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# Solution of the Linear Inverse Problem in Magnetic Interpretation with Application to Oceanic Magnetic Anomalies

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*Summary*

The problem of interpreting a magnetic anomaly usually reduces to either (1) determining the distribution of magnetization given the shape of the body and the direction of magnetization, or (2) determining the shape of one interface, given the magnetization and other interfaces. These involve solution of an integral equation, linear in case (1) and non-linear in case (2). The paper gives two solutions of the linear problem applicable either to gravity or magnetic interpretation, which may also be used as a starting point for the iterative solution of the non-linear problem.

*Method* (1): The Fourier convolution theorem has been used to derive a two-dimensional magnetic version of the 'equivalent layer' theorem, which is the simplest case of the inverse problem. This enables a given magnetic anomaly to be replaced by a coating of magnetic moment per unit area of specified direction over the horizontal plane of the measured anomalies. The 'equivalent layer' can be continued downwards by existing methods. A computer program applicable to the method is available at Durham.

*Method* (2): A more versatile approach is to approximate the linear integral equation by a summation, resulting in a set of linear equations. If the number of observations are equal to the unknown magnetization parameters the equations are solved directly; if there are more observations than unknowns, least squares is used. In either case, the matrix schemes available on most computers provide the main tool for the method.

Method (2) has been applied to the interpretation of oceanic magnetic anomalies in terms of two-dimensional rectangular blocks confined between two specified depths, the magnetization varying laterally from block to block. The matrix inversion needs to be done once only for specified depths, block width and direction of magnetization. It provides a weighting function for the observed anomaly and repeated application of the convolution gives the underlying distribution of magnetization. Applications of the method to the Juan de Fuca Ridge, suggests that unacceptably high contrasts in magnetization are required if the main source is below the oceanic layer 2, supporting the view that rocks in layer 2 cause a substantial part of the anomalies.

## 1. Introduction

This paper describes two methods for the solution of the inverse problem in the interpretation of magnetic anomalies. Method (1) uses the Fourier convolution theorem and in its simplest form it leads to a magnetic equivalent layer theorem. Method (2) depends on the solution of linear equations and has much wider application. The methods described apply to two-dimensional magnetic interpretation but are equally applicable to gravity interpretation and may be extended to three-dimensional interpretation without further complication apart from increase in computing time.

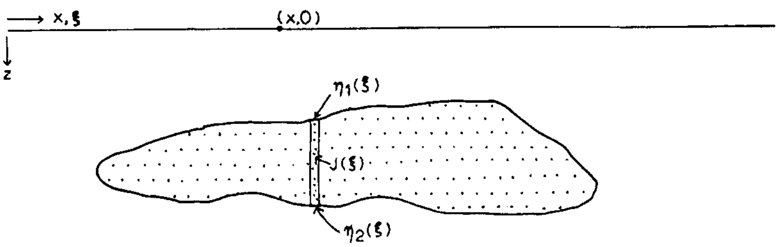


FIG. 1

Consider a magnetic anomaly measured along the horizontal $x$-axis, caused by an underlying two-dimensional distribution of magnetization with its strike direction perpendicular to the $x$-axis and with a constant (specified) direction of magnetization (Fig. 1). The $z$-axis points vertically downwards and the observed anomaly at $(x, 0)$ is $A(x)$. The magnetization is parallel to $\mathbf{m}(\cos\mu, \sin\mu)$ and the measured component of anomaly is in the direction $\mathbf{s}(\cos\sigma, \sin\sigma)$, both of which are assumed to lie within the $xz$-plane; if they do not a simple transformation may be used to bring them into this plane (Bott, Smith & Stacey 1966).

Suppose the distribution of magnetization causing the anomaly may be represented by a closed body or system of closed bodies whose surfaces are cut either twice or not at all by any vertical line. Let $\xi$ be the $x$-co-ordinate of a point on or in the system of bodies and let the upper and lower surfaces be at depths $z=\eta_1(\xi)$ and $z=\eta_2(\xi)$ respectively. The intensity of magnetization $J(\xi)$ is assumed to be a function of $\xi$ alone within the bodies and zero without. This situation gives rise to two types of interpretation problem:

(i) if the shape of the magnetized body or bodies is given, problem (i) is to deduce $J(\xi)$ from $A(x)$;

(ii) if $J(\xi)$ and either $\eta_1(\xi)$ or $\eta_2(\xi)$ are given, problem (ii) is to deduce the shape of the undefined surface.

