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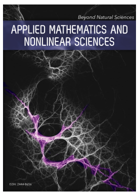

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Solutions and conservation laws of a generalized second extended (3+1)-dimensional Jimbo-Miwa equation

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Abstract

In this paper we study a nonlinear multi-dimensional partial differential equation, namely, a generalized second extended (3+1)-dimensional Jimbo-Miwa equation. We perform symmetry reductions of this equation until it reduces to a nonlinear fourth-order ordinary differential equation. The general solution of this ordinary differential equation is obtained in terms of the Weierstrass zeta function. Also travelling wave solutions are derived using the simplest equation method. Finally, the conservation laws of the underlying equation are computed by employing the conservation theorem due to Ibragimov, which include conservation of energy and conservation of momentum laws.

Keywords: A generalized second extended (3+1)-dimensional Jimbo-Miwa equation, Lie point symmetries, exact solutions, simplest equation method, conservation laws
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1 Introduction

Most natural phenomena of the real world are modelled by nonlinear partial differential equations (NLPDEs). Such equations can seldom be solved by an analytic method. In contrast the linear differential equations have a particularly good algebraic structure to their solutions, which makes them solvable. Unfortunately, for NLPDEs there is no general theory which can be applied to obtain exact closed-form solutions. However, scientists have developed geometric methods and dynamical systems theory which play prominent roles in the study of differential equations. Such theories deal with the long-term qualitative behaviour of dynamical systems and do not focus on finding precise solutions to the equations. Nevertheless, various methods have also been established by the researchers which provide exact solutions to NLPDEs. Some of these methods

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are Hirota's bilinear transformation method [1], the inverse scattering method [2], Kudryashov's method [3, 4], the sine-cosine method [5], the tanh-coth method [6], the simplest equation method [7, 8], the tanh-function method [9], the Darboux transformation [10], the $(G'/G)-$expansion method [11, 12], the Bäcklund transformation [13], and Lie symmetry methods [14–19].

One of the NLPDEs is the (3 + 1)-dimensional Jimbo-Miwa equation

$$u_{xxxy}+3u_yu_{xx}+3u_xu_{xy}+2u_{yt}-3u_{xz}=0, \tag{1}$$

which is the second member of a Kadomtsev-Petviashvili hierarchy. This equation has been studied extensively by researchers because of the fact that it can be used to describe some fascinating (3+1)-dimensional waves in physics. See for example [20–23] and references therein.

Recently equation (1) has been extended to the equation [24]

$$u_{xxxy}+3\left(u_yu_x\right)_x+2(u_{xt}+u_{yt}+u_{zt})-3u_{xz}=0, \tag{2}$$

where the term u_{yt} was extended to $u_{xt}+u_{yt}+u_{zt}$ and because of this reason it is called the extended (3+1)-dimensional Jimbo-Miwa equation. Applying the simplified Hirota's method multiple soliton solutions of (2) were derived and it was shown that the dispersion relations and the phase shifts of (2) were distinct compared to the dispersion and shifts of (1). By using bilinear forms Sun and Chen [25] obtained the lump solutions and their dynamics of (1) and (2). Furthermore, the lump-kink solution which contains interaction between a lump and a kink wave were also obtained in [25].

In this paper we consider a generalized version of the second extended (3+1)-dimensional Jimbo-Miwa equation, namely

$$u_{xxxy}+k\left(u_yu_x\right)_x+h(u_{xt}+u_{yt}+u_{zt})-ku_{xz}=0, \tag{3}$$

where h and k are constants. We obtain exact solutions of (3) using symmetry reductions along with simplest equation method. Furthermore, we derive conservation laws for (3) using the conservation theorem due to Ibragimov.

Lie symmetry theory, originally developed by Marius Sophus Lie (1842-1899), a Norwegian mathematician, around the middle of the nineteenth century, is based upon the study of the invariance under one parameter Lie group of point transformations [14–19]. The theory is highly algorithmic and is one of the most powerful methods to find exact solutions of differential equations be it linear or nonlinear. It has been applied to many scientific fields such as classical mechanics, relativity, control theory, quantum mechanics, numerical analysis, to name but a few.

