \quad \text{on} \quad \Gamma\,; \tag{2.21}$$

here

$$l_i = a_i x + b_i y + c_i\,, \quad i = 0, 1, \ldots, k\,, \tag{2.22}$$

where a_i, b_i, c_i are constants depending on the choice of the points B_i on Γ_i. However, in this case the function $\hat{u}(x, y)$ which is the solution of (2.14), (2.20) need not differ from the solution $u(x, y)$ of (2.14), (2.21) only by a linear function all over G, because in general the constants a_i, b_i, c_i are not the same on every Γ_i. The difference of such two solutions may be a singular biharmonic function of the character shown in Ex. 1.1, thus producing a "false" stress-tensor (with a non single-valued displacement).

From the heuristic point of view it is to be expected that if we replace on Γ_i ($i = 1, \ldots, k$) the functions (2.21) by properly chosen functions

$$g_{i0} + a_i x + b_i y + c_i\,, \quad g_{i1} + \frac{\partial}{\partial \nu}(a_i x + b_i y + c_i)\,, \quad i = 1, \ldots, k \tag{2.23}$$

(which thus correspond to the same loading) then the (weak or very weak) solution of the problem

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{2.24}$$

$$u = g_{00}\,, \quad \frac{\partial u}{\partial \nu} = g_{01} \quad \text{on} \quad \Gamma_0\,, \tag{2.25}$$

$$\begin{gathered} u = g_{i0} + a_i x + b_i y + c_i\,, \\ \frac{\partial u}{\partial \nu} = g_{i1} + \frac{\partial}{\partial \nu}(a_i x + b_i y + c_i) \quad \text{on} \quad \Gamma_i\,, \quad i = 1, \ldots, k\,, \end{gathered} \tag{2.26}$$

will be an Airy function.

Definition 2.2. An Airy function $U(x, y)$ which is the solution of (2.24)–(2.26) is called the *Airy function corresponding to the given loading* (characterized by the functions g_{i0}, g_{i1}, $i = 0, 1, \ldots, k$).

In more detail, we shall speak of the *weak* or *very weak Airy function corresponding to the given loading*, according to whether the function $U(x, y)$ is a weak or very weak solution of (2.24)–(2.26), respectively.

Now, we are prepared to give

Formulation of the problem: *To find an Airy function corresponding to the given loading.*

In more detail: The functions g_{i0}, g_{i1} $(i = 0, 1, \ldots, k)$ being given, find such constants a_i, b_i, c_i $(i = 1, \ldots, k)$ that the solution of (2.24)–(2.26) be an Airy function (and find this function, of course).

Remark 2.5. Some heuristic considerations lie in the background of the formulation of our problem. However we show that this formulation is "reasonable" also from the purely mathematical point of view. Namely, we show that:

(i) If the functions g_{i0}, g_{i1} belong to the space $W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$, $i = 0, 1, \ldots, k$, there exists precisely one Airy function corresponding to the given loading. In particular, if $g_{00} = 0$, $g_{01} = 0$ and g_{i0}, g_{i1} are respectively of the form

$$A_i x + B_i y + C_i \quad \text{or} \quad \frac{\partial}{\partial \nu}(A_i x + B_i y + C_i) \quad \text{on} \quad \Gamma_i, \qquad i = 1, \ldots, k$$

(A_i, B_i, C_i constants), then $U(x, y) \equiv 0$. [12]

(ii) If we replace the functions g_{00}, g_{01} also on Γ_0 by some functions

$$g_{00} + a_0 x + b_0 y + c_0, \quad g_{01} + \frac{\partial}{\partial \nu}(a_0 x + b_0 y + c_0),$$

then the new Airy function $U(x, y)$ corresponding to the given loading will differ from the original one precisely by the linear function $a_0 x + b_0 y + c_0$ all over G.

To start with, we introduce the so-called elementary singular biharmonic functions:[13]

Let i be a fixed integer, $1 \leqq i \leqq k$. Let $r_{i1}(x, y)$, $r_{i2}(x, y)$, $r_{i3}(x, y)$ be the weak solutions of the problems

$$\Delta^2 u = 0 \quad \text{in} \quad G, \tag{2.27}$$

$$u = 1, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_i, \tag{2.28}$$

$$u = 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_j, \quad j \neq i \text{ [14]} \tag{2.29}$$

[12]) Thus the only Airy function corresponding to the given loading in Ex. 1.1 is the zero function.

[13]) Note that these functions are of auxiliary character, though they play an essential role in the theoretical considerations. As will be seen in the next chapter, they are completely eliminated from the numerical process when constructing effectively the required Airy function.

[14]) Including Γ_0, i.e. $j = 0, 1, \ldots, i - 1, i + 1, \ldots, k$.

or

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{2.30}$$

$$u = x\,, \quad \frac{\partial u}{\partial \nu} = \frac{\partial x}{\partial \nu} = \nu_1 \quad \text{on} \quad \Gamma_i\,, \tag{2.31}$$

$$u = 0\,, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_j\,, \quad j \neq i \tag{2.32}$$

or

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{2.33}$$

$$u = y\,, \quad \frac{\partial u}{\partial \nu} = \frac{\partial y}{\partial \nu} = \nu_y \quad \text{on} \quad \Gamma_i\,, \tag{2.34}$$

$$u = 0\,, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_j\,, \quad j \neq i\,, \tag{2.35}$$

respectively.

Remark 2.6. If we denote

$$l_{i1}(x, y) \equiv 1\,, \quad l_{i2}(x, y) \equiv x\,, \quad l_{i3}(x, y) \equiv y\,, \quad i = 1, \ldots, k\,, \tag{2.36}$$

then the functions $r_{i1}(x, y)$, $r_{i2}(x, y)$, $r_{i3}(x, y)$ assume on Γ_i the values of the functions $l_{i1}(x, y)$, $l_{i2}(x, y)$, $l_{i3}(x, y)$, and their derivatives with respect to the outward normal assume on Γ_i the values of the outward normal derivatives of these functions. Let $u(x, y)$ be the (weak or very weak) solution of (2.14), (2.15). Then the function

$$U(x, y) = u(x, y) - \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij}\, r_{ij}(x, y)\,, \tag{2.37}$$

where α_{ij} are arbitrary (real) constants, is a solution of the biharmonic equation with boundary conditions of the form (2.25), (2.26). Consequently, if we succeed in finding the coefficients α_{ij} in such a way that $U(x, y)$ is an Airy function, then this function will be an Airy function, corresponding to the given loading, and thus *it will be the required solution.*

Remark 2.7. Note that the weak solution of each of the above three problems exists (uniquely, of course). Actually, each of the functions (2.36) is very smooth in $\bar{G}$ so that it is sufficient to multiply it by a function also sufficiently smooth in $\bar{G}$ and equal to one in a neighbourhood of Γ_i and to zero in the neighbourhoods of the remaining boundary curves, to get a function from the space $W_2^{(2)}(G)$ which satisfies the given boundary conditions (and thus ensures the existence of the weak solution).

Lemma 2.3. *Each of the functions* $r_{ij}(x, y)$ $(i = 1, \ldots, k, j = 1, 2, 3)$ *is a singular biharmonic function* (thus producing a multi-valued displacement).

This lemma is a special case of the following one:

Lemma 2.4. *An arbitrary linear combination*

$$\sum_{i=1}^{k} \sum_{j=1}^{3} a_{ij} r_{ij}(x, y) \tag{2.38}$$

of the functions $r_{ij}(x, y)$ *is a singular biharmonic function provided that at least one of the coefficients* a_{ij} *is different from zero.*

The proof of this lemma is not trivial and is postponed to Chap. 5. [15])

Definition 2.3. The functions $r_{ij}(x, y)$ will be called *basic singular biharmonic functions.*

As said above, these functions will be used to obtain the desired Airy function corresponding to the given loading.

Thus, let the problem (2.14), (2.15), i.e. the problem

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{2.39}$$

$$u = g_{i0}\,, \quad \frac{\partial u}{\partial \nu} = g_{i1} \quad \text{on} \quad \Gamma_i\,, \qquad i = 0, 1, \ldots, k \tag{2.40}$$

with $(g_{i0}, g_{i1}) \in W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$ be given and let $u(x, y)$ be its very weak solution.[16]) Since this function is biharmonic in G, the functions

$$\sigma_x = \frac{\partial^2 u}{\partial y^2}\,, \quad \sigma_y = \frac{\partial^2 u}{\partial x^2}\,, \quad \tau_{xy} = -\frac{\partial^2 u}{\partial x\, \partial y} \tag{2.41}$$

are sufficiently smooth in G and fulfil the relations (2.1), (2.2). According to Lemma 2.2 it is possible to construct the corresponding stress-functions of the form (2.10),

---

[15]) If the boundary Γ is sufficiently smooth, then the functions $r_{ij}(x, y)$ can be shown to be sufficiently smooth in $\bar{G}$, and the proof is relatively simple — it is possible to apply the idea of the proof of the classical Kirchhoff theorem 2.12.1 from [4]. If Γ is only Lipschitzian, this is not the case because then it is not possible to ensure the necessary smoothness of the functions appearing in the proof up to the boundary, so that we have to proceed in another way.

[16]) The functions g_{i0}, g_{i1} belonging to the space $W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$ for $i = 0, 1, \ldots, k$, the existence (and uniqueness) of the very weak solution is ensured. This very weak solution may turn into a weak one (if there exists such a function $w \in W_2^{(2)}(G)$ that we have $w = g_{i0}$, $\partial w/\partial \nu = g_{i1}$ on Γ_i in the sense of trances). We are not going to draw always attention to this fact, and *we shall speak mostly of the very weak solution only,* tacitly admitting the possibility that it may be a weak solution. Only when the concept of the weak solution plays an important role in our considerations, we shall point out this fact.

(2.11) so that the relations (2.7), (2.8), (2.9) hold. In particular, according to Remark 2.3, keeping the points z_i fixed, the coefficients A_i, B_i, C_i $(i = 1, \ldots, k)$ in (2.10), (2.11) are uniquely determined by the functions (2.41), and thus also by the functions (2.40).