We have

$$A(x) = \int_{-\infty}^{+\infty} J(\xi)\, K\Big(\eta_1, \eta_2, \beta, (x-\xi)\Big)\, d\xi, \tag{1}$$

where $K$ is a kernel which depends on $\eta_1$, $\eta_2$ and $\beta = (\mu+\sigma)$. Since $K$ is independent of $J$, problem (i) involves the solution of a linear integral equation. Problem (ii) is non-linear since $\eta_1$ and $\eta_2$ are included in $K$. Most problems in gravity and magnetic interpretation can be reduced to one of these two. The solution of the linear problem forms the substance of this paper. The non-linear problem can usually only be solved by iterative methods (e.g. Bott 1960, Tanner 1967), but the linear solution provides a powerful tool for making the successive approximations (Tanner 1967).

## 2. Fourier convolution solution of the linear problem

This method follows Kreisel's (1949) solution of integral equations with kernels. Suppose the distribution of magnetization (or density) causing a given anomaly is either concentrated on a plane at fixed depth $\eta$ or lies between planes at fixed depths $\eta_1$ and $\eta_2$. $J(\xi)$ is the magnetic moment (or mass) per unit area, or unit volume, respectively. Equation (1) can be written

$$A(x)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{+\infty} J(\xi)k(x-\xi)d\xi=J*k, \tag{2}$$

where the asterisk denotes Fourier convolution. It is usually necessary to replace $A(x)$ by a smoothed anomaly $A'(x')$ such that

$$A'(x')=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{+\infty} A(x)\,\omega(x'-x)dx=A*\omega, \tag{3}$$

where $\omega$ is an appropriate weighting function.

Denote the Fourier transform of $A(x)$ by $\bar{A}(s)$ etc. using the definition

$$\bar{A}(s)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{+\infty} A(x)e^{-ixs}dx,$$

and apply the Fourier convolution theorem to (2) and (3). This gives

$$\bar{A}'(s)=\bar{k}(s)\bar{J}(s)$$

and

$$\bar{A}'(s)=\bar{\omega}(s)\bar{A}(s).$$

Thus

$$\bar{J}(s)=\bar{\omega}(s)\bar{A}(s)/\bar{k}(s). \tag{4}$$

$\omega$ must be chosen to ensure that $\bar{J}$ is a Fourier transform. Invert (4) to give

$$J(\xi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi s}\frac{\bar{\omega}(s)}{\bar{k}(s)}\int_{-\infty}^{+\infty} e^{-ixs}A(x)dx\,ds. \tag{5}$$

Equation (5) formally enables $J$ to be calculated from $A$. Assume that the order of integration may be changed. Thus

$$\left.\begin{aligned} J(\xi)&=\int_{-\infty}^{+\infty} A(x)g(x-\xi)dx,\\ \text{where}\quad g(t)&=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-its}\frac{\bar{\omega}(s)}{\bar{k}(s)}ds, \text{ and } t=(x-\xi).\end{aligned}\right\} \tag{6}$$

Equation (6) is applicable either to gravity or magnetic interpretation provided that the mass or magnetization is either concentrated on a plane at constant depth or is uniformly distributed between two fixed depths, and that they tend to zero as $|x|\to\infty$. The '$\sin x/x$' method of gravity interpretation (Tomoda & Aki 1955) and

Bullard & Cooper's method (1948) may be derived from (6) by inserting the relevant weighting function. In the following section it is used to develop a similar method for magnetic interpretation.

