SOLUTION OF TIME-VARYING SINGULAR NONLINEAR SYSTEMS BY SINGLE-TERM WALSH SERIES

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A method for finding the solution of time-varying singular nonlinear systems by using single-term Walsh series is proposed. The properties of single-term Walsh series are given and are utilized to find the solution of time-varying singular nonlinear systems.

1. Introduction

Singular nonlinear systems has been of interest to some investigators [4, 12], however no closed-form solution was given in [4, 12]. In some analysis of neural networks, both singular systems [8] and bilinear systems [16] have been used. For singular bilinear systems, Lewis et al. [11] applied the Walsh function (WF) approach for time-invariant singular bilinear systems and Hsiao and Wang [9] used the Haar wavelets for the solution of time-varying singular nonlinear systems.

Walsh functions (WFs) have received considerable attention in dealing with various problems of dynamic systems. Chen and Hsiao [5, 6, 7] applied the WF technique to the analysis, optimal control, and synthesis of linear systems. WFs have also found wide applications in signal processing, communication, and pattern recognition [13]. Rao et al. [14] presented a method of extending computation beyond the limit of the initial normal interval in Walsh series analysis of dynamical systems. In [14] various time functions in the system were first expanded in terms of their truncated WF with unknown coefficients. Using the Kronecker product [10], the unknown coefficient of the rate variable was obtained by finding the inverse of a square matrix. It was shown that this method involve some numerical difficulties if the dimension of this matrix is large. To remove the inconveniences in WF technique, the single-term Walsh series (STWS) was introduced in [14], and Balachandran and Murugesan [1, 2, 3] applied STWS technique to the analysis of the linear and nonlinear singular systems. The STWS method provides block-pulse and discrete solutions to any length of time.

In the present paper, we use the STWS approach for the solution of time-varying singular nonlinear systems. As compared to [9], our method is simpler and consumes less computer time.

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The paper is organized as follows: in Section 2 we describe the basic properties of the WF and STWS required for our subsequent development. Section 3 is devoted to the formulation of the time-varying singular nonlinear systems. In Section 4 we apply the proposed numerical method to the time-varying singular nonlinear systems and in Section 5, we report our numerical finding and demonstrate the accuracy of the proposed method.

2. Properties of WF and STWS

2.1. Walsh functions. A function $f(t)$, integrable in $[0,1)$, may be approximated using WF as

$$f(t)=\sum_{i=0}^{\infty} f_i\phi_i(t), \tag{2.1}$$

where $\phi_i(t)$ is the ith WF and f_i is the corresponding coefficient. In practice, only the first m terms are considered, where m is an integral power of 2. Then from (2.1), we get

$$f(t)=\sum_{i=0}^{m-1} f_i\phi_i(t)=F^T\Phi(t), \tag{2.2}$$

where

$$F=\left(f_0,f_1,\dots,f_{m-1}\right)^T, \qquad \Phi(t)=\left(\phi_0(t),\phi_1(t),\dots,\phi_{m-1}(t)\right)^T. \tag{2.3}$$

The coefficients f_i are chosen to minimize the mean integral square error

$$\epsilon=\int_0^1\left(f(t)-F^T\Phi(t)\right)^2 dt, \tag{2.4}$$

and are given by

$$f_i=\int_0^1 f(t)\phi_i(t)dt. \tag{2.5}$$

The integration of the vector $\Phi(t)$ defined in (2.3) can be approximated by

$$\int_0^t \Phi(t')dt' \simeq E\Phi(t), \tag{2.6}$$

where E is the $m\times m$ operational matrix for integration with $E_{1\times 1}=1/2$ and is given in [16].

2.2. Single-term Walsh series. With the STWS approach, in the first interval, the given function is expanded as STWS in the normalized interval $\tau\in[0,1)$, which corresponds to $t\in[0,1/m)$ by defining $\tau=mt$, m being any integer. In STWS, the matrix E in (2.6) becomes $E=1/2$.

Let $\dot{x}(\tau)$ and $x(\tau)$ be expanded by STWS series in the first interval as

$$\dot{x}(\tau) = V^{(1)}\phi_0(\tau), \qquad x(\tau) = X^{(1)}\phi_0(\tau), \tag{2.7}$$

and in the kth interval as

$$\dot{x}(\tau) = V^{(k)}\phi_0(\tau), \qquad x(\tau) = X^{(k)}\phi_0(\tau). \tag{2.8}$$

Integrating (2.7) with $E = 1/2$, we get

$$X^{(1)} = \frac{1}{2}V^{(1)} + x(0), \tag{2.9}$$

where $x(0)$ is the initial condition. According to Sannuti [15], we have

$$V^{(1)} = \int_0^1 \dot{x}(\tau)d\tau = x(1) - x(0). \tag{2.10}$$

In general, for any interval k, $k = 1, 2, \ldots$, we obtain

$$X^{(k)} = \frac{1}{2}V^{(k)} + x(k-1), \tag{2.11}$$

$$x(k) = V^{(k)} + x(k-1). \tag{2.12}$$

In (2.11) and (2.12), $X^{(k)}$ and $x(k)$ give the block-pulse and the discrete values of the state, respectively.