Let G' be a $(k + 1)$-tuply connected region lying inside G $(\bar{G}' \subset G)$, with a smooth boundary $\Gamma' = \Gamma'_0 \cup \Gamma'_1 \cup \ldots \cup \Gamma'_k$ (Fig. 2), and let z_i, $i = 1, 2, \ldots, k$ be points contained in the interior of the curves Γ_i as well as in the interior of Γ'_i. Choose one of the curves $\Gamma'_1, \ldots, \Gamma'_k$, say Γ'_p, and on this curve choose an arbitrary point z. Putting (2.10), (2.11) into (2.9), it is possible to compute the "complex" displacement

$$d_1 + \mathrm{i}d_2 \tag{2.42}$$

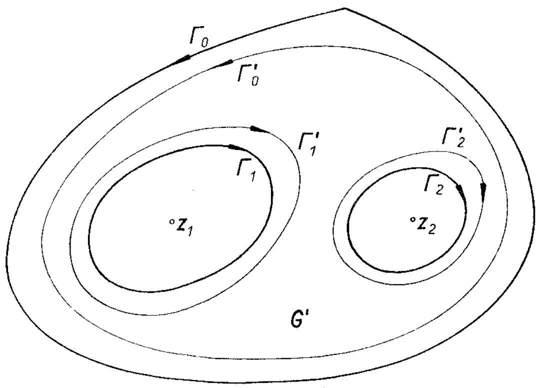

Fig. 2.

at this point, Now, let us run along this curve in the positive sense of its orientation and return back to the same point z. All the terms in (2.10), (2.11) remain unchanged, with the exception of the terms containg $\ln(z - z_p)$. From (2.9) we get by an easy computation that, when running along the curve Γ'_p, the complex displacement (2.42) will change by the value

$$-\frac{\pi}{\mu}\mathrm{i}[(\varkappa + 1) A_p z + \varkappa B_p + \bar{C}_p] . \tag{2.43}$$

We write it in a simpler form

$$-\frac{\pi}{\mu}\mathrm{i}(\gamma_{p1} z + \gamma_{p2} + \mathrm{i}\gamma_{p3}) , \tag{2.44}$$

putting

$$(\varkappa + 1) A_p = \gamma_{p1} , \quad \mathrm{Re}(\varkappa B_p + \bar{C}_p) = \gamma_{p2} , \quad \mathrm{Im}(\varkappa B_p + \bar{C}_p) = \gamma_{p3} . \tag{2.45}$$

(Thus the numbers $\gamma_{p1}, \gamma_{p2}, \gamma_{p3}$ are real.)

It follows from (2.44) that the vector of displacement corresponding to the functions (2.41), i.e. to the solution $u(x, y)$ of the problem (2.39), (2.40), will be a single-valued function in G if and only if

$$(2.46) \qquad \gamma_{p1} = 0, \; \gamma_{p2} = 0, \; \gamma_{p3} = 0 \quad \text{for every} \quad p = 1, \dots, k .$$

Precisely in this case the solution $u(x, y)$ of (2.39), (2.40) will be an Airy function.

In general, this is not the case. This means that the function $u(x, y)$ will "produce" $3k$ (real) numbers

$$(2.47) \qquad \gamma_{i1}, \gamma_{i2}, \gamma_{i3} ,$$

at least one of them being different from zero. A question arises whether it is possible to find such numbers α_{ij} that the function (2.37), i.e. the function

$$(2.48) \qquad U(x, y) = u(x, y) - \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij} \, r_{ij}(x, y)$$

be an Airy function.

Similarly as the function $u(x, y)$ produces the numbers (2.47), each of the functions $r_{ij}(x, y)$ $(i = 1, \dots, k, j = 1, 2, 3)$, being biharmonic in G, produces $3k$ numbers – let us denote them by β_{ijpq} $(p = 1, \dots, k, q = 1, 2, 3)$. As said above, we try to find such a linear combination

$$(2.49) \qquad v(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij} \, r_{ij}(x, y)$$

of the functions $r_{ij}(x, y)$ that the function

$$(2.50) \qquad U(x, y) = u(x, y) - v(x, y)$$

be an Airy function, i.e. (see (2.46)) that all the $3k$ numbers corresponding to it be equal to zero. This condition leads to the system of $3k$ equations

$$(2.51) \qquad \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij} \beta_{ijpq} = \gamma_{pq} , \quad p = 1, \dots, k , \quad q = 1, 2, 3$$

for $3k$ unknowns α_{ij}, $i = 1, \dots, k, j = 1, 2, 3$.

Lemma 2.5. *The determinant D of the system (2.51) is different from zero; consequently, this system is uniquely solvable.*

The proof is easy: If we had $D = 0$, then the corresponding homogeneous system would have also a nonzero solution; but then the function $u(x, y)$ and according to (2.50) also the function $v(x, y)$ would be Airy functions, while at the same time at least one of the α_{ij} would be different from zero. This is contradiction to Lemma2.4.

Remark 2.8. (Uniqueness.) In this way, the α_{ij} $(i = 1, \ldots, k,\ j = 1, 2, 3)$ satisfying (2.51) are found and thus (cf. Remark 2.6) the existence of a very weak Airy function of the form

$$U(x, y) = u(x, y) - \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij}\, r_{ij}(x, y) \tag{2.52}$$

corresponding to the given loading, i.e. of an Airy function which solves the problem (2.24)–(2.26) is proved. Before formulating the corresponding existence theorem, let us clarify the question of uniqueness. In fact, it is not a priori clear whether the function (2.52) with α_{ij} computed from (2.51) is the only Airy function corresponding to the given loading, because

(i) the numbers B_i, C_i in (2.10), (2.11) depend on the choice of the points z_i $(i = 1, \ldots, k)$. Thus also the right-hand sides γ_{pq} in (2.51) may depend on the choice of these points, as well as the numbers β_{ijpq} in (2.51). Consequently, also the numbers α_{ij} may depend on the choice of the points z_i;

(ii) it is not a priori evident that there exist no other very weak Airy functions corresponding to the given loading, with other a_i, b_i, c_i in (2.26) than are those corresponding to the function (2.52).

We shall show that (2.52) with α_{ij} computed from (2.51) is the only Airy function corresponding to the given loading (independently of the choice of the points z_i in (2.10), (2.11)).

Thus let $\hat{U}(x, y)$ be another Airy function satisfying (in the very weak sense) (2.24)–(2.26), possibly with other constants $\hat{a}_i$, $\hat{b}_i$, $\hat{c}_i$ in (2.26). Denoting $\hat{U}(x, y) - U(x, y) = \bar{U}(x, y)$ and $\hat{a}_i - a_i = \bar{a}_i$, $\hat{b}_i - b_i = \bar{b}_i$, $\hat{c}_i - c_i = \bar{c}_i$, the function $\bar{U}(x, y)$ is the solution of the problem

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{2.53}$$

$$u = 0\,, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0\,, \tag{2.54}$$

$$u = \bar{a}_i x + \bar{b}_i y + \bar{c}_i\,, \quad \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(\bar{a}_i x + \bar{b}_i y + \bar{c}_i) \quad \text{on} \quad \Gamma_i\,, \quad i = 1, \ldots, k\,. \tag{2.55}$$

In virtue of the definition of the functions $r_{ij}(x, y)$ (see (2.27)–(2.35)), the solution of the problem (2.53)–(2.55) can be written in the form

$$\sum_{i=1}^{k} \bar{a}_i\, r_{i2}(x, y) + \sum_{i=1}^{k} \bar{b}_i\, r_{i3}(x, y) + \sum_{i=1}^{k} \bar{c}_i\, r_{i1}(x, y)\,. \tag{2.56}$$

Due to the uniqueness of the very weak solution of this problem, we have

$$\bar{U}(x, y) = \sum_{i=1}^{k} \bar{a}_i\, r_{i2}(x, y) + \sum_{i=1}^{k} \bar{b}_i\, r_{i3}(x, y) + \sum_{i=1}^{k} \bar{c}_i\, r_{i1}(x, y)\,. \tag{2.57}$$

The function (2.57), being the difference of two Airy functions, is also an Airy function. According to Lemma 2.4, this is possible only if all the coefficients $\bar{a}_i$, $\bar{b}_i$, $\bar{c}_i$ are equal to zero, so that $\bar{U}(x, y) \equiv 0$.

Thus we have

Theorem 2.1. *Let G be a bounded $(k + 1)$ – tuply connected region with a Lipschitzian boundary. Let*

$$g_{i0} \in W_2^{(1)}(\Gamma_i)\,, \quad g_{i1} \in L_2(\Gamma_i)\,, \quad i = 0, 1, \ldots, k\,. \tag{2.58}$$

Then there exists precisely one very weak Airy function $U(x, y)$ corresponding to the given loading, i.e. solving the problem (2.24)–(2.26). This function can be written in the form

$$U(x, y) = u(x, y) - \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij}\, r_{ij}(x, y)\,, \tag{2.59}$$

where $u(x, y)$ is the very weak solution of the problem (2.39), (2.40), $r_{ij}(x, y)$, $i = 1, \ldots, k$, $j = 1, 2, 3$ are the basic singular biharmonic functions defined as solutions of the problems (2.27)–(2.35) and α_{ij}, $l = 1, \ldots, k, j = 1, 2, 3$ are uniquely determined from the system (2.51) (independently of the choice of the points z_i $(i = 1, \ldots, k)$ in (2.10), (2.11)).

Remark 2.9. If we replace the original "starting points" A_i on Γ_i $(i = 0, 1, \ldots, k)$ by new starting points B_i, then, as mentioned above, the functions

$$g_{i0}, g_{i1} \quad \text{on} \quad \Gamma_i \tag{2.60}$$

turn into functions of the form

$$\bar{g}_{i0} = g_{i0} + a_i x + b_i y + c_i\,, \quad \bar{g}_{i1} = g_{i1} + \frac{\partial}{\partial \nu}(a_i x + b_i y + c_i)\,. \tag{2.61}$$

Let first $B_0 = A_0$ so that the starting point on Γ_0 remains the same and thus $a_0 = b_0 = c_0 = 0$. According to the result just obtained, there exists precisely one (very weak) Airy function corresponding to the loading (2.61) with $a_0 = b_0 = c_0 = 0$. It follows immediately that this function will be equal precisely to the Airy function (2.59) from Theorem 2.1. In fact, the difference of these two functions should be an Airy function fulfilling conditions of the form (2.54)–(2.55), and such a function is identically zero as shown in Remark 2.8.