## 3. Application of the Fourier convolution method to the magnetic equivalent layer problem

Suppose the distribution of dipoles causing a given two-dimensional magnetic anomaly is concentrated on a horizontal plane at depth $z=\eta$ and that $\beta=(\mu+\sigma)$ is constant and specified. Put $t=(x-\xi)$. Then

$$\left.\begin{aligned} k(t)&=2\sqrt{(2\pi)}\left\{\cos\beta\frac{\partial}{\partial\eta}\left(\frac{\eta}{t^2+\eta^2}\right)+\sin\beta\frac{\partial}{\partial t}\left(\frac{\eta}{t^2+\eta^2}\right)\right\}, \\ \text{and}\qquad \bar{k}(s)&=-2\pi e^{-|s|\eta}(is\sin\beta+|s|\cos\beta). \end{aligned}\right\} \tag{7}$$

Substitute (7) in (6):

$$J(\xi)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}e^{-its}e^{|s|\eta}\frac{(is\sin\beta-|s|\cos\beta)}{s^2}\bar{\omega}(s)ds\right\}A(x)dx.$$

Thus

$$J(\xi)=\int_{-\infty}^{+\infty}\Big(g_1(t)\sin\beta+g_2(t)\cos\beta\Big)A(x)dx, \tag{8}$$

where

$$g_1(t)=\frac{1}{2\pi^2}\int_0^\infty\bar{\omega}(s)e^{s\eta}\frac{\sin ts}{s}ds, \tag{8a}$$

and

$$g_2(t)=-\frac{1}{2\pi^2}\int_0^\infty\bar{\omega}(s)e^{s\eta}\frac{\cos ts}{s}ds. \tag{8b}$$

Equations (8), (8a) and (8b) give the required equivalent layer theorem. We proceed to evaluate the integrals (8a) and (8b) for the special case $\eta=0$. Let $\bar{\omega}(s)=1$ for $s_1\leqslant|s|\leqslant s_2$ and $\bar{\omega}(s)=0$ elsewhere. Then

$$g_1(t)=\frac{1}{2\pi^2}\int_{s_1}^{s_2}\frac{\sin ts}{s}ds=\frac{1}{2\pi^2}\{Si(ts_2)-Si(ts_1)\},$$

and

$$g_2(t)=-\frac{1}{2\pi^2}\int_{s_1}^{s_2}\frac{\cos ts}{s}ds=\frac{1}{2\pi^2}\{Ci(|t|s_1)-Ci(|t|s_2)\}.$$

As $s_2\to\infty$,

$$g_1(t)\to\frac{1}{2\pi^2}\left\{\frac{\pi}{2}\operatorname{sign}(t)-Si(ts_1)\right\},$$

and

$$g_2(t)\to\frac{1}{2\pi^2}Ci(|t|s_1).$$

Thus

$$g_2(t)=\frac{1}{2\pi^2}\left\{\gamma+\log s_1+\log|t|-\frac{(s_1t)^2}{2.2!}+\dots\right\}, \tag{9}$$

where $\gamma$ is Euler's constant. The r.h.s. of (9) has a singularity at $s_1 = 0$ but since $\int_{-\infty}^{+\infty} A(x)dx = 0$ for any two-dimensional magnetic anomaly and the singularity is independent of $t$ the term $(\gamma + \log s_1) A(x)$ may be removed from (8). Thus as $s_1 \to 0$ we have

$$g_1(t) \to \frac{1}{4\pi} \operatorname{sign}(t),$$

and

$$g_2(t) \to \frac{1}{2\pi^2} \log|t|.$$

Thus

$$J(\xi) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} \left\{ \log|x-\xi| \cos\beta + \frac{\pi}{2} \operatorname{sign}(x-\xi) \sin\beta \right\} A(x) dx. \tag{10}$$

Equation (10) enables a given magnetic anomaly to be interpreted in terms of a coating of magnetic moment per unit area of specified direction over the horizontal plane on which the anomaly is measured. The formula can also be obtained from the two-dimensional version of the pseudo-gravity calculation (Bott, Smith & Stacey 1966).

The coating of magnetization on a plane underlying the observations could be obtained by inserting an appropriate weighting function and evaluating the kernel functions (8a) and (8b). A simpler approach is to continue the measured anomaly downwards and then apply (10); alternatively $J(\xi)$ as obtained using (10) may be continued downwards since it satisfies the two-dimensional form of Laplace's equation.

A computer program for evaluating (10) is available at Durham. It was written by R. A. Stacey in connection with the pseudo-gravity method for determining the direction of magnetization of a body causing a magnetic anomaly (Bott, Smith & Stacey 1966). Computer methods are also available for the downward continuation.

