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SOLUTION OF THE HALL FIELD BOUNDARY VALUE PROBLEM BY FOURIER SERIES

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1. INTRODUCTION

The two-dimensional current density field when respecting the Hall effect is characterized by an isogonal system of equipotential and flow lines (i.e. not by an orthogonal one, as in classical problems on the current density field). Thus, owing to the Hall effect, the solution of the current density field, brought by technical applications, leads to an unusual boundary value problem.

A way has been already shown of solving this problem by means of conformal mapping [1, 2, 3]. In the present paper, a way is shown how to solve this problem by developments into Fourier series. The problem will be explained on the case of the field in a semiinfinite strip (semi-slab), its frontal side being represented as an equipotential plane. In this case, the problem can be reduced to very simple relatilons for the coefficients of the Fourier series.

New methods of complex variable functions are used for the calculation of the Fourier coefficients. In comparison with conventional calculations of the Fourier analysis, our treatment is less toilsome, and we shall show its further advantages in future papers when dealing with regions for which the calculation by the conformal mapping would be substantially more complicated.

2. BASIC EQUATIONS

Consider a two-dimensional current density field corresponding to a semiconducting plate in a uniform magnetic field (Fig. 1). We assume that the material of the plate is homogeneous and isotropic. We shall employ the following symbols:

$\boldsymbol{J}$	current density	R	Hall constant
$\boldsymbol{E}$	electric field intensity	$\boldsymbol{B}$	magnetic field intensity
σ	conductivity (for the zero magnetic field)	d	thickness of the plate
		I	current along the plate

Θ Hall angle
μ mobility
φ electric field potential
b width of the semi-strip
$\boldsymbol{k}$ unit vector in z-direction

The presence of the Hall effect causes that the vectors of the current density $\boldsymbol{J}$ and the electric field intensity $\boldsymbol{E}$ are not parallel but make the angle Θ (Fig. 1) obeying the relations

$$\boldsymbol{E} = \frac{1}{\sigma}\boldsymbol{J} + R(\boldsymbol{J} \times \boldsymbol{B})$$

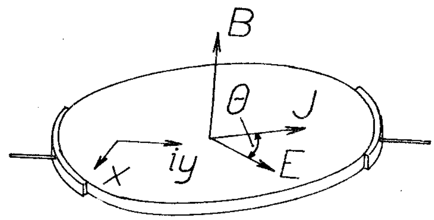

Fig. 1. Geometry of basic quantities.

(1) $$\sigma\boldsymbol{E} = \boldsymbol{J} + \operatorname{tg}\Theta(\boldsymbol{J} \times \boldsymbol{k})$$

(2) $$\operatorname{tg}\Theta = \sigma R|\boldsymbol{B}| = \mu|\boldsymbol{B}|$$

Of course, Maxwell equations

$$\operatorname{curl}\boldsymbol{E} = 0\,; \quad \operatorname{div}\boldsymbol{J} = 0$$

must be satisfied; therefore, the electric field has a scalar potential and a vector potential may be defined for the current density field. The two-dimensionality of the problem enables us to use the complex variable $z = x + iy$ and to employ properties of the complex potential defined by a regular function [4, 5]

$$F(z) = U(x, y) + iV(x, y)\,.$$

Two functions $U(x, y)$ and $V(x, y)$ forming a regular function are harmonic conjugate. They fulfil the Cauchy-Riemann conditions and both satisfy the Laplace equation for two variables. The imaginary part of the function $F(z)$, i.e. $V(x, y)$ will be named "fictitious potential" defined by the relation

(3) $$\boldsymbol{J} = -\operatorname{grad} V\,.$$

The lines of the constant fictitious potential are thus perpendicular to the vector $\boldsymbol{J}$.

The electric potential φ is defined by the relation

(4) $$\boldsymbol{E} = -\operatorname{grad}\varphi\,.$$

If we neglected the Hall effect, the electric and fictitious potential would be identical.

Taking the Hall effect into account, the equations (3) or the equations (5), (4) and (1) yield the relations

$$(5)\qquad \boldsymbol{J} = J_x + iJ_y = \sigma\left(-\frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y}\right) = -i\sigma\overline{F'(z)}$$

$$\sigma\mathbf{E} = -\sigma \operatorname{grad}\varphi = \boldsymbol{J} + \operatorname{tg}\Theta(\boldsymbol{J}\times\boldsymbol{k}) =$$
$$= (1 - i\operatorname{tg}\Theta)(J_x + iJ_y) = -i\sigma\overline{[(1 + i\operatorname{tg}\Theta)F'(z)]}\,.$$