If we change the starting point also on Γ_0 so that

$$\bar{g}_{00} = g_{00} + a_0 x + b_0 y + c_0\,, \quad \bar{g}_{01} = g_{01} + \frac{\partial}{\partial \nu}(a_0 x + b_0 y + c_0)\,,$$

then we can look for the Airy function corresponding to the loading (2.61) in the form

$$U_0(x, y) = a_0 x + b_0 y + c_0 + \tilde{U}(x, y)$$

and transform this problem in this way into the previous one. The function $U_0(x, y)$ will then differ from the function from Theorem 2.1 by the expression $a_0x + b_0y + c_0$ all over G.

Thus we can summarize:

If the points A_i $(i = 1, \ldots, k)$ with $s = 0$ on Γ_i are replaced by new points B_i, then the Airy function corresponding to the given loading remains unchanged. If also A_0 is replaced by B_0, then the Airy function differs from the original one by an expression of the form $a_0x + b_0y + c_0$ all over G.

CHAPTER 3. THE METHOD OF LEAST SQUARES ON THE BOUNDARY

For the case of a *simply connected* region,[1]) the method of least squares on the boundary is described in detail in [1]: Consider the first biharmonic problem

$$\Delta^2 u = 0 \quad \text{in} \quad G, \tag{3.1}$$

$$u = g_0, \quad \frac{\partial u}{\partial \nu} = g_1 \quad \text{on } \Gamma \tag{3.2}$$

with $(g_0, g_1) \in W_2^{(1)}(\Gamma) \times L_2(\Gamma)$. Let the system of basic biharmonic polynomials be given. (For details see [1]; we have

$$z_1(x, y) \equiv 1,$$
$$z_2(x, y) = x, \quad z_3(x, y) = y,$$
$$z_4(x, y) = x^2 - y^2, \quad z_5(x, y) = 2xy, \quad z_6(x, y) = -y^2,$$
$$z_7(x, y) = x^3 - 3xy^2, \quad z_8(x, y) = 3x^2y - y^3,$$
$$z_9(x, y) = -3xy^2, \quad z_{10}(x, y) = -y^3,$$

etc.; for every fixed $n \geqq 2$ there are precisely $4n - 2$ polynomials of degrees $\leqq n$.) The approximate solution of (3.1), (3.2) is assumed in the form

$$u_n(x, y) = \sum_{i=1}^{4n-2} a_{ni} z_i(x, y), \quad n \geqq 2, \text{[2])} \tag{3.3}$$

---

[1]) Bounded regions with Lipschitzian boundaries are always considered.

[2]) The assumption $n \geqq 2$ is introduced for formal reasons only: For example, there are three polynomials of degree $\leqq 1$, namely $z_1(x, y)$, $z_2(x, y)$, $z_3(x, y)$, while putting $n = 1$ into (3.3) we obtain only two terms. Of course, in principle it is possible to consider linear terms only. But it is of no particular use, because these terms yield only the zero stress in G, cf. (1.5), p. 353.

where the coefficients a_{ni} are determined from the condition of least squares on the boundary, i.e. from the condition that

$$(3.4)\qquad Fu_n = \int_\Gamma (u_n - g_0)^2 \, ds + \int_\Gamma \left(\frac{\partial u_n}{\partial s} - \frac{dg_0}{ds}\right)^2 ds + \int_\Gamma \left(\frac{\partial u_n}{\partial \nu} - g_1\right)^2 ds = \min .$$

among all the expressions of the form

$$(3.5)\qquad Fv_n = \int_\Gamma (v_n - g_0)^2 \, ds + \int_\Gamma \left(\frac{\partial v_n}{\partial s} - \frac{dg_0}{ds}\right)^2 ds + \int_\Gamma \left(\frac{\partial v_n}{\partial \nu} - g_1\right)^2 ds ,$$

where

$$(3.6)\qquad v_n(x, y) = \sum_{i=1}^{4n-2} b_{ni} \, z_i(x, y)$$

and ν is the outward normal to the boundary. (Thus the functional (3.5) has to assume its minimal value on the set of functions (3.6) precisely for the function (3.3)). The condition (3.4) leads to a system of linear algebraic equations for the unknowns a_{ni}. In [1], the unique solvability of this system is proved as well as the convergence, in $L_2(G)$, of the sequence $\{u_n(x, y)\}$ to the very weak solution $u(x, y)$ of (3.1), (3.2).

In the case of a *multiply connected* region, the situation is more complicated. The first difficulty lies in the fact that the very weak[3]) solution $u(x, y)$ of (3.1), (3.2)[4]) need not be the required Airy function $U(x, y)$; according to Theorem 2.1 we know that we have

$$(3.7)\qquad U(x, y) = u(x, y) - \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij} \, r_{ij}(x, y) ,$$

where $r_{ij}(x, y)$ are the basic singular biharmonic functions introduced on p. 367 and α_{ij} are constants uniquely determined by the function u.[5])

The second difficulty is the following: In [1], i.e. for a simply connected region, we were able to prove by means of the well-known relation

$$(3.8)\qquad u(x, y) = \operatorname{Re}(\bar{z}\varphi + \chi)$$

that every biharmonic function in G sufficiently smooth in $\bar{G}$, can be approximated in $W_2^{(2)}(G)$ with an arbitrary accuracy by biharmonic polynomials. This is not the case if G is multiply connected. The reason is that the corresponding holomorphic functions cannot be approximated only by polynomials, but that also rational

[3]) Cf. the footnote 16, p. 369.

[4]) We write here briefly g_0 or g_1 on Γ instead of g_{0i}, or g_{1i} on Γ_i, $i = 0, 1, \ldots, k$, respectively.

[5]) The functions $r_{ij}(x, y)$ need not be very simple in general, as functions considered in G; but our method uses only their values on the boundary in the computation, and these values are eminently simple — cf. the definition of these functions.

functions in z are to be taken into account; moreover, the proof of Lemma 4.6 on density (in Part II of the present paper) shows that some simple logarithmic functions should be considered. For these reasons, we assume the approximate solution in the form

$$u_{st}(x, y) = U_{st}(x, y) + \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{stij} \, r_{ij}(x, y), \tag{3.9}$$

where

$$U_{st}(x, y) = \sum_{p=1}^{4s-2} a_{stp} \, z_p(x, y) + \sum_{i=1}^{k} \sum_{q=1}^{4t} b_{stiq} \, v_{iq}(x, y) + \tag{3.10}$$
$$+ \sum_{i=1}^{k} c_{sti} \ln \left[(x - x_i)^2 + (y - y_i)^2\right] .^{6)}$$

Here s, t are positive integers, $s \geqq 2$, $z_p(x, y)$ are basic biharmonic polynomials, (x_i, y_i) are arbitrarily chosen but fixed points lying in the interior of Γ_i, $i = 1, \ldots, k$ (thus outside $\bar{G}$)[7]) and $v_{iq}(x, y)$ are rational biharmonic functions corresponding (cf. (3.8)), to the above mentioned rational functions in z:

$$v_{i,4l+1}(x, y) = \operatorname{Re}\left[\frac{\bar{z}}{(z - z_i)^{l+1}}\right], \quad v_{i,4l+2}(x, y) = \operatorname{Im}\left[\frac{\bar{z}}{(z - z_i)^{l+1}}\right],$$
$$v_{i,4l+3}(x, y) = \operatorname{Re}\left[\frac{1}{(z - z_i)^{l+1}}\right], \quad v_{i,4l+4}(x, y) = \operatorname{Im}\left[\frac{1}{(z - z_i)^{l+1}}\right], \tag{3.11}$$
$$i = 1, \ldots, k, \quad l = 0, 1, 2, \ldots, \quad z_i = x_i + \mathrm{i} y_i .$$

For example, if G is an annulus with its centre at the origin, then we have $k = 1$ and we can choose $z_1 = 0$. The functions (3.11) then become

$$v_{11} = \frac{x^2 - y^2}{x^2 + y^2}, \quad v_{12} = -\frac{2xy}{x^2 + y^2}, \quad v_{13} = \frac{x}{x^2 + y^2}, \quad v_{14} = -\frac{y}{x^2 + y^2},$$

$$v_{15} = \frac{x^3 - 3xy^2}{(x^2 + y^2)^2}, \quad v_{16} = \frac{y^3 - 3x^2 y}{(x^2 + y^2)^2}, \quad v_{17} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad v_{18} = -\frac{2xy}{(x^2 + y^2)^2},$$

etc. There is only one logarithmic function in this case, $\ln (x^2 + y^2)$.

The second term in (3.9) corresponds to the second term in (3.7) and represents the "singular part" of the approximation, while the first term in (3.9) represents its "Airy part".

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[6]) In general, t is different for different i ($i = 1, \ldots, k$). From the mathematical point of view this presents no difficulties. (See the proof of the convergence theorem in the next chapter.)

[7]) x_i and y_i are the real and the imaginary parts of the points z_i considered in the preceding chapter.

Thus, let the problem (3.1), (3.2) be given for a $(k+1)$-tupply connected region G, where $g_0 = g_{i0}$ and $g_1 = g_{i1}$ on Γ_i, $i = 0, 1, \ldots, k$,

$$(3.12) \qquad g_{i0} \in W_2^{(1)}(\Gamma_i)\,, \quad g_{i1} \in L_2(\Gamma_i)$$

(cf. the footnote 4 on p. 375). As said before, we are going to look for an approximate solution $u_{st}(x, y)$ in the form (3.9), where the coefficients α_{stij}, a_{stp}, b_{stiq}, c_{sti} are to be determined from the condition of least squares on the boundary, analogous to the condition (3.4):

$$(3.13) \qquad Fu_{st} = \int_\Gamma (u_{st} - g_0)^2\,\mathrm{d}s + \int_\Gamma \left(\frac{\partial u_{st}}{\partial s} - \frac{\mathrm{d}g_0}{\mathrm{d}s}\right)^2 \mathrm{d}s + \int_\Gamma \left(\frac{\partial u_{st}}{\partial \nu} - g_1\right)^2 \mathrm{d}s = \min.\ ^{8)}$$

among all expressions of the form

$$(3.14) \quad F\tilde{u}_{st} = \int_\Gamma (\tilde{u}_{st} - g_0)^2\,\mathrm{d}s + \int_\Gamma \left(\frac{\partial \tilde{u}_{st}}{\partial s} - \frac{\mathrm{d}g_0}{\mathrm{d}s}\right)^2 \mathrm{d}s + \int_\Gamma \left(\frac{\partial \tilde{u}_{st}}{\partial \nu} - g_1\right)^2 \mathrm{d}s\,,$$

where

$$(3.15) \quad \tilde{u}_{st} = \sum_{p=1}^{4s-2} \tilde{a}_{stp}\, z_p(x, y) + \sum_{i=1}^{k} \sum_{q=1}^{4t} \tilde{b}_{stiq}\, v_{iq}(x, y) + \sum_{i=1}^{k} \tilde{c}_{sti} \ln\left[(x - x_i)^2 + (y - y_i)^2\right] + \sum_{i=1}^{k} \sum_{j=1}^{3} \tilde{\alpha}_{stij}\, r_{ij}(x, y)$$

with $\tilde{a}_{stp}$, $\tilde{b}_{stiq}$, $\tilde{c}_{sti}$, $\tilde{\alpha}_{stij}$ arbitrary.