## 4. The matrix solution of the linear problem

An alternative method of solving the linear version of the inverse problem in gravity or magnetic interpretation is to subdivide the body into a finite number of volume elements, each being sufficiently small to allow the assumption of uniform density or magnetization. The direction of magnetization needs to be specified. Suppose the value of the anomaly $A_i(x, z)$ is known at $n$ points and there are $m$ volume elements ($n \geqslant m$). Then the integral (1) may be replaced by a finite summation, giving

$$A_i = \sum_{j=1}^{m} K_{ij} J_j \quad (i = 1, 2, \ldots, n). \tag{11}$$

$K_{ij}$ is the anomaly at the $i$th field point caused by the $j$th volume element assuming unit density or magnetization; it is a function only of the shape of the volume element and its position relative to the field point, and $\beta$ in the magnetic case.

If $m = n$, the solution is obtained by solving a set of $n$ linear equations. This gives, in matrix notation,

$$\mathbf{J} = \mathbf{K}^{-1} \mathbf{A}. \tag{12}$$

If $n > m$, (11) is solved by least squares (Tanner 1967), giving

$$\mathbf{J} = (\mathbf{K}^T \mathbf{K})^{-1} \mathbf{K}^T \mathbf{A} \tag{13}$$

where $\mathbf{K}^T$ is the transpose of $\mathbf{K}$.