Conservation laws can be described as fundamental laws of nature, which have extensive applications in various fields of scientific study such as physics, chemistry, biology, engineering, and so on. They have many uses in the study of differential equations. Conservation laws have been used to prove global existence theorems and shock wave solutions to hyperbolic systems. They have been applied to problems of stability and have been used in scattering theory and elasticity [17, 26–29].

The paper is organized as follows. In Section 2 we first perform symmetry reductions of the generalized second extended (3+1)-dimensional Jimbo-Miwa equation (3) and reduce it to a nonlinear fourth-order ordinary differential equation. Thereafter we find the general solution of the ordinary differential equation in terms of the Weierstrass zeta function. We also find travelling wave solutions of (3) using the simplest equation method. Conservation laws of (3) are obtained by employing the conservation theorem due to Ibragimov in Section 3. Finally we present concluding remarks in Section 4.

2 Exact solutions of (3)

In this section we present exact solutions to the generalized second extended (3+1)-dimensional Jimbo-Miwa equation (3).

2.1 Lie point symmetries and symmetry reductions of (3)

We apply the algorithm for computing Lie point symmetries of (3) and then use them to perform symmetry reductions several times until we arrive at an ordinary differential equation (ODE).

The vector field of the form

$$X=\xi^1\frac{\partial}{\partial x}+\xi^2\frac{\partial}{\partial y}+\xi^3\frac{\partial}{\partial z}+\xi^4\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u},$$

where ξ^i, $i=1,2,3,4$ and η depend on x, y, z, t and u, will generate a symmetry group of (3) provided

$$\mathrm{pr}^{(4)}X\left(u_{xxxy}+k\left(u_yu_x\right)_x+h(u_{xt}+u_{yt}+u_{zt})-ku_{xz}\right)|_{(3)}=0, \tag{4}$$

where $\mathrm{pr}^{(4)}X$ is the fourth prolongation of X [17]. Expanding the determining equation (4) and splitting on derivatives of u, we obtain an overdetermined system of linear homogeneous partial differential equations. Solving this resultant system one obtains the values of ξ^i, $i=1,2,3,4$ and η. Consequently, we have the following nine Lie point symmetries of (3):

$$\begin{aligned}
&X_1=\frac{\partial}{\partial t},\ X_2=\frac{\partial}{\partial x},\ X_3=\frac{\partial}{\partial y},\ X_4=\frac{\partial}{\partial z},\ X_5=f_1(t)\frac{\partial}{\partial u},\ X_6=f_2(z)\frac{\partial}{\partial u},\\
&X_7=-3ht\frac{\partial}{\partial t}+(2kt-hx+hz)\frac{\partial}{\partial x}+(2hx+hy-4kz+hu)\frac{\partial}{\partial u},\\
&X_8=ht\frac{\partial}{\partial t}-(kt+hz)\frac{\partial}{\partial x}-hz\frac{\partial}{\partial y}-hz\frac{\partial}{\partial z}+(kt+2hz)\frac{\partial}{\partial u},\\
&X_9=ht\frac{\partial}{\partial t}-kt\frac{\partial}{\partial x}+(hy-hz)\frac{\partial}{\partial y}+(kt-hy+2hz)\frac{\partial}{\partial u}.
\end{aligned} \tag{5}$$

We now make use of the four translation symmetries and perform symmetry reductions. Solving the associated Lagrange system for $X=X_1+\alpha X_2+X_3+X_4$, where α is a constant, we obtain four invariants

$$w=z-y,\quad f=t-y,\quad g=x-\alpha y,\quad \theta=u. \tag{6}$$

Using these invariants the generalized seconded extended (3+1)-dimensional Jimbo-Miwa equation (3) transforms to

$$\begin{aligned}
&\theta_{fggg}+\alpha\theta_{gggg}+\theta_{gggw}+k\theta_{gg}\left(\theta_f+\alpha\theta_g+\theta_w\right)+k\theta_{gw}+k\theta_g\left(\theta_{fg}+\alpha\theta_{gg}+\theta_{gw}\right)\\
&+h\left((\alpha-1)\theta_{fg}+\theta_{ff}\right)=0,
\end{aligned} \tag{7}$$