Obviously, (3.14) is a quadratic functional on the set M of all functions of the form (3.15). Substituting (3.15) for $\tilde{u}_{st}$, it becomes a quadratic function in the variables $\tilde{a}_{stp}$, $\tilde{b}_{stiq}$, $\tilde{c}_{sti}$, $\tilde{\alpha}_{stij}$. Necessary (and obviously also sufficient) conditions for (3.13) to be fulfilled are then

$$(3.16) \qquad \frac{\partial F}{\partial \tilde{a}_{st1}}\left(a_{stp}, b_{stiq}, c_{sti}, \alpha_{stij}\right) = 0\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial F}{\partial \tilde{\alpha}_{stk3}}\left(a_{stp}, b_{stiq}, c_{sti}, \alpha_{stij}\right) = 0\,.$$

For example, we have

$$\frac{\partial}{\partial \tilde{a}_{st1}} \int_\Gamma (u_{st} - g_0)^2\,\mathrm{d}s = 2\int_\Gamma (u_{st} - g_0)\, z_1\,\mathrm{d}s\,,$$

---

[8]) Here the index s in u_{st} has nothing common with the length of arc in ∂s and $\mathrm{d}s$, of course.

etc. If we define, for every pair of functions $u, v \in M$

$$(3.17) \qquad (u, v)_\Gamma = \int_\Gamma uv \, ds + \int_\Gamma \frac{\partial u}{\partial s}\frac{\partial v}{\partial s} ds + \int_\Gamma \frac{\partial u}{\partial v}\frac{\partial v}{\partial v} ds$$

(for every such pair (3.17) has sense) and denote briefly

$$(3.18) \qquad (g, u)_\Gamma = \int_\Gamma g_0 u \, ds + \int_\Gamma \frac{dg_0}{ds}\frac{\partial u}{\partial s} ds + \int_\Gamma g_1 \frac{\partial u}{\partial v} ds ,$$

the equations (3.16) become (after dividing them by the factor 2)

$$(3.19) \qquad \begin{aligned} (u_{st}, z_1)_\Gamma &= (g, z_1)_\Gamma , \\ &\dots\dots\dots\dots \\ (u_{st}, r_{k3})_\Gamma &= (g, r_{k3})_\Gamma . \end{aligned}$$ [9]

Substituting (3.9) with (3.10) for u_{st} into (3.19), we get the following system of linear algebraic equations for the unknowns $a_{stp}, b_{stiq}, c_{sti}, \alpha_{stij}$:

$$(3.20) \qquad \begin{aligned} &\sum_{p=1}^{4s-2} (z_p, z_m)_\Gamma a_{stp} + \sum_{i=1}^{k} \sum_{q=1}^{4t} (v_{iq}, z_m)_\Gamma b_{stiq} + \\ &+ \sum_{i=1}^{k} (\ln [(x - x_i)^2 + (y - y_i)^2], z_m)_\Gamma c_{sti} + \\ &+ \sum_{i=1}^{k} \sum_{j=1}^{3} (r_{ij}, z_m)_\Gamma \alpha_{stij} = (g, z_m)_\Gamma , \quad m = 1, \dots, 4s - 2 , \\ &\sum_{p=1}^{4s-2} (z_p, v_{ln})_\Gamma a_{stp} + \sum_{i=1}^{k} \sum_{q=1}^{4t} (v_{iq}, v_{ln})_\Gamma b_{stiq} + \\ &+ \sum_{i=1}^{k} (\ln [(x - x_i)^2 + (y - y_i)^2], v_{ln})_\Gamma c_{sti} + \\ &+ \sum_{i=1}^{k} \sum_{j=1}^{3} (r_{ij}, v_{ln})_\Gamma \alpha_{stij} = (g, v_{ln})_\Gamma , \quad l = 1, \dots, k , \quad n = 1, \dots, 4t , \end{aligned}$$

---

[9]) Of course, these equations have their individual character according to functions appearing in them. For example, $z_1(x, y) \equiv 1$ so that according to (3.17), (3.18), the first equation in (3.19) becomes

$$\int_\Gamma u_{st} \, ds = \int_\Gamma g_0 \, ds ;$$

further, the boundary values of the function r_{k3} are equal to y, $\partial y/\partial s = v_x$, $\partial y/\partial v = v_y$ on Γ_k and vanish elsewhere on Γ_i, $i = 0, 1, \dots, k - 1$. Thus the last of the equations (3.19) reads

$$\int_{\Gamma_k} u_{st} y \, ds + \int_{\Gamma_k} \frac{\partial u_{st}}{\partial s} v_x \, ds + \int_{\Gamma_k} \frac{\partial u_{st}}{\partial v} v_y \, ds = \int_{\Gamma_k} g_0 y \, ds + \int_{\Gamma_k} \frac{dg_0}{ds} v_x \, ds + \int_{\Gamma_k} g_1 v_y \, ds ,$$

etc.

$$\sum_{p=1}^{4s-2} (z_p, \ln[(x - x_l)^2 + (y - y_l)^2])_\Gamma \, a_{stp} +$$
$$+ \sum_{i=1}^{k} \sum_{q=1}^{4t} (v_{iq}, \ln[(x - x_l)^2 + (y - y_l)^2])_\Gamma \, b_{stiq} +$$
$$+ \sum_{i=1}^{k} (\ln[(x - x_i)^2 + (y - y_i)^2], \ln[(x - x_l)^2 + (y - y_l)^2])_\Gamma \, c_{sti} +$$
$$+ \sum_{i=1}^{k} \sum_{j=1}^{3} (r_{ij}, \ln[(x - x_l)^2 + (y - y_l)^2]_\Gamma \, \alpha_{stij} =$$
$$= (g, \ln[(x - x_l)^2 + (y - y_l)^2])_\Gamma, \quad l = 1, \ldots, k,$$
$$\sum_{p=1}^{4s-2} (z_p, r_{lh})_\Gamma \, a_{stp} + \sum_{i=1}^{k} \sum_{q=1}^{4t} (v_{iq}, r_{lh})_\Gamma \, b_{stiq} +$$
$$+ \sum_{i=1}^{k} (\ln[(x - x_i)^2 + (y - y_i)^2], r_{lh})_\Gamma \, c_{sti} +$$
$$+ \sum_{i=1}^{k} \sum_{j=1}^{3} (r_{ij}, r_{lh})_\Gamma \, \alpha_{stij} = (g, r_{lh})_\Gamma, \quad l = 1, \ldots, k, \quad h = 1, 2, 3.$$

The system (3.20) represents a system of linear algebraic equations (with a symmetric matrix) for the unknowns $a_{st1}, \ldots, \alpha_{stk3}$. For example, if G is a doubly connected region and if we choose $s = 3$, $t = 1$, we get 18 equations for 18 unknowns $a_{321}, \ldots, a_{32,10}, b_{3211}, \ldots, b_{3214}, c_{321}, \alpha_{3211}, \alpha_{3212}, \alpha_{3213}$.

Theorem 3.1. *The system* (3.20) *is uniquely solvable.*

For the proof see Chap. 5.

Theorem 3.2. *For $s \to \infty$, $t \to \infty$, the functions $u_{st}(x, y)$ with $a_{stp}, b_{stiq}, c_{sti}, \alpha_{stij}$ determined by* (3.20) *converge in $L_2(G)$ to the very weak solution $u(x, y)$ of the problem* (3.1), (3.2).[10]) *At the same time, the functions $U_{st}(x, y)$ converge in $L_2(G)$ to the "Airy part" $U(x, y)$ of $u(x, y)$* (see (3.7)). *Moreover, this convergence, and even the convergence of the corresponding derivatives of arbitrary order, is locally uniform on G: If G' is an arbitrary region such that $G' \subset \bar{G}' \subset G$, then $U_{st}(x, y) \to \to U(x, y)$ uniformly on G', and the same holds for the convergence of partial derivative of $U_{st}(x, y)$ of an arbitrary order to the corresponding derivative of $U(x, y)$.*[11])

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[10]) More precisely: To every $\varepsilon > 0$ there exist such positive integers s_0 and t_0 that

$$s > s_0, \quad t > t_0 \Rightarrow \|u - u_{st}\|_{L_2(G)} < \varepsilon.$$

[11]) Particularly, we have

$$\sigma_{xst} \to \sigma_x, \quad \sigma_{yst} \to \sigma_y, \quad \tau_{xyst} \to \tau_{xy}$$

The proof of this theorem is the subject of the next chapter.

In the conclusion of the present chapter, we give three examples showing the application of our method. The first example is very simple and has an illustrative character only. In the third example we show how to proceed if the convention is not fulfilled concerning the requirement of static and moment equilibriums of the loading on every Γ_i separately.

Example 3.1. Consider an annulus G with its centre at the origin and with the inner radius $r_1 = 1$ and the outer radius $r_0 = 2$, loaded as shown in Fig. 3.