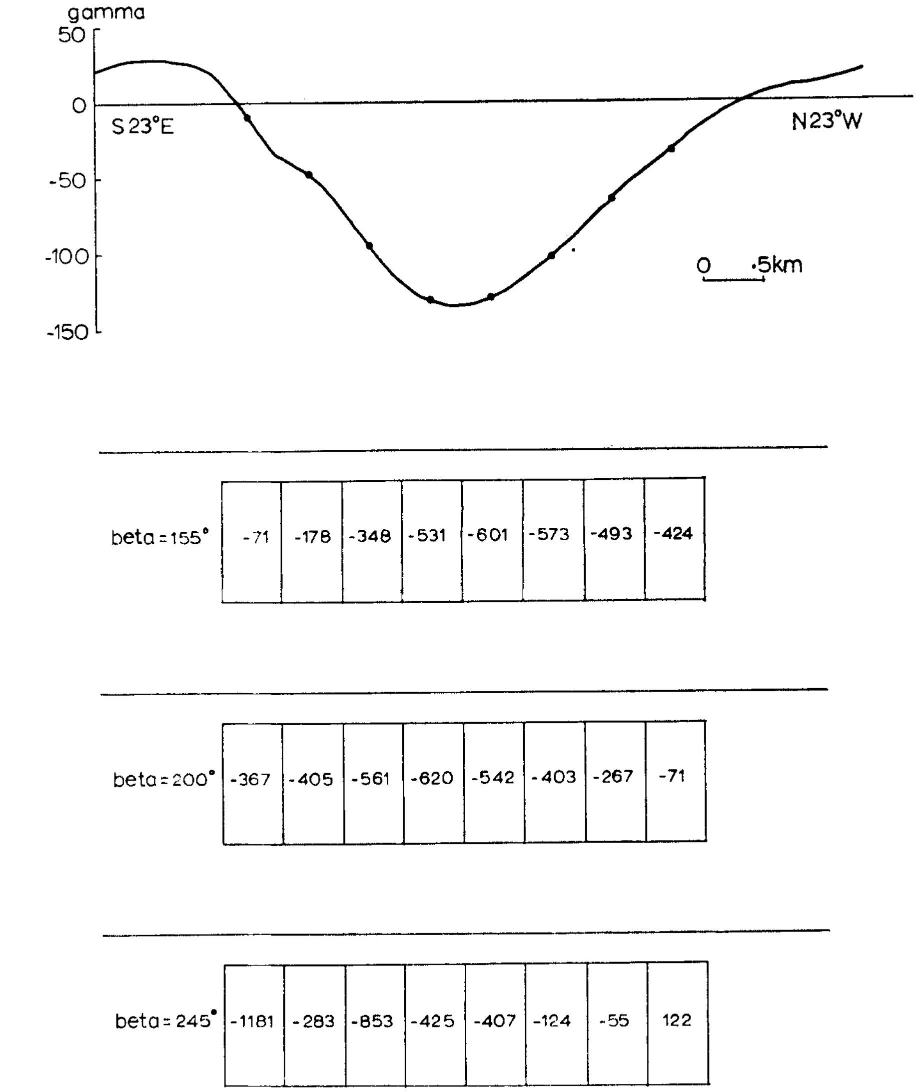


FIG. 2. Interpretation of a magnetic anomaly in the north Irish Sea by method (2), using the same number of blocks and field points. Three different directions of magnetization are assumed. Magnetization in units of $10^{-5}$ e.m.u./cm$^3$.

This method is particularly versatile since it can be applied to irregularly shaped bodies which are not amenable to the Fourier method and can also give least squares solutions. The field points do not need to be on the same horizontal plane. The method provides a tool for the solution of the non-linear problem by successive approximation (Tanner 1967).

An example of application of the method to a magnetic anomaly in the North Irish Sea (Bott 1966) is shown in Figs. 2 and 3. The magnetic body is represented by eight rectangular blocks of specified dimensions. Fig. 2 shows the solution for three different values of $\beta$ using eight field points. Fig. 3 shows the solution by least squares using 19 field points. A correction for the best-fitting background anomaly may be included in the least squares solution.

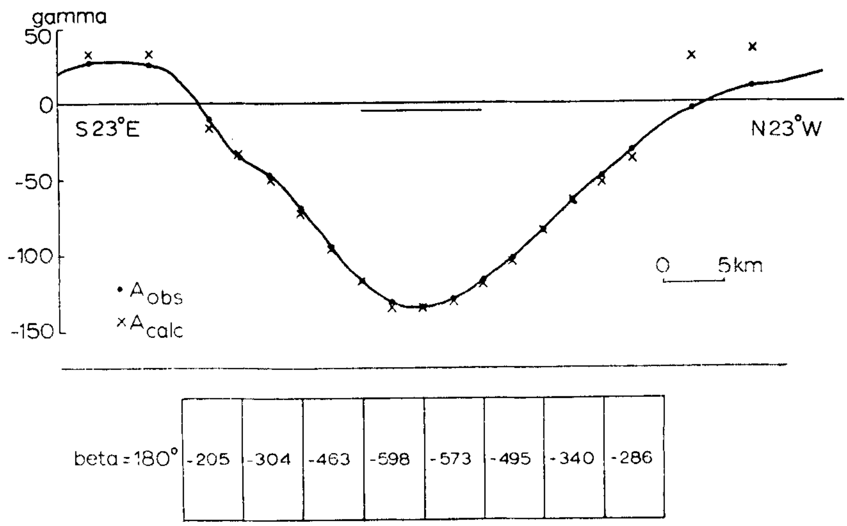


FIG. 3. Interpretation of the magnetic anomaly of Fig. 2 by least squares, using more field points than blocks.

## 5. Application of the matrix method to interpretation of oceanic magnetic anomalies

To demonstrate the potential of the matrix methods, they are here applied to one of the more difficult problems of magnetic interpretation. This is to interpret oceanic magnetic anomaly profiles some hundreds of kilometres in length in terms of an underlying distribution of magnetization confined between two fixed depths, $\eta_1$ and $\eta_2$. The magnetic layer is approximately represented by a series of two-dimensional rectangular blocks of constant width $\Delta x$ and constant thickness $(\eta_2 - \eta_1)$, each being uniformly magnetized in a fixed direction. The regional gradient is subtracted from the observed anomaly, which is then digitized at a series of equally spaced field points situated over the centre of each block. The problem is to determine the intensity of magnetization and its sign for each block from the anomaly values.

We adopt the model shown in Fig. 4 to obtain the magnetization of the block $B_0$. Beyond a certain specified distance on either side of $B_0$, blocks are grouped together into larger units to reduce computation. Blocks beyond a second specified distance from $B_0$ are ignored. The kernel matrix **K** for the model is obtained and inverted. The central row of $\mathbf{K}^{-1}$ is picked out and is used to give a series of weighting values which when convolved with the observed anomalies gives the intensity of magnetization of $B_0$. The advantage of this method is that the matrix inversion needs to be done once only since the one set of weighting coefficients can be successively applied to each block except for blocks near each end of the profile.