(The bar over a complex number means the operation of complex conjugation.) Thus, whilst the fictitious potential is

$$V(x, y) = \operatorname{Im}\{F(z)\}\,,$$

the electric potential is given by the relation

$$\varphi(x, y) = \operatorname{Im}\{(1 + i\operatorname{tg}\Theta)F(z)\} =$$
$$= \operatorname{tg}\Theta U(x, y) + V(x, y)\,.$$

The function

$$U(x, y) = \operatorname{Re}\{F(z)\}$$

is identical with the so-called vector potential of the current density field. Using it we may express the current density by the relation

$$\boldsymbol{J} = \sigma \operatorname{curl} U = \sigma\left(\frac{\partial U}{\partial y} - i\frac{\partial U}{\partial x}\right).$$

All these quantities are illustrated in Fig. 2. A section of the current density field (with two marked points, A and B) is shown schematically; the flow lines are repre-

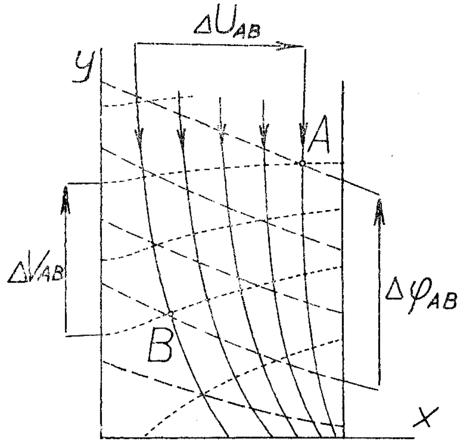

Fig. 2. Scheme of the current density field near to the front of the semi-strip. The dashed curves represent the electric equipotentials and the dotted curves corresponding to constant values of the fictitious potential.

sented by full curves provided by arrows; the dashed lines and the dotted lines represent the electric and fictitious equipotentials, respectively. The quantities illustrated in Fig. 2 have the following meaning:

$\Delta\varphi_{AB}$ is the potential difference between the points A and B;

$\sigma\Delta U_{AB}$ is the current related to the unit height of the plate, passing between the points A and B. It its value is positive, the current flows from the left to the right side passing through the hypothetical line oriented from B to A;

ΔV_{AB} is the difference of the fictitious potential which means a non-measurable voltage drop corresponding to the resulting current density field.

3. FORMULATION AND SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR THE SEMI-STRIP

Consider the semi-strip oriented as shown in Fig. 3a. Let us assume, first, that its width is equal to π (later this assumption will be omitted). For such a semi-strip, it is convenient to write

$$F(z) = c_0 z + c + \sum_{n=1}^{\infty} c_n e^{inz}\,.$$

Our task consists in expressing the coefficients c_n in dependence on the current I and the angle Θ. The function $F(t)$ has a part which – if continued out of the semi-strip – is periodic with the period 2π. We shall bear in mind the analytical continuation of $F(z)$ over the whole upper complex half-plane assuming that a semiconducting plate (of the same material as the original plate with the finite width) extends over the whole half-plane. The boundary ($y = 0$) of this plate is one electrode, the second electrode being at infinity ($y \to \infty$). To provide physical grounds for the analytical

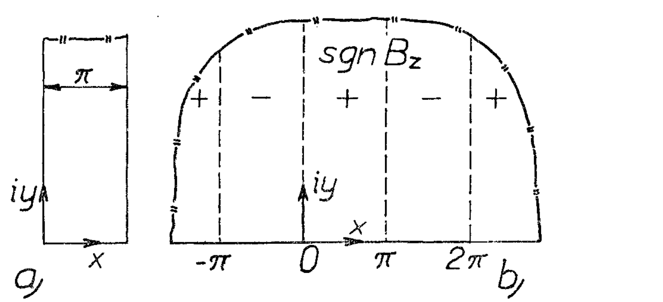

Fig. 3. Construction of the analytical continuation of the solution on the half-plane.

continuation, we assume that the plate is situated in a perpendicular magnetic field of constant magnitude which, nevertheless, changes its sign in a semi-strips of width π as shown in Fig. 3b.