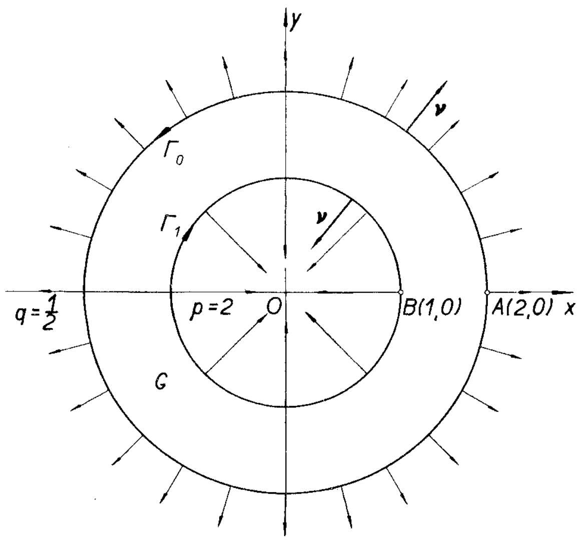

Fig. 3.

As mentioned above, the example is only illustrative. With the aid of the theory based on the use of functions of a complex variable (see e.g. [4]) one easily finds that the required Airy function is of the form

$$U(x, y) = \ln(x^2 + y^2) + ax + by + c\,,$$

where a, b, c are arbitrary constants.

---

uniformly on G', where

$$\sigma_{xst} = \frac{\partial^2 U_{st}}{\partial y^2}, \quad \sigma_{yst} = \frac{\partial^2 U_{st}}{\partial x^2}, \quad \tau_{xyst} = -\frac{\partial^2 U_{st}}{\partial x\, \partial y},$$

and

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x\, \partial y}$$

are the components of the stress-tensor corresponding to the given loading.

On the inner and outer boundary curves Γ_1 and Γ_0 we choose the parameter s of the length of arc with $s = 0$ at the point $B(1, 0)$ and $A(2, 0)$, respectively. Thus we have

$$0 \leqq s < 2\pi \text{ on } \Gamma_1, \quad 0 \leqq s < 4\pi \text{ on } \Gamma_0 .$$

The orientation of Γ_0, Γ_1 is clear from Fig. 3. For the components of the unit outward normal ν we have

$$\nu_x = \cos\frac{s}{2}, \quad \nu_y = \sin\frac{s}{2} \text{ on } \Gamma_0,$$

$$\nu_x = -\cos s, \quad \nu_y = \sin s \text{ on } \Gamma_1 .$$

First we construct the functions g_0, g_1 on Γ, in more detail the functions

$$g_{00}, g_{01} \text{ on } \Gamma_0, \quad g_{10}, g_{11} \text{ on } \Gamma_1,$$

according to (1.7)–(1.9), p. 353. On Γ_0, we have

$$X(s) = \frac{1}{2}\cos\frac{s}{2}, \quad Y(s) = \frac{1}{2}\sin\frac{s}{2}$$

and according to (1.7)

$$\frac{\partial u}{\partial x} = -\int_0^s Y(t)\,\mathrm{d}t = \cos\frac{s}{2} - 1, \quad \frac{\partial u}{\partial y} = \int_0^s X(t)\,\mathrm{d}t = \sin\frac{s}{2}.$$

It follows (cf. (1.8), (1.9))

$$u = \int_0^s \frac{\partial u}{\partial s}(t)\,\mathrm{d}t = \int_0^s \left(-\frac{\partial u}{\partial x}\nu_y + \frac{\partial u}{\partial y}\nu_x\right)\mathrm{d}t = \int_0^s \sin\frac{t}{2}\,\mathrm{d}t = 2\left(1 - \cos\frac{s}{2}\right),$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x}\nu_x + \frac{\partial u}{\partial y}\nu_y = 1 - \cos\frac{s}{2}.$$

Similarly, on Γ_1 we get (we have to pay attention to its orientation)

$$X(s) = -2\cos s, \quad Y(s) = 2\sin s,$$

$$\frac{\partial u}{\partial x} = -\int_0^s Y(t)\,\mathrm{d}t = 2(\cos s - 1), \quad \frac{\partial u}{\partial y} = \int_0^s X(t)\,\mathrm{d}t = -2\sin s,$$

$$u = \int_0^s \frac{\partial u}{\partial s}(t)\,\mathrm{d}t = \int_0^s \left(-\frac{\partial u}{\partial x}\nu_y + \frac{\partial u}{\partial y}\nu_x\right)\mathrm{d}t = \int_0^s 2\sin t\,\mathrm{d}t\, 2(1 - \cos s),$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x}\nu_x + \frac{\partial u}{\partial y}\nu_y = -2(1 - \cos s).$$

Thus, the problem (3.1), (3.2) reads in our case

$$\Delta^2 u = 0 \text{ in } G\,, \tag{3.21}$$

$$u = g_{00} = 2\left(1 - \cos\frac{s}{2}\right) = 2 - x \text{ on } \Gamma_0\,, \tag{3.22}$$

$$\frac{\partial u}{\partial \nu} = g_{01} = 1 - \cos\frac{s}{2} = 1 - \frac{x}{2} \text{ on } \Gamma_0\,, \tag{3.23}$$

$$u = g_{10} = 2(1 - \cos s) = 2(1 - x) \text{ on } \Gamma_1\,, \tag{3.24}$$

$$\frac{\partial u}{\partial \nu} = g_{11} = -2(1 - \cos s) = -2(1 - x) \text{ on } \Gamma_1\,. \tag{3.25}$$

Obviously, the loading is in the static and moment equilibriums on both curves and $(g_0, g_1) \in W_2^{(1)}(\Gamma) \times L_2(\Gamma)$ – the functions are actually much smoother.

Using the method of least squares on the boundary, let us choose in this illustrative example $s = 1$, $t = 1$, $x_1 = y_1 = 0$ in (3.9) so that we have

$$u_{11}(x, y) = a_{111} \,.\, 1 + a_{112}x + a_{113}y + b_{1111}\frac{x^2 - y^2}{x^2 + y^2} + b_{1112}\frac{-2xy}{x^2 + y^2} +$$
$$+ b_{1113}\frac{x}{x^2 + y^2} + b_{1114}\frac{-y}{x^2 + y^2} + c_{111}\ln(x^2 + y^2) + \alpha_{1111}\, r_{11}(x, y) +$$
$$+ \alpha_{1112}\, r_{12}(x, y) + \alpha_{1113}\, r_{13}(x, y)\,.$$

Let us remind that we have

$$r_{11} \equiv 1\,, \quad \frac{\partial r_{11}}{\partial \nu} = 0\,, \quad r_{12} \equiv x\,, \quad \frac{\partial r_{12}}{\partial \nu} = \nu_x\,, \quad r_{13} \equiv y\,, \quad \frac{\partial r_{13}}{\partial \nu} = \nu_y \text{ on } \Gamma_1 \tag{3.26}$$

and

$$r_{11} \equiv 0\,, \quad \frac{\partial r_{11}}{\partial \nu} \equiv 0\,, \quad r_{12} \equiv 0\,, \quad \frac{\partial r_{12}}{\partial \nu} \equiv 0\,, \quad r_{13} \equiv 0\,, \quad \frac{\partial r_{13}}{\partial \nu} \equiv 0 \text{ on } \Gamma_0\,. \tag{3.27}$$

To obtain the values of $a_{111}, \ldots, \alpha_{1113}$, we use the system (3.20) (see Tab. 3.1) which represents 11 equations for 11 unknowns. Constructing this system, we have for example

$$(z_1, z_1)_\Gamma = \int_{\Gamma_0} z_1^2\, ds + \int_{\Gamma_0}\left(\frac{\partial z_1}{\partial s}\right)^2 ds + \int_{\Gamma_0}\left(\frac{\partial z_1}{\partial \nu}\right)^2 ds + \int_{\Gamma_1} z_1^2\, ds +$$
$$+ \int_{\Gamma_1}\left(\frac{\partial z_1}{\partial s}\right)^2 ds + \int_{\Gamma_1}\left(\frac{\partial z_1}{\partial \nu}\right)^2 ds = \int_0^{4\pi} 1^2\, ds + \int_0^{4\pi} 0^2\, ds + \int_0^{4\pi} 0^2\, ds +$$
$$= \int_0^{2\pi} 1^2\, ds + \int_0^{2\pi} 0^2\, ds + \int_0^{2\pi} 0^2\, ds = 6\pi$$

Tab. 3.1

	a_{111}	a_{112}	a_{113}	b_{1111}	b_{1112}	b_{1113}	b_{1114}	c_{111}	α_{1111}	α_{1112}	α_{1113}	
(1)	6π	0	0	0	0	0	0	$4\pi \ln 4$	2π	0	0	12π
(2)	0	15π	0	0	0	3π	0	0	0	3π	0	-18π
(3)	0	0	15π	0	0	0	-3π	0	0	0	3π	0
(4)	0	0	0	6π	0	0	0	0	0	0	0	0
(5)	0	0	0	0	6π	0	0	0	0	0	0	0
(6)	0	3π	0	0	0	6π	0	0	0	$-\pi$	0	-4π
(7)	0	0	-3π	0	0	0	6π	0	0	0	π	0
(8)	$4\pi \ln 4$	0	0	0	0	0	0	$4\pi \ln^2 4 + 12\pi$	0	0	0	$8\pi \ln 4 + 12\pi$
(9)	2π	0	0	0	0	0	0	0	2π	0	0	4π
(10)	0	3π	0	0	0	$-\pi$	0	0	0	3π	0	-6π
(11)	0	0	3π	0	0	0	π	0	0	0	3π	0

since $z_1(x, y) \equiv 1$ so that $\partial z_1/\partial s \equiv 0$, $\partial z_1/\partial v \equiv 0$,

$$(z_1, z_2)_\Gamma = \int_{\Gamma_0} z_1 z_2 \, ds + \int_{\Gamma_0} \frac{\partial z_1}{\partial s}\frac{\partial z_2}{\partial s} ds + \int_{\Gamma_0} \frac{\partial z_1}{\partial v}\frac{\partial z_2}{\partial v} ds +$$

$$+ \int_{\Gamma_1} z_1 z_2 \, ds + \int_{\Gamma_1} \frac{\partial z_1}{\partial s}\frac{\partial z_2}{\partial s} ds + \int_{\Gamma_1} \frac{\partial z_1}{\partial v}\frac{\partial z_2}{\partial v} =$$

$$= \int_0^{4\pi} 1 \,.\, 2 \cos \frac{s}{2} ds + \int_0^{4\pi} 0 \,.\left(-\sin \frac{s}{2}\right) ds + \int_0^{4\pi} 0 \,.\, \cos \frac{s}{2} ds +$$

$$+ \int_0^{2\pi} 1 \,.\, \cos s \, ds + \int_0^{2\pi} 0 \,.\, (-\sin s) \, ds + \int_0^{2\pi} 0 \,.\, (-\cos s) \, ds = 0 \,,$$

because

$$z_2(x, y) = x = \begin{cases} 2 \cos \dfrac{s}{2} \text{ on } \Gamma_0 \,, \\ \cos s \text{ on } \Gamma_1 \,; \end{cases}$$

further

$$(z_1, r_{11})_\Gamma = \int_{\Gamma_0} \left(z_1 r_{11} + \frac{\partial z_1}{\partial s}\frac{\partial r_{11}}{\partial s} + \frac{\partial z_1}{\partial v}\frac{\partial r_{11}}{\partial v} \right) ds +$$

$$+ \int_{\Gamma_1} \left(z_1 r_{11} + \frac{\partial z_1}{\partial s}\frac{\partial r_{11}}{\partial s} + \frac{\partial z_1}{\partial v}\frac{\partial r_{11}}{\partial v} \right) ds =$$

$$= 0 + \int_0^{2\pi} (1 \,.\, 1 + 0 \,.\, 0 + 0 \,.\, 0) \, ds = 2\pi$$

because of (3.27) and (3.26), etc.