This method has been applied to a total field magnetic profile across the Juan de Fuca Ridge in the north-eastern Pacific as presented by Vine & Tuzo Wilson (1965).

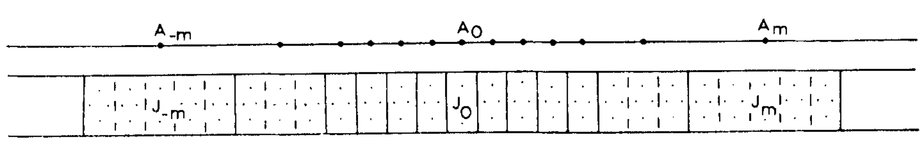


FIG. 4. The model used for interpretation of oceanic magnetic anomalies.

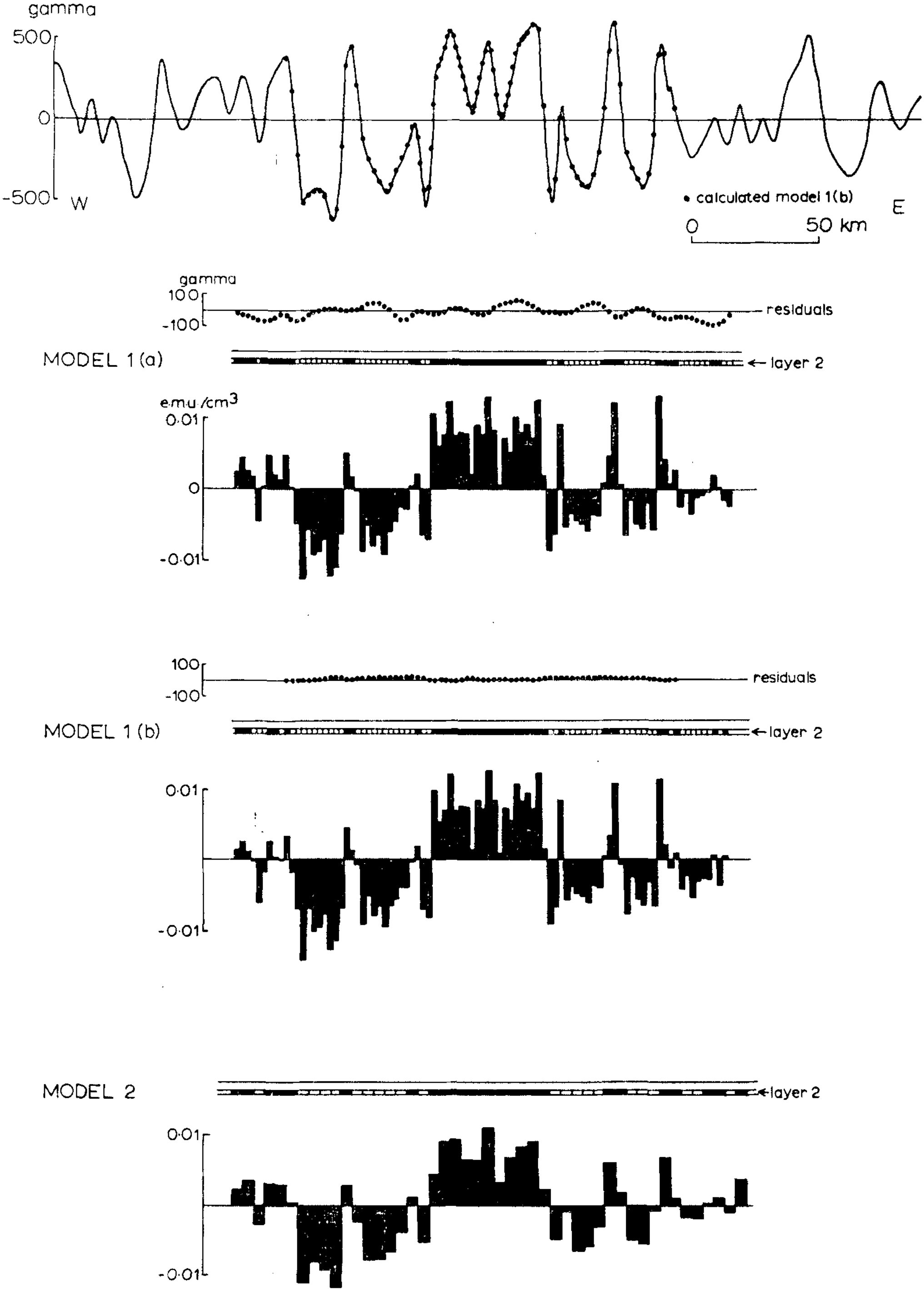


FIG. 5. The interpretation of a magnetic profile across the Juan de Fuca Ridge in terms of a distribution of magnetization confined to layer 2 of the oceanic crust. Model 1 (a) uses blocks 2·15 km wide and is the result of a single iteration; model 1 (b) is obtained after a second iteration. Model 2 has blocks 4·3 km wide and one iteration only is used. The calculated anomalies for model 1 (b) are shown on the observed profile, and residuals (observed minus calculated) are shown above models 1 (a) and 1 (b).