In such a case, both potentials $U(x, y)$ and $V(x, y)$ satisfy the Laplace equation. The electric potential $\varphi(x, y)$ is then also a uniquely defined and continuous function, but it cannot be expressed in the whole complex half-plane by a single harmonic function. On the boundary where the magnetic field changes its sign, the vector **E** possesses a non-zero surface divergence. Introducing Dirichlet's factor sgn (B_z), we may express the electric potential, for instance, by the integral

(8)

$$\varphi(z) = \varphi(z_0) - \int_{z_0}^{z} \mathbf{E} \, . \, \mathrm{d}\mathbf{s} = \varphi(z_0) - \int_{z_0}^{z} \operatorname{Im} \{[(1 + i \operatorname{sgn}(B_z) \operatorname{tg} \Theta) F'(z)] \, \mathrm{d}z\} \, .$$

The required solution is interesting only in the section of the complex half-plane, namely in the semi-strip with the same direction of the magnetic field. In view of the physical nature of the problem, the solution exists and is unique. It must fulfil the condition

$$U(\pm n\pi, y) = \operatorname{Re} \{F(\pm n\pi, y)\} = \pm \frac{nI}{d\sigma}$$

demanding that the parallel lines $y = \pm n\pi$ (where n is integer) represent the current lines, and the total current flowing out of one semi-strip equals I.

The second boundary value condition demands the boundary of the half-plane to be equipotential. The integral (6) taken along the real axis must be thus identically zero:

$$\varphi(x, 0) - \varphi(x_0, 0) = \int_{x_0}^{x} \operatorname{Im} \{[1 + i \operatorname{sgn}(B_z) \operatorname{tg} \Theta] \, F'(z)\} \, \mathrm{d}x = 0 \, .$$

Therefore, the integrand must be zero:

$$\operatorname{Im} \{[1 + i \operatorname{sgn}(B_z) \operatorname{tg} \Theta] \, F'(z)\}_{y=0} = V'(x, 0) + \operatorname{sgn}(B_z) \operatorname{tg} \Theta \, U'(x, 0) = 0$$

which means that

$$\frac{V'(x, 0)}{U'(x, 0)} = -\operatorname{tg} \Theta \operatorname{sgn}(B_z) \, .$$

However, as it is well known, the quantity

$$\operatorname{arctg} \frac{V'(x, 0)}{U'(x, 0)} = [\arg F'(z)]_{y=0}$$

represents the argument of the derivative withr espect to the complex variable. With the aid of this relation, we may write the boundary value condition for the real axis in the form

$$[\arg F'(z)]_{y=0} = -\Theta \operatorname{sgn}(B_z) \, .$$

Thus, the argument of $F'(z)$ changes by jumps at those points of the boundary of the complex half-plane where the magnetic field changes its sign, and its is constant along the segments between these points.

This condition can be expressed within the framenwork of the theory of the conformal mapping. The function $F(z)$ is to mediate the mapping of the upper half-plane onto the region whose boundary is given by the piece-wise linear line in Fig. 4a,b. The points 0, $\pm 2n\pi$ are mapped onto the points of refraction with the vertex angle

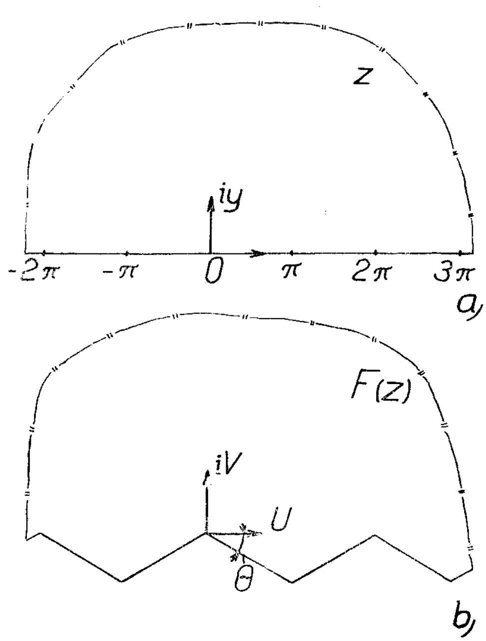

Fig. 4. Formulation of the boundary value condition by conformal mapping.

$\pi + 2\Theta$, and the points and the points $\pm(2n - 1)\pi$ onto the remaining points of with the vertex angle $\pi - 2\Theta$. As it is proved in the theory of complex variable [4, 5] in order that the function $F'(z)$ have the above mentioned property, it is necessary for it to have poles or zeros at the points of refraction. The function $\ln F'(z)$ may be characterized at these points by the so-called logarithmic residuum, i.e. the residuum of the function

$$[\ln F'(z)]' = \frac{F''(z)}{F'(z)}$$

which possesses only poles of the first order at the points of refraction. The value of the logarithmic residuum q is connected with the refraction angle α measured on the internal side of the region in question by the relation

$$q = \frac{\alpha - \pi}{\pi}.$$

If we value of the logarithmic residuum is positive (negative), the order of the corresponding zero point (pole) of the function $F'(z)$ is determined by it.