From Tab. 3.1 it is evident that the solution of the given system reduces to the solution of four very simple systems. From (4) it follows $b_{1111} = 0$, from (3), (7), (11) we get $a_{113} = 0$, $b_{1114} = 0$, $\alpha_{1113} = 0$; similarly we get $b_{1112} = 0$, $c_{111} = 1$, $\alpha_{1111} = \ln 4$, $a_{111} = 2 - \ln 4$, $a_{112} = -1$, $\alpha_{1112} = -1$, $b_{1113} = 0$.

Thus the result is

$$u_{11}(x, y) = \ln(x^2 + y^2) - x + 2 - \ln 4 + r_{11}(x, y)\ln 4 - r_{12}(x, y). \tag{3.28}$$

Hence the required approximation $U_{11}(x, y)$ of the Airy function is

$$U_{11}(x, y) = \ln(x^2 + y^2) - x + 2 - \ln 4.$$

In our case it represents the exact solution.

The reader can check immediately that the function (3.28) fulfils all the boundary conditions (3.22)–(3.25) (exactly, in our case).

The components of the corresponding stress-tensor are

$$\sigma_x = \frac{\partial^2 U_{11}}{\partial y^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2},$$

$$\sigma_y = \frac{\partial^2 U_{11}}{\partial x^2} = -\frac{2(x^2 - y^2)}{(x^2 + y^2)^2},$$

$$\tau_{xy} = \frac{4xy}{(x^2 + y^2)^2}.$$

If we introduce polar coordinates by

$$x = r\cos\omega, \quad y = r\sin\omega,$$

we get

$$\sigma_x = \frac{2}{r^2}(\cos^2\omega - \sin^2\omega) = \frac{2}{r^2}\cos 2\omega,$$

$$\sigma_y = -\sigma_x = -\frac{2}{r^2}\cos 2\omega,$$

$$\tau_{xy} = \frac{4}{r^2}\sin\omega\cos\omega = \frac{2}{r^2}\sin 2\omega.$$

Example 3.2. Let us investigate a rectangular wall-beam with a circular hole, loaded as shown in Fig. 4. The parameter s of the length of arc is chosen so that on Γ_0 we have $s = 0$ at the point $B(a, -b)$ and $0 \leqq s < 4a + 4b$, on Γ_1 we have $s = 0$ at the point $(r, 0)$ and $0 \leqq s < 2\pi r$. The orientation of the curves Γ_0, Γ_1 is evident from Fig. 4.

First, let us determine the functions g_0, g_1.

On Γ_1, we have (taking the orientation of this curve into account)

$$x = r\cos\frac{s}{r}, \quad y = -r\sin\frac{s}{r}, \quad v_x = -\cos\frac{s}{r}, \quad v_y = \sin\frac{s}{r}.$$

Further,

$$X(s) = p\cos\frac{s}{r}, \quad Y(s) = -p\sin\frac{s}{r},$$

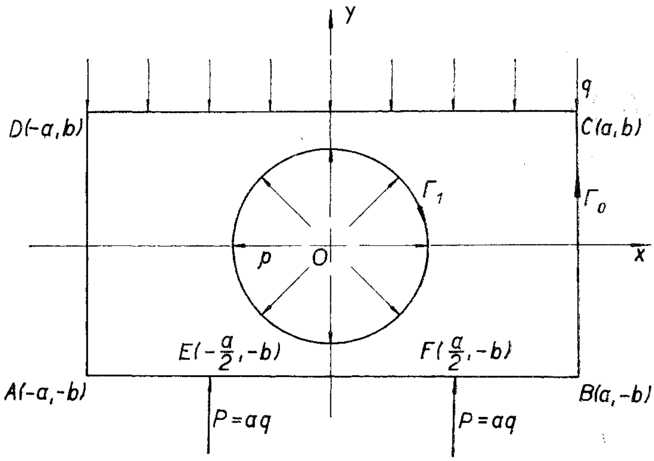

Fig. 4.

so that according to (1.7),

$$\frac{\partial u}{\partial x} = -\int_0^s Y(t)\,dt = pr\left(1 - \cos\frac{s}{r}\right),$$

$$\frac{\partial u}{\partial y} = \int_0^s X(t)\,dt = pr\sin\frac{s}{r}$$

and by (1.8), (1.9),

(3.29) $$u = g_{10} = pr^2\left(\cos\frac{s}{r} - 1\right) = pr(x - r),$$

$$\frac{\partial u}{\partial v} = g_{11} = pr\left(1 - \cos\frac{s}{r}\right) = p(r - x).$$

On Γ_0 we have first $X(s) = 0$ so that

$$\frac{\partial u}{\partial y} \equiv 0.$$

Further by (1.7),

$$\frac{\partial u}{\partial x} = -\int_0^s Y(t)\,\mathrm{d}t = 0 \quad \text{for} \quad 0 \leqq s < 2b\,, \quad \text{i.e. on } BC\,,$$

$$\begin{aligned}
&q(s-2b) && \text{for} \quad 2b \leqq s < 2b + 2a\,, && \text{i.e. on } CD\,,\\
&2qa && \text{for} \quad 2a + 2b \leqq s < 2a + 4b\,, && \text{i.e. on } DA\,,\\
&2qa && \text{for} \quad 4b + 2a \leqq s < 4b + \tfrac{5}{2}a\,, && \text{i.e. on } AE\,,\\
&qa && \text{for} \quad 4b + \tfrac{5}{2}a \leqq s < 4b + \tfrac{7}{2}a\,, && \text{i.e. on } EF\,,\\
&0 && \text{for} \quad 4b + \tfrac{7}{2}a \leqq s < 4b + 4a\,, && \text{i.e. on } FB\,.
\end{aligned}$$

Using (1.8), (1.9), we get

(3.30)
$$\begin{aligned}
u = g_{00} &= 0 && \text{on } BC\,,\\
&= -\frac{q}{2}(s-2b)^2 && \text{on } CD\,,\\
&= -2qa^2 && \text{on } DA\,,\\
&= -2qa^2 + 2qa(s - 2a - 4b) && \text{on } AE\,,\\
&= -\ qa^2 + \ qa(s - 4b - \tfrac{5}{2}a) && \text{on } EF\,,\\
&= 0 && \text{on } FB\,,
\end{aligned}$$

(3.31)
$$\begin{aligned}
\frac{\partial u}{\partial v} = g_{01} &= 0 && \text{on } BC\,,\\
&= 0 && \text{on } CD\,,\\
&= -2qa && \text{on } DA\,,\\
&= 0 && \text{on } AB\,.
\end{aligned}$$

In our numerical example, let us choose

$$a = 2\,, \quad b = 2\,, \quad r = 1\,, \quad p = 1\,, \quad q = 2\,.$$

Thus we solve the problem

(3.32)
$$\Delta^2 u = 0 \quad \text{in} \quad G\,,$$

(3.33)
$$\begin{aligned}
u = g_{00} &= 0 && \text{on } BC\,,\\
&= -(2-x)^2 && \text{on } CD\,,\\
&= -16 && \text{on } DA\,,\\
&= 8x && \text{on } AE\,,\\
&= -4 + 4x && \text{on } EF\,,\\
&= 0 && \text{on } FB \quad \text{on} \quad \Gamma_0\,,
\end{aligned}$$

$$(3.34) \qquad \frac{\partial u}{\partial v} = g_{01} = \begin{cases} 0 & \text{on } BC\,, \\ 0 & \text{on } CD\,, \\ -8 & \text{on } DA\,, \\ 0 & \text{on } AB \end{cases} \quad \text{on } \Gamma_0\,,$$

$$(3.35) \qquad u = g_{10} = x - 1\,, \qquad \text{on } \Gamma_1\,,$$

$$(3.36) \qquad \frac{\partial u}{\partial v} = g_{11} = 1 - x\,, \qquad \text{on } \Gamma_1\,.$$

Evidently, the loading on both the curves Γ_0 and Γ_1 is in the static and moment equilibriums and $(g_0, g_1) \in W_2^{(1)}(\Gamma) \times L_2(\Gamma)$.