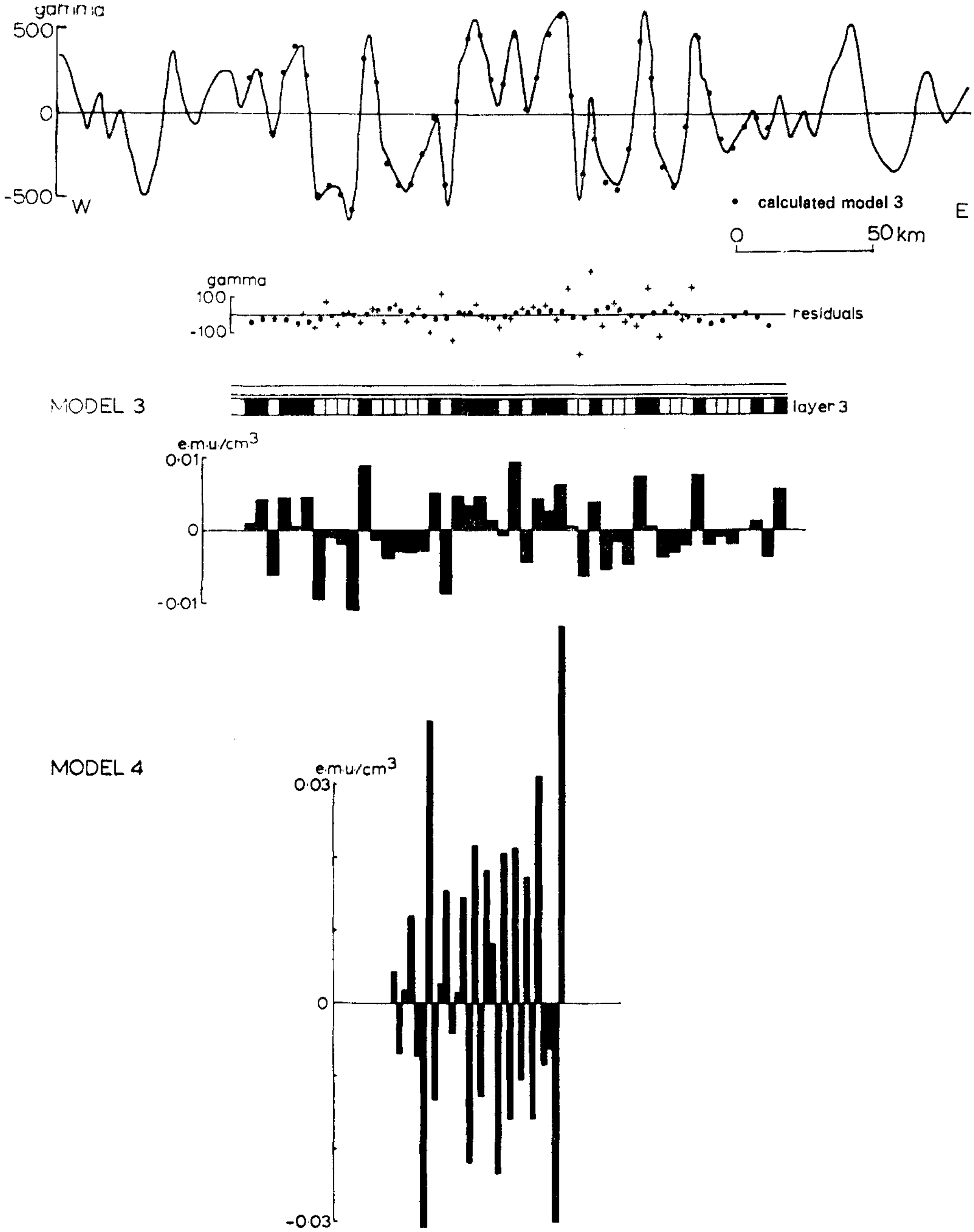


FIG. 6. Interpretation of the profile of Fig. 5 in terms of a distribution of magnetization restricted to the oceanic layer 3. Model 3 is based on blocks 4·3 km wide and the residuals are shown (crosses for intermediate points). Model 4 uses 2·15 km blocks.

The midpoint of the profile is 47·0° N, 129·2° W, and it is about 350 km long on a true bearing of 110°. The profile is perpendicular to the well-established magnetic lineation, justifying the two-dimensional approach to interpretation.

The oceanic crust is subdivided into three layers. Layer 1 consists of a thin veneer of unconsolidated sediments which do not cause magnetic anomalies. Layer 2 is typically about 1·5 km thick and it is usually thought to consist of either volcanic rocks or consolidated sediments. Following Vine & Tuzo Wilson, it is here assumed that layer 2 extends between depths of 3·3 and 5 km. Layer 3, the main oceanic crustal layer, is taken to lie between 5 and 11 km. The program is here used to estimate the distribution of magnetization in either layer 2 or layer 3 required to cause the observed anomalies.

Fig. 5 shows the result of applying the method to layer 2. The observed anomaly has been digitized at 2·15 km intervals and layer 2 is subdivided into rectangular blocks 2·15 km wide. Model 1(a) shows the result of a single application of the method. Using a computer program for the calculation of magnetic anomalies over two-dimensional bodies, the residuals (observed minus calculated anomalies) were obtained and are shown. A second iteration improves the estimate and gives model 1(b) which has negligible residuals. The calculated anomalies for model 1(b) are shown on the observed profile: over the central part of the profile the anomalies have been calculated at intermediate points and the agreement is still excellent. Thus model 1(b) gives a complete and accurate model of the magnetization within layer 2 required to explain the