In our case, the function $F'(z)$ has zero points at the points 0, $\pm 2n\pi$ of the order

(7) $$q = \frac{2\Theta}{\pi}$$

whilst it has poles of the same order at the points $(2n - 1)\pi$. Except the points just consistered, the function $F'(z)$ has no other singular points, as it is regular at the infinity.

Let us introduce a new variable t by the relation

(8) $$t = e^{iz} .$$

Thus, the upper complex half-plane is mapped into a Riemann surface with infinitely many sheets within the unit circle (Fig. 5). One sheet of the Riemann surface corresponds to the vertical semi-strip of the width 2π and the circumference of the unit circle corresponds to the real axis. The points 0 and 2π are mapped into the point 1 of the plane t, the point π into the point -1.

It folows from the definition of the logarithmic residuum that by the conformal mapping its value does not change at the points where the mapping is regular. Therefore, the function

(9) $$G'(t) = F'(-i \ln t)$$

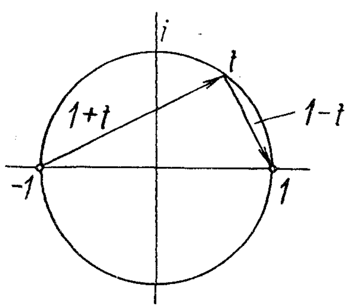

Fig. 5. Transformation of the half-plane from Fig. 4a by the mapping function $t = e^{iz}$.

has the same logarithmic residue as $F'(z)$ at the corresponding points. The function $\ln G'(t)$ must have first-order poles at the points 1 and -1 with residua q and $-q$, respectively.

This is fulfilled by the function

$$\ln G'(t) = q \ln (1 - t) - q \ln (1 + t) + \ln c_0 = \ln c_0 \left(\frac{1 - t}{1 + t}\right)^q =$$

$$= \ln \left| c_0 \left(\frac{1 - t}{1 + t}\right)^q \right| + i \arg \left[c_0 \left(\frac{1 - t}{1 + t}\right)^q \right]$$

where q is related with Θ by (7). Therefore

$$G'(t) = c_0 \left(\frac{1 - t}{1 + t}\right)^q . \tag{10}$$

In Fig. 5, the numerator and denominator are given in the complex t-plane. We may find out that the function $G'(t)$ possesses the argument $-\Theta$ on the upper half-circle and the argument Θ on the lower half-circle. The constant c_0 in the expression (10) is therefore real.

Substituting into (9) (10) from relation (8) we obtain

$$F'(z) = c_0 \operatorname{tg}^q \frac{z}{2} .$$

The function $G'(t)$ may be developed into a series with respect to t in the interior of the unit circle. Both the numerator and the denominator can be developed according to the binomial theorem:

$$(1 - t)^q = 1 - \binom{q}{1} t + \binom{q}{2} t^2 - \binom{q}{3} t^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n \binom{q}{n} t^n ,$$

$$(1 + t)^{-q} = 1 + \binom{-q}{1} t + \binom{-q}{2} t^2 + \binom{-q}{3} t^3 + \ldots = \sum_{n=0}^{\infty} \binom{-q}{n} t^n .$$

The binomial series converge absolutely and uniformly so that the development of the function $G'(t)$ may be obtained by multiplying both the preceding series. The coefficient at the n-th power will be

$$c'_n = \sum_{k=0}^{n} (-1)^k \binom{q}{k} \binom{-q}{n - k} \tag{11}$$

and thus the development of $G'(t)$ into the power series will be

$$G'(t) = c_0 \sum_{n=0}^{\infty} c'_n t^n .$$

Let us take into account the substitution (8):

$$F'(z) = c_0 \sum_{n=0}^{\infty} c'_n e^{inz} . \tag{12}$$

Now, generalizing our consideration to a semi-strip of a general width b and introducing a new variable

$$w = \frac{zb}{\pi}$$

we determine the constant c_0 from the current density at the point $y \to \infty$; to this purpose we consider only the absolute term in the development (12), i.e., the current density field is homogeneous and has the current density

$$[J]_{y\to\infty} = \frac{I}{bd}\,.$$

Taking into account the relation (12), we obtain the expression

$$c_0 = \frac{I}{\sigma bd}\,.$$

As the series (12) converges within the semi-strip uniformly, we may obtain the function $F(w)$ by integrating the series term by term. If we rearrange the formula (11) for the Fourier series coefficients then $F(w)$ will be

(13) $$c_n = \frac{c_n'}{n} = \frac{1}{n}\sum_{k=0}^{n}\binom{q}{k}\binom{-q}{n-k}(-1)^k\,,$$

$$F(w) = \frac{I}{\sigma bd}\left\langle x + iy + c + \frac{b}{\pi}\sum_{n=1}^{\infty} c_n \exp\left(-\frac{n\pi y}{b}\right)\left[\sin\frac{n\pi x}{b} - i\cos\frac{n\pi x}{b}\right]\right\rangle.$$