To solve approximately this problem by the method of least squares on the boundary, put in (3.9) $s = 3$, $t = 1$. Choosing $x_1 = y_1 = 0$, we get

$$(3.37) \qquad u_{31}(x, y) = a_{311}\,.\,1 + a_{312}x + a_{313}y + a_{314}(x^2 - y^2) + \\ + a_{315}\,.\,2xy + a_{316}(-y^2) + a_{317}(x^3 - 3xy^2) + a_{318}(3x^2y - y^3) + \\ + a_{319}(-3xy^2) + a_{3110}(-y^3) + \\ + b_{3111}\frac{x^2 - y^2}{x^2 + y^2} + b_{3112}\frac{-2xy}{x^2 + y^2} + b_{3113}\frac{x}{x^2 + y^2} + \\ + b_{3114}\frac{-y}{x^2 + y^2} + c_{3111}\ln(x^2 + y^2) + \\ + \alpha_{3111}r_{11}(x, y) + \alpha_{3112}r_2(x, y) + \alpha_{3113}r_{13}(x, y)\,.$$

The system (3.20) for the unknowns $a_{311}, \ldots, \alpha_{3112}$ is given in Tab. 3.2. We have for example

$$(z_5, z_4)_\Gamma = \int_{\Gamma_0}\left(z_5z_4 + \frac{\partial z_5}{\partial s}\frac{\partial z_4}{\partial s} + \frac{\partial z_5}{\partial v}\frac{\partial z_4}{\partial v}\right)\mathrm{d}s + \\ + \int_{\Gamma_1}\left(z_5z_4 + \frac{\partial z_5}{\partial s}\frac{\partial z_4}{\partial s} + \frac{\partial z_5}{\partial v}\frac{\partial z_4}{\partial v}\right)\mathrm{d}s\,,$$

$$\int_{\Gamma_0} z_5z_4\,\mathrm{d}s = \int_{-2}^{2} 4y(4 - y^2)\,\mathrm{d}y + \int_{-2}^{2} 4x(x^2 - 4)\,\mathrm{d}x + \int_{-2}^{2}(-4y)(4 - y^2)\,\mathrm{d}y + \\ + \int_{-2}^{2}(-4x)(x^2 - 4)\,\mathrm{d}x = 0\,,$$

Tab. 3.2

	a_{311}	a_{312}	a_{313}	a_{314}	a_{315}	a_{316}	a_{317}	a_{318}	a_{319}
(1)	22·2832	0	0	0	0	−45·8083	0	0	0
(2)	0	68·0914	0	0	0	0	−102·4000	0	−395·7810
(3)	0	0	68·0914	0	0	0	0	102·4000	0
(4)	0	0	0	506·1410	0	274·3638	0	0	0
(5)	0	0	0	0	710·9410	0	0	0	0
(6)	−45·8083	0	0	274·3638	0	339·1892	0	0	0
(7)	0	−102·4000	0	0	0	0	5764·8331	0	6291·1677
(8)	0	0	102·4000	0	0	0	0	5764·8331	0
(9)	0	−395·7810	0	0	0	0	6291·1677	0	6965·0143
(10)	0	0	−293·3810	0	0	0	0	391·3384	0
(11)	0	0	0	71·6387	0	41·3171	0	0	0
(12)	0	0	0	0	−178·3806	0	0	0	0
(13)	0	11·1416	0	0	0	0	78·5251	0	−21·7501
(14)	0	0	−11·1416	0	0	0	0	−5·4750	0
(15)	26·4038	0	0	0	0	−181·0217	0	0	0
(16)	0	9·4248	0	0	0	0	0	0	−11·7810
(17)	0	0	9·4248	0	0	0	0	0	0
(18)	6·2832	0	0	0	0	0	0	0	0

$$\int_{\Gamma_0} \frac{\partial z_5}{\partial s} \frac{\partial z_4}{\partial s} \, \mathrm{d}s = \int_{-2}^{2} \frac{\partial z_5}{\partial y}(2, y) \frac{\partial z_4}{\partial y}(2, y) \, \mathrm{d}y + \int_{-2}^{2} \left[-\frac{\partial z_5}{\partial x}(x, 2) \right] \left[-\frac{\partial z_4}{\partial x}(x, 2) \right] \mathrm{d}x +$$
$$+ \int_{-2}^{2} \left[-\frac{\partial z_5}{\partial y}(-2, y) \right] \left[-\frac{\partial z_4}{\partial y}(-2, y) \right] \mathrm{d}y +$$
$$+ \int_{-2}^{2} \frac{\partial z_5}{\partial x}(x, -2) \frac{\partial z_4}{\partial x}(x, -2) \, \mathrm{d}x = \int_{-2}^{2} 4 \, . \, (-2y) \, \mathrm{d}y + \int_{-2}^{2} (-4) \, . \, (-2x) \, \mathrm{d}x +$$
$$+ \int_{-2}^{2} 4 \, . \, 2y \, \mathrm{d}y + \int_{-2}^{2} (-4) \, . \, 2x \, \mathrm{d}x = 0 \, ,$$

$$\int_{\Gamma_0} \frac{\partial z_5}{\partial v} \frac{\partial z_4}{\partial v} \, \mathrm{d}s = \int_{-2}^{2} \frac{\partial z_5}{\partial x}(2, y) \frac{\partial z_4}{\partial x}(2, y) \, \mathrm{d}y + \int_{-2}^{2} \frac{\partial z_5}{\partial y}(x, 2) \frac{\partial z_4}{\partial y}(x, 2) \, \mathrm{d}x +$$
$$+ \int_{-2}^{2} \left[-\frac{\partial z_5}{\partial x}(-2, y) \right] \left[-\frac{\partial z_4}{\partial x}(-2, y) \right] \mathrm{d}y + \int_{-2}^{2} \left[-\frac{\partial z_5}{\partial y}(x, -2) \right] .$$
$$. \left[-\frac{\partial z_4}{\partial y}(x, -2) \right] \mathrm{d}x = \int_{-2}^{2} 2y \, . \, 4\mathrm{d}y + \int_{-2}^{2} 2x \, . \, (-4) \, \mathrm{d}x + \int_{-2}^{2} (-2y) \, . \, 4\mathrm{d}y +$$
$$+ \int_{-2}^{2} (-2x) \, . \, (-4) \, \mathrm{d}x = 0 \, ,$$

a_{3110}	b_{3111}	b_{3112}	b_{3113}	b_{3114}	c_{311}	α_{3111}	α_{3112}	α_{3113}	
0	0	0	0	0	26·4038	0	0	6·2832	−111·6165
0	0	0	11·1416	0	0	9·4248	0	0	244·0914
−293·3810	0	0	0	−11·1416	0	0	9·4248	0	−2·6667
0	71·6387	0	0	0	0	0	0	0	−245·4667
0	0	−178·3806	0	0	0	0	0	0	0
0	41·3171	0	0	0	−181·0217	0	0	0	260·0914
0	0	0	78·5251	0	0	0	0	0	−409·6000
391·3384	0	0	0	−5·4750	0	0	0	0	17·8667
0	0	0	−21·7501	0	0	−11·7810	0	0	−1547·7810
1990·7120	0	0	0	11·5627	0	0	−11·7810	0	10·6667
0	28·8584	0	0	0	0	0	0	0	−37·6865
0	0	29·0295	0	0	0	0	0	0	0
0	0	0	11·8883	0	0	3·1416	0	0	35·1416
11·5627	0	0	0	11·8883	0	0	−3·1416	0	1·3982
0	0	0	0	0	47·1715	0	0	0	−218·8012
0	0	0	3·1416	0	0	9·4248	0	0	9·4248
−11·7810	0	0	0	−3·1416	0	0	9·4248	0	0
0	0	0	0	0	0	0	0	6·2832	−6·2832

$$\int_{\Gamma_1} z_5 z_4 \, \mathrm{d}s = \int_0^{2\pi} (-2 \cos s \sin s)(\cos^2 s - \sin^2 s) \, \mathrm{d}s = 0 \,,$$

$$\int_{\Gamma_1} \frac{\partial z_5}{\partial s} \frac{\partial z_4}{\partial s} \, \mathrm{d}s = \int_{\Gamma_1} \left(- \frac{\partial z_5}{\partial x} v_y + \frac{\partial z_5}{\partial y} v_x \right) \left(- \frac{\partial z_4}{\partial x} v_y + \frac{\partial z_4}{\partial y} v_x \right) \mathrm{d}s =$$

$$= 8 \int_0^{2\pi} (-\sin^3 s \cos s + \sin s \cos^3 s) \, \mathrm{d}s = 0 \,,$$

$$\int_{\Gamma_1} \frac{\partial z_5}{\partial v} \frac{\partial z_4}{\partial v} \, \mathrm{d}s = \int_{\Gamma_1} \left(\frac{\partial z_5}{\partial x} v_x + \frac{\partial z_5}{\partial y} v_y \right) \left(\frac{\partial z_4}{\partial x} v_x + \frac{\partial z_4}{\partial y} v_y \right) \mathrm{d}s =$$

$$= 8 \int_0^{2\pi} (-\cos^3 s \sin s + \cos s \sin^3 s) \, \mathrm{d}s = 0$$

so that

$$(z_5, z_4)_\Gamma = 0 \,.$$

(A little routine yields this result more rapidly, if the simple form of the functions $z_4 = x^2 - y^2$, $z_5 = 2xy$ is considered.)

The matrix of the system is obviously symmetric. Similarly as in Example 3.1, the system "degenerates" into simpler systems containing no more than 6 unknowns in our case. The result is

$$u_{31} = -4{\cdot}3600 + 4{\cdot}000x - 0{\cdot}0732y -$$
$$- 0{\cdot}6745(x^2 - y^2) + 0{\cdot}0000 \,.\, 2xy + 0{\cdot}4046 \,.\, (-y^2) + 0{\cdot}0000(x^3 - 3xy^2) +$$
$$+ 0{\cdot}0049(3x^2y - y^3) - 0{\cdot}0000 \,.\, (-3xy^2) - 0{\cdot}0063 \,.\, (-y^3) -$$
$$- 0{\cdot}2108 \,.\, \frac{x^2 - y^2}{x^2 + y^2} + 0{\cdot}0000 \,.\, \frac{-2xy}{x^2 + y^2} - 0{\cdot}0000 \,.\, \frac{x}{x^2 + y^2} +$$
$$+ 0{\cdot}0820 \,.\, \frac{-y}{x^2 + y^2} - 0{\cdot}6455 \ln (x^2 + y^2) - 3{\cdot}0000 r_{11}(x, y) +$$
$$+ 0{\cdot}0926 r_{12}(x, y) + 3{\cdot}3599 r_{13}(x, y)$$

so that the required Airy function is

$$U_{31} = - 4{\cdot}3600 + 4{\cdot}000x - 0{\cdot}0732y -$$
$$- 0{\cdot}6745x^2 + 0{\cdot}2699y^2 + 0{\cdot}0147x^2y + 0{\cdot}0014y^3 -$$
$$- 0{\cdot}2108 \,.\, \frac{x^2 - y^2}{x^2 + y^2} - 0{\cdot}0820 \,.\, \frac{y}{x^2 + y^2} - 0{\cdot}6455 \ln (x^2 + y^2) \,.$$

In figs. 5 – 7, the components

$$\sigma_x = \frac{\partial^2 U_{31}}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U_{31}}{\partial x^2}, \quad \tau_{xy} = - \frac{\partial^2 U_{31}}{\partial x \, \partial y}$$

of the (approximate) stress-tensor are sketched in the cross-section $y = -1{\cdot}5$, $-2 \leqq \leqq x \leqq 2$.