3. Problem statement

Consider a time-varying singular nonlinear system of the following form:

$$E(t)\dot{x}(t) = f\left(t, x(t), u(t)\right), \quad x(0) = x_0, \tag{3.1}$$

where the singular matrix $E(t) \in \mathbb{R}^{n\times n}$, the nonlinear function $f \in \mathbb{R}^n$, the state $x(t) \in \mathbb{R}^n$, and the control $u(t) \in \mathbb{R}^q$. The response $x(t)$ is required to be found.

4. Solution of time-varying singular nonlinear systems via STWS

Normalizing (3.1) by defining $\tau = mt$, we get

$$mE(\tau)\dot{x}(\tau) = f\left(\tau, x(\tau), u(\tau)\right), \quad x(0) = x_0. \tag{4.1}$$

Let $E(\tau)$ be expressed by STWS in the kth interval as

$$E(\tau) = E^{(k)}\phi_0(\tau), \tag{4.2}$$

where $E^{(k)} \in \mathbb{R}^{n\times n}$. By using (2.8) and (2.11), we get

$$x(\tau) = \left(\frac{1}{2}V^{(k)} + x(k-1)\right)\phi_0(\tau). \tag{4.3}$$

To solve (4.1), we first substitute (4.3) in $f(\tau, x(\tau), u(\tau))$; we then express the resulting equation by STWS as

$$f\left(\tau, \left(\frac{1}{2}V^{(k)} + x(k-1)\right)\phi_0(\tau), u(\tau)\right) = F^{(k)}\phi_0(\tau). \tag{4.4}$$

Using (4.1), (4.2), (4.3), and (4.4), we get

$$mE^{(k)}V^{(k)} = F^{(k)}. \tag{4.5}$$

By solving (4.5), the components of $V^{(k)}$ can be obtained. By substituting $V^{(k)}$ in (2.11) and (2.12), we obtain block-pulse and discrete approximations of the state, respectively. Further, using (2.7), we get

$$x(\tau) = \int_0^{\tau} \dot{x}(\tau')d\tau' + x(0) = V^{(1)}\tau + x(0). \tag{4.6}$$

Thus, we can obtain a continuous approximation of the state as

$$x(\tau) = V^{(k)}\tau + x(k-1). \tag{4.7}$$

5. Numerical examples

Three examples are given in this section. These examples were considered by Hsiao and Wang [9] by using Haar wavelets. Our method differs from their approach and thus these examples could be used as a basis for comparison.

Example 5.1. Consider a time-varying nonlinear singular system of the following form [9]:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & t^2 \\ 0 & 0 & 0 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} tx_1(t) + x_2(t) \\ \exp(t)x_1(t)x_2(t) \\ x_2(t)\big(x_1(t) + x_3(t)\big) \end{bmatrix} + \begin{bmatrix} 0 \\ 2t^2\exp(-t) \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}. \tag{5.1}$$

The exact solution is (see [9])

$$x(t) = \begin{bmatrix} 2\exp(-t)(1-2t) \\ t^2\exp(-t) \\ -2\exp(-t)(1-2t) \end{bmatrix}. \tag{5.2}$$

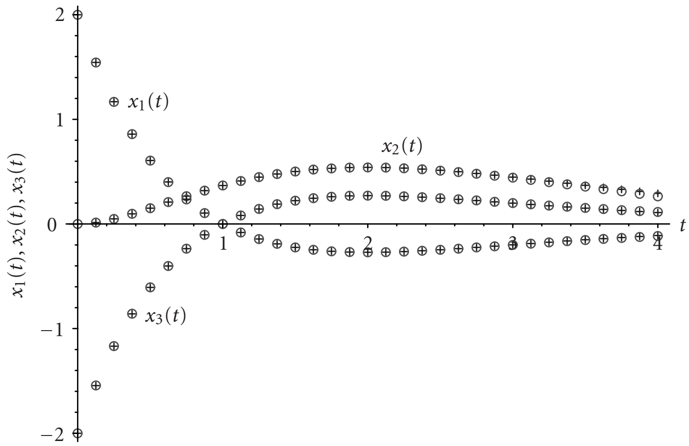

Figure 5.1. STWS with $m = 32$ (circles) and exact solution: ++++ of Example 5.1.