The constant c depends on the choice of the zero boundary value of the complex potential. Let us examine the field in the semi-strip $x \in \langle 0, b\rangle$, and choose the complex potential at the origin equal to zero. This choice means that the electric potential will be zero on the boundary of the semi-strip and the solution will be

$$F(w) = \frac{I}{\sigma bd}\left\langle x + iy + \frac{b}{\pi}\sum_{n=1}^{\infty} c_n\left\{i + \exp\left(-\frac{n\pi y}{b}\right)\left[\sin\frac{n\pi x}{b} - i\cos\frac{n\pi x}{b}\right]\right\}\right\rangle.$$

For the potential in the semi-strip we have

(14)
$$\varphi(x, y) = \frac{I}{\sigma bd}\left\langle x \operatorname{tg}\Theta + y + \frac{b}{\pi}\sum_{n=1}^{\infty} c_n\left[1 - \exp\left(-\frac{n\pi y}{b}\right)\frac{\cos\left(\frac{n\pi x}{b} + \Theta\right)}{\cos\Theta}\right]\right\rangle.$$

Similarly, for the vector and fictitious potential in this semi-strip, we have

(15a) $$U(x, y) = \frac{I}{\sigma bd}\left\langle x + \frac{b}{\pi}\sum_{n=1}^{\infty} c_n \exp\left(-\frac{n\pi y}{b}\right)\sin\frac{n\pi x}{b}\right\rangle,$$

(15b) $$V(x, y) = \frac{I}{\sigma bd}\left\langle y + \frac{b}{\pi}\sum_{n=1}^{\infty} c_n\left[1 - \exp\left(-\frac{n\pi y}{b}\right)\cos\frac{n\pi x}{b}\right]\right\rangle.$$

As it can be proved, the solution of the field converges in the whole region of the semi-strip including its boundary except the point $(b, 0)$. The value at this point may be found as a limit of an arbitrary series converging to this point from inside of this region or along its boundary, e.g. as a limit from the left

$$\varphi(b, 0) = \lim_{x \to b} \varphi(x, 0) .$$

In the Institute of Technical Kybernetics of the Slovak Academy of Sciences a computing programme has been designed in GIER-ALGOL language for the computation of $\varphi(x, y, \Theta)$. In this programme n Fourier coefficients have been calculated, limited by the condition $|c_n| > \Theta/150$. According to this condition the validity of this solution was verified by checking its boundary condition on the frontal side of the semi-strip $(y = 0)$. The solution was computed for $\Theta = \pi/16, \pi/8, \pi/4, 3\pi/8, 7\pi/16, 15\pi/32$. The convergence gets. worse with increasing Θ. The computer took into account $n = 10, 16, 29, 63, 103, 138$ Fourier coefficients. The solution converges most slowly near to the point $(b; 0)$. For $(0{,}98b; 0)$, the angle $\Theta = 15\pi/32$ gave the maximum relative error $\Delta\varphi/\Delta\varphi_{\text{HALL}}$ approximately 3%. The relative error of the value of φ was less then 10^{-2}.

The calculation of the values of the potential in the whole region is the most difficult task comparing with similar problems. The physical constants connected with the integral form of the field, e.g. the effect of magnetoresistence, can be determined with a much greater accuracy.

4. CONCLUSION

The aim of this article is to show the possibility of solving the boundary value problem occuring in problems on the current density field respecting the Hall effect. The method suggested in the article was applied to a semiinfinite strip for which the problem leads, after using a transformation, to a fieild with two singular points, which is the minimum number of singular points in problems of this kind. The advantages of this method can be appreciated mainly in more complicated problems. The number of singular points will be greater and it will be necessary to work with Laurent series. In the future, we intend to publish papers where this method will be applied to a general symmetric Hall generator and to the Hall generator in a step-like magnetic field. Problems of this kind are of importance in the solutions of slotted surfaces of electric machines [6]. Geometric shapes similar to a parallelogram appear in the solutions of anisotropic materials.

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Súhrn

RIEŠENIE OKRAJOVEJ ÚLOHY HALLOVHO POĽA FOURIEROVÝMI RADMI

Jaroslav Schilder

V článku je opísaná metóda umožňujúca riešenie okrajovej úlohy Hallovho poľa Fourierovými radmi. Metóda používajúca funkcie komplexnej premennej je ilustrovaná na príklade Hallovho poľa na okraji polopásu.

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