In the following example we show how to proceed if the requirement of equilibrium of the loading on each of the boundary curves separately is not fulfilled.

Example 3.3. For illustration, let us consider the same annulus G as in Ex. 3.1, with its centre at the origin and with the inner and outer radius $r_1 = 1$, $r_0 = 2$, respectively, loaded as shown in Fig. 8. The orientation of the boundary curves Γ_0, Γ_1 as well as the choice of the points $A \in \Gamma_0$, $B \in \Gamma_1$ with $s = 0$ is also obvious from the figure.

We have

$$x = 2 \cos \frac{s}{2}, \quad y = 2 \sin \frac{s}{2}, \quad v_x = \cos \frac{s}{2}, \quad v_y = \sin \frac{s}{2} \text{ on } \Gamma_0 \,,$$

$$x = \cos s \,, \quad y = -\sin s \,, \quad v_x = -\cos s \,, \quad v_y = \sin s \text{ on } \Gamma_1 \,.$$

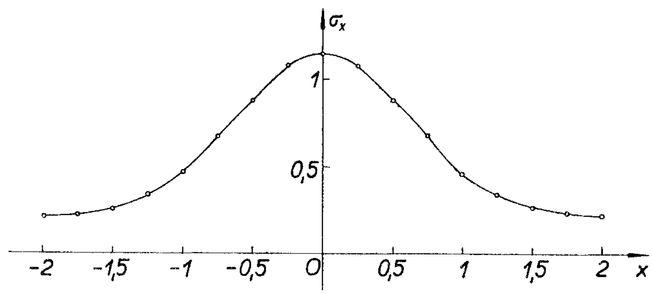

Fig. 5.

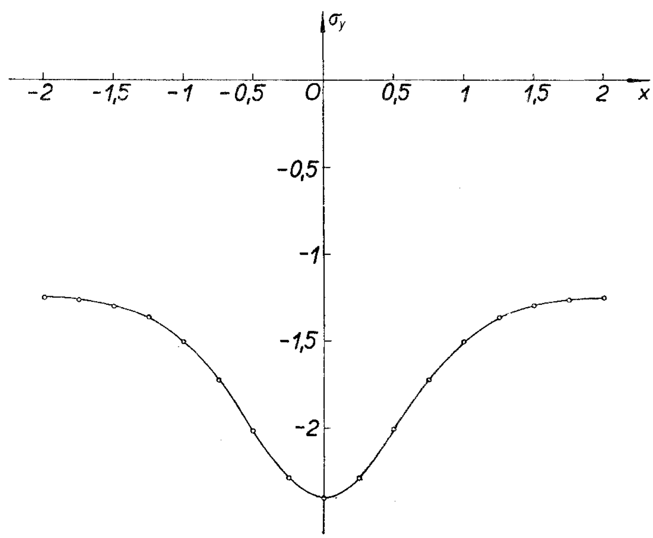

Fig. 6.

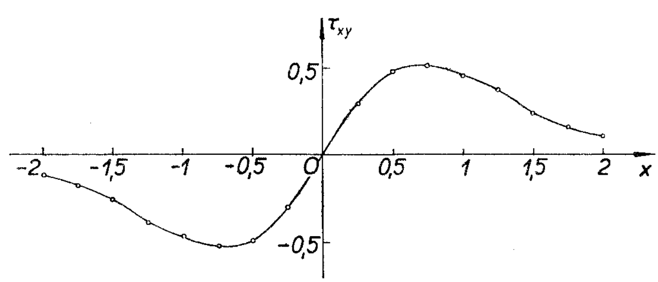

Fig. 7.

A simple computation yields

$$\text{on } \Gamma_0\colon \quad \frac{\partial u}{\partial x} = -\int_0^s Y(t)\,\mathrm{d}t = \begin{cases} 0 & \text{for} \quad 0 \leqq s \leqq 3\pi\,, \\ -P & \text{for} \quad 3\pi < s < 4\pi\,, \end{cases}$$

$$\frac{\partial u}{\partial y} = 0\,,$$

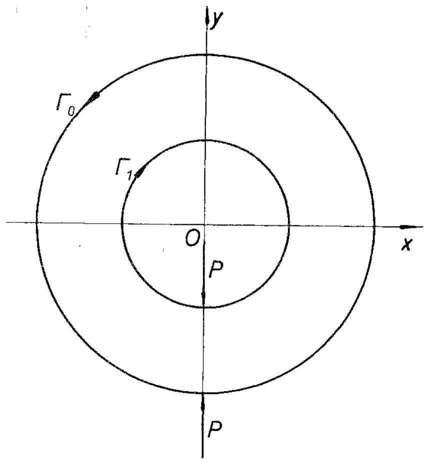

Fig. 8.

$$(3.38) \qquad \frac{\partial u}{\partial v} = \begin{cases} 0 & \text{for} \quad 0 \leqq s \leqq 3\pi\,, \\ -P\cos\frac{s}{2} & \text{for} \quad 3\pi < s < 4\pi\,; \end{cases}$$

$$u = \begin{cases} 0 & \text{for} \quad 0 \leqq s \leqq 3\pi\,, \\ -2P\cos\frac{s}{2} & \text{for} \quad 3\pi < s < 4\pi\,; \end{cases}$$

Similarly, on Γ_1 we get

$$(3.39) \qquad \frac{\partial u}{\partial v} = \begin{cases} 0 & \text{for} \quad 0 \leqq s \leqq \frac{\pi}{2}\,, \\ -P\cos s & \text{for} \quad \frac{\pi}{2} < s < 2\pi\,, \end{cases}$$

$$u = \begin{cases} 0 & \text{for} \quad 0 \leqq s \leqq \frac{\pi}{2}\,, \\ P\cos s & \text{for} \quad \frac{\pi}{2} < s < 2\pi\,. \end{cases}$$

The loading on Γ_0 as well as on Γ_1 is not in a static equilibrium (as evident from Fig.8). It follows that, for example, the function $u(s)$ is continuous neither at the point A on Γ_0 nor at the point B on Γ_1. From (3.38), (3.39) we easily compute

$$\lim_{s\to 4\pi-} u(s) - u(0) = -2P \text{ on } \Gamma_0, \tag{3.40}$$

$$\lim_{s\to 2\pi-} u(s) - u(0) = P \quad \text{on} \quad \Gamma_1 .$$

Similar relations can be derived for $\partial u/\partial v$.

To be able to apply the method of least squares on the boundary in its original form, we can use the well-known result from [4], Sec. 2.10:[11]) The functions

$$\varphi(z) = -\frac{X + iY}{2\pi(1+\varkappa)} \ln(z - z_1), \tag{3.41}$$

$$\psi(z) = \frac{\varkappa(X - iY)}{2\pi(1+\varkappa)} \ln(z - z_1)$$

produce a state of stress in G (with a single-valued displacement) with the main vector (X, Y) and a zero moment on Γ_1. Here z_1 is an arbitrary point lying in the interior of Γ_1, $\varkappa = (\lambda + 3\mu)/(\lambda + \mu)$, where λ and μ are the Lamé constants. In our example we can choose for example $z_1 = 0$ and then put (see Fig. 8)

$$X = 0, \quad Y = -P$$

so that

$$\varphi(z) = \frac{iP}{2\pi(1+\varkappa)} \ln z, \quad \psi(z) = \frac{i\varkappa P}{2\pi(1+\varkappa)} \ln z .$$

To get the Airy function corresponding to these functions by

$$u = \operatorname{Re}(\bar{z}\varphi + \chi) \tag{3.42}$$

we compute

$$\chi(z) = \frac{i\varkappa P}{2\pi(1+\varkappa)} z(\ln z - 1)$$

and writing $\ln z = \ln r + i\omega$, we get by (3.42)

$$u = \frac{1}{2\pi(1+\varkappa)} \operatorname{Re}\left[\bar{z}\, iP \ln z + i\varkappa P z(\ln z - 1)\right] = \tag{3.43}$$

$$= \frac{P}{2\pi(1+\varkappa)} \left[-(1+\varkappa)\, x\omega + \varkappa y + (1-\varkappa)\, y \ln r\right].$$

[11]) Actually we try to find a "particular solution" which will produce the same static inequilibrium as is the given one, not changing the given moment which is equal to zero in our case.

From (3.43) it follows easily – by putting $x = 2$, $y = 0$ or $x = 1$, $y = 0$ (the coordinates of the points A or B), respectively, and taking into account that ω increases or decreases by 2π or -2π when we run along Γ_0 or Γ_1 in the positive sense of its orientation – that the function (3.43) fulfils (3.40). Similarly, it is possible to verify analogous relations for $\partial u/\partial v$. Thus we can take the function (3.43) as a "particular solution" of our problem and perform the algorithm of our least squares method with the function

$$u_{st}(x, y) = u_{\text{part.}}(x, y) + U_{st}(x, y) + \sum_{j=1}^{3} \alpha_{st1j} r_{1j}(x, y),$$

where $u_{\text{part.}}(x, y)$ is the function (3.43) and $U_{st}(x, y)$ and $r_{1j}(x, y)$ are the functions (3.10) and the singular functions introduced above, respectively. Note that the values of the function (3.43) on Γ_0 as well as on Γ_1 can be easily computed which is of importance for further numerical calculations.

From Ex. 3.3 it is easy to see how to proceed in a general case. The idea is to "remove" the inequilibrium on the inner boundary curves by finding a proper "particular solution". Most frequently, it is convenient to use functions of the type (3.41), with suitably chosen points z_i.[12]) In other problems it may appear that another type of a particular solution is more convenient.

In the next issue of this journal, Part II of the present paper will appear, containing proofs of Theorems 3.1 and 3.2 from p. 379 as well as of Lemma 2.4, p. 369.

References

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[12]) Note that we could have chosen the point z_1 in another way than we did.