To solve (5.1) by STWS, we first express t, t^2, $\exp(t)$, and $2t^2\exp(-t)$ by STWS, then we substitute these values together with $V_i^{(k)} = \dot{x}_i(t)$ and $x_i(t) = (1/2)V_i^{(k)} + x_i(k-1)$, $i = 1,2,3$, in (5.1). By solving the resulting equation, $V^{(k)} = [V_1^{(k)}, V_2^{(k)}, V_3^{(k)}]^T$ can be calculated. By using (2.11), (2.12), and (4.7), block-pulse, discrete, and continuous approximations of state $x(t)$ are obtained.

The comparison between STWS solution with $m = 32$ and the exact solution for $t \in [0,4)$ is shown in Figure 5.1.

Example 5.2. Consider the following time-invariant singular nonlinear system [9]:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} x_3(t) - x_5(t) \\ x_2(t) + x_3(t) - x_4(t) - x_5(t) \\ \left(x_1(t) + x_2(t) - 1\right)^2 - x_3(t) \\ -x_4(t) \\ x_2(t)\left(x_1(t) + x_2(t)\right) - x_5(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{5.3}$$

The results obtained by STWS with $m = 16$ and $m = 110$ together with those obtained by Haar wavelets with $m = 512$ are presented in Tables 5.1, 5.2, and 5.3, respectively.

Example 5.3. Consider a time-invariant nonlinear singular system of the following form [9]:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ x_3^3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}. \tag{5.4}$$

The results obtained by STWS with $m = 24$ and $m = 32$ and those obtained by Haar wavelets with $m = 32$ and $m = 128$ are presented in Tables 5.4 and 5.5, respectively.

Table 5.1. Estimated values of $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$, and $x_5(t)$ by STWS with $m = 16$.

Time	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$	$x_5(t)$
0.0	0.0000	0.0000	1.0000	0.0000	0.0000
0.5	−0.3297	0.5771	0.5671	0.0000	0.1443
1.0	−0.4776	0.8003	0.4596	0.0000	0.2599
1.5	−0.5486	0.9023	0.4186	0.0000	0.3209
2.0	−0.5835	0.9514	0.4004	0.0000	0.3516
2.5	−0.6008	0.9755	0.3919	0.0000	0.3674
3.0	−0.6094	0.9875	0.3877	0.0000	0.3752
3.5	−0.6137	0.9935	0.3856	0.0000	0.3791
4.0	−0.6159	0.9965	0.3846	0.0000	0.3811

Table 5.2. Estimated values of $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$, and $x_5(t)$ by STWS with $m = 110$.

Time	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$	$x_5(t)$
0.0	0.0000	0.0000	1.0000	0.0000	0.0000
0.5	−0.3295	0.5774	0.5658	0.0000	0.1431
1.0	−0.4775	0.8006	0.4582	0.0000	0.2587
1.5	−0.5485	0.9027	0.4171	0.0000	0.3197
2.0	−0.5834	0.9518	0.3989	0.0000	0.3507
2.5	−0.6007	0.9760	0.3903	0.0000	0.3663
3.0	−0.6094	0.9880	0.3861	0.0000	0.3741
3.5	−0.6137	0.9940	0.3841	0.0000	0.3780
4.0	−0.6159	0.9970	0.3830	0.0000	0.3800

Table 5.3. Estimated values of $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$, and $x_5(t)$ by Haar wavelets with $m = 512$.

Time	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$	$x_5(t)$
0.0	0.0000	0.0000	1.0000	0.0000	0.0000
0.5	−0.3295	0.5774	0.5658	0.0000	0.1431
1.0	−0.4775	0.8006	0.4582	0.0000	0.2587
1.5	−0.5485	0.9027	0.4171	0.0000	0.3197
2.0	−0.5834	0.9518	0.3989	0.0000	0.3507
2.5	−0.6007	0.9760	0.3903	0.0000	0.3663
3.0	−0.6094	0.9880	0.3861	0.0000	0.3741
3.5	−0.6137	0.9940	0.3841	0.0000	0.3780
4.0	−0.6159	0.9970	0.3830	0.0000	0.3800

Table 5.4. Estimated values of $x_1(t)$, $x_2(t)$, and $x_3(t)$ by STWS with $m = 24$ and $m = 32$.