observations. Model 1(a) does not differ greatly and is also an acceptable model. Model 2 has been computed for blocks 4·3 km wide. The agreement is good at field points over the centre of each block, but the residuals are larger at the intermediate points, suggesting that these wider blocks are inadequate as a complete explanation of the observations.

Fig. 6 applies the method to the hypothesis that the anomalies arise from layer 3. Model 3 uses blocks 4·3 km wide and has been obtained by a single iteration. The residuals have been computed and agree well at the field points above the centre of each block, but show much larger discrepancies at intermediate points (shown by crosses). This shows that there are short wavelength components in the observed anomaly which cannot be explained using wide blocks. Model 4 has been computed for 2·15 km wide blocks; the agreement between observed and calculated anomalies is good but variations in magnetization of the order of 0·08 e.m.u./$\text{cm}^3$ appear and the simple pattern is lost.

It is concluded that magnetic rocks within, or partly within, layer 2 can explain the observed anomalies. If the magnetic rocks were confined to layer 3, excessively strong contrasts in magnetization would be needed. The models for layer 2 show a conspicuous symmetry about the centre of the Juan de Fuca Ridge, supporting the hypothesis of Vine & Matthews (1963) that the anomalies are caused during the process of ocean floor spreading in both directions from the centre. There is, however, no indication of excessively strong magnetization for the central block, contrary to the model presented by Vine & Tuzo Wilson. A layer of uniform magnetization could be added to the model without affecting the calculated anomalies: it is therefore not possible to decide from these anomalies whether the magnetization is alternatively normal and reversed, or alternatively strong and weak.

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