Time	STWS with $m = 24$			STWS with $m = 32$		
	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_1(t)$	$x_2(t)$	$x_3(t)$
0.0	2.00000	−1.00000	−2.00000	2.00000	−1.00000	−2.00000
0.125	1.75175	−0.75175	−1.97189	1.75175	−0.75175	−1.97189
0.25	1.50706	−0.50706	−1.94309	1.50706	−0.50706	−1.94309
0.375	1.26601	−0.26601	−1.91353	1.26601	−0.26601	−1.91353
0.5	1.02871	−0.02871	−1.88316	1.02871	−0.02871	−1.88316
0.625	0.79526	0.20474	−1.85187	0.79526	0.20474	−1.85187
0.75	0.56578	0.43422	−1.81960	0.56578	0.43422	−1.81960
0.875	0.34040	0.65960	−1.78622	0.34040	0.65960	−1.78622
1.0	0.11927	0.88073	−1.75160	0.11927	0.88073	−1.75160

Table 5.5. Estimated values of $x_1(t)$, $x_2(t)$, and $x_3(t)$ by Haar wavelets with $m = 32$ and $m = 128$.

Time	Haar wavelets with $m = 32$			Haar wavelets with $m = 128$		
	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_1(t)$	$x_2(t)$	$x_3(t)$
0.0	2.0000	−1.0000	−2.0000	2.0000	−1.0000	−2.0000
0.125	1.7517	−0.7517	−1.9719	1.7517	−0.7517	−1.9719
0.25	1.5071	−0.5071	−1.9431	1.5071	−0.5071	−1.9431
0.375	1.2660	−0.2660	−1.9135	1.2660	−0.2660	−1.9135
0.5	1.0287	−0.0287	−1.8832	1.0287	−0.0287	−1.8832
0.625	0.7953	0.2047	−1.8519	0.7953	0.2047	−1.8519
0.75	0.5658	0.4342	−1.8196	0.5658	0.4342	−1.8196
0.875	0.3404	0.6596	−1.7862	0.34040	0.6596	−1.7862
1.0	0.1193	0.8807	−1.7516	0.1193	0.8807	−1.7516

6. Conclusion

The properties of STWS are used to solve the time-varying singular nonlinear systems. The key idea is to transform the time-varying functions into STWS. The method can be implemented using a digital computer. It occupies less memory space and consumes less computer time than the method in [9]. Illustrative examples were included to demonstrate the validity and applicability of the technique.

References

[1] K. Balachandran and K. Murugesan, *Analysis of different systems via single term Walsh series method*, Int. J. Comput. Math. **33** (1990), 171–180.

[2] ———, *Analysis of nonlinear singular systems via STWS method*, Int. J. Comput. Math. **36** (1990), 9–12.

[3] ———, *Numerical solution of a singular nonlinear system from fluid dynamics*, Int. J. Comput. Math. **38** (1991), 211–218.
[4] S. L. Campbell, *Bilinear nonlinear descriptor control systems*, CRSC Tech. Rep. 102386-01, Department of Mathematics, North Carolina State University, North Carolina, 1987.
[5] C. F. Chen and C.-H. Hsiao, *A state-space approach to Walsh series solution of linear systems*, Internat. J. Systems Sci. **6** (1975), no. 9, 833–858.
[6] ———, *Time-domain synthesis via Walsh functions*, Proc. IEE **122** (1975), 565–570.
[7] ———, *Walsh series analysis in optimal control*, Internat. J. Control **21** (1975), 881–897.
[8] N. Declaris and A. Rindos, *Semi state analysis of neural networks in Aplysia California*, Proc. 27th MSCS, 1984, pp. 686–689.
[9] C.-H. Hsiao and W.-J. Wang, *State analysis of time-varying singular nonlinear systems via Haar wavelets*, Math. Comput. Simulation **51** (1999), no. 1-2, 91–100.
[10] P. Lancaster, *Theory of Matrices*, Academic Press, New York, 1969.
[11] F. L. Lewis, B. G. Mertzios, and W. Marszalek, *Analysis of singular bilinear systems using Walsh functions*, Proc. IEE-D **138** (1991), no. 2, 89–92.
[12] R. W. Newcomb, *The semistate description of nonlinear time-variable circuits*, IEEE Trans. Circuits and Systems **28** (1981), no. 1, 62–71.
[13] G. P. Rao, *Piecewise Constant Orthogonal Functions and Their Application to Systems and Control*, Lecture Notes in Control and Information Sciences, vol. 55, Springer-Verlag, Berlin, 1983.
[14] G. P. Rao, K. R. Palanisamy, and T. Srinivasan, *Extension of computation beyond the limit of initial normal interval in Walsh series analysis of dynamical systems*, IEEE Trans. Automat. Control **25** (1980), 317–319.
[15] P. Sannuti, *Analysis and synthesis of dynamic systems via block-pulse functions*, Proc. IEE **124** (1977), 569–571.
[16] N. Wiener, *Cybernetics*, MIT Press, Cambridge, 1965.

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