which is a nonlinear PDE in three independent variables. Equation (7) has the following seven Lie point symmetries:

$$\begin{aligned}
&\Gamma_1=\frac{\partial}{\partial w},\ \Gamma_2=\frac{\partial}{\partial f},\ \Gamma_3=\frac{\partial}{\partial g},\ \Gamma_4=\frac{\partial}{\partial\theta},\ \Gamma_5=(w-f)\frac{\partial}{\partial\theta},\\
&\Gamma_6=w\frac{\partial}{\partial w}+(2w-f)\frac{\partial}{\partial f}+(2\alpha w-g)\frac{\partial}{\partial g}+(2g-2\alpha w+\theta)\frac{\partial}{\partial\theta},\\
&\Gamma_7=6hkw\frac{\partial}{\partial w}+6hkw\frac{\partial}{\partial f}+k(7a\alpha h+\alpha bh+ah+ak-bh-bk-2dh)\frac{\partial}{\partial g}\\
&\qquad+(a\alpha^2h^2-2a\alpha h^2-2a\alpha hk+ah^2+2ahk+ak^2+4dhk+2h\theta k)\frac{\partial}{\partial\theta}.
\end{aligned}$$

Utilizing the symmetry $\Gamma=\Gamma_1+\Gamma_2+\beta\Gamma_3$, where β is a constant, we reduce equation (7) to a PDE in two independent variables. From the associated Lagrange system for Γ, we obtain three invariants

$$r=g-\beta f,\quad s=w-f,\quad \phi=\theta \tag{8}$$

and these invariants transform equation (7) to

$$(\beta-\alpha)\phi_{rrrr}+\left(\alpha\beta h-\beta^2 h-\beta h\right)\phi_{rr}+\alpha h\phi_{rs}-2\beta h\phi_{rs}-h\phi_{rs}-h\phi_{ss}-2\alpha k\phi_r\phi_{rr}$$
$$+2\beta k\phi_r\phi_{rr}-k\phi_{rs}=0, \tag{9}$$

which is a nonlinear PDE in two independent variables. We perform further symmetry reduction on equation (9). This equation has five symmetries including the two translation symmetries $\Sigma_1=\partial/\partial r$ and $\Sigma_2=\partial/\partial s$. The combination $\Sigma=\nu\Sigma_1+\Sigma_2$, yields the two invariants

$$q=r-\nu s, \quad F=\phi,$$

which give rise to a group-invariant solution $\phi=F(q)$ and consequently, equation (9) is transformed into the fourth-order nonlinear ODE

$$AF''''(q)+BF'(q)F''(q)+CF''(q)=0, \tag{10}$$

where $A=\alpha-\beta$, $B=2k(\alpha-\beta)$, $C=h(\beta-\nu)(-\alpha+\beta-\nu+1)-k\nu$ and $q=x+(\beta-\alpha)y-\nu z+(\nu-\beta)t$. Integration of the above equation twice with respect to q gives

$$\frac{A}{2}F''^2+\frac{B}{6}F'^3+\frac{C}{2}F'^2+C_1F'+C_2=0,$$

where C_1 and C_2 are integration constants. Letting $H=F'$, the above equation becomes

$$H'^2=-\frac{B}{3A}H^3-\frac{C}{A}H^2-\frac{2C_1}{A}H-\frac{2C_2}{A}.$$

Now using the transformation

$$H(q)=-\frac{12A}{B}\wp(q)-\frac{C}{B}, \tag{11}$$

we obtain equation for the Weierstrass elliptic function [30]

$$\wp'^2=4\wp^3-g_1\wp-g_2,$$

where

$$g_1=\frac{C^2-2BC_1}{12A^2},\ g_2=\frac{C^3+3B(BC_2-CC_1)}{216A^3}.$$

Thus integrating equation (11) and reverting to our original variables we obtain the solution of (3), which is given by

$$u(x,y,z,t)=\frac{12A}{B}\zeta\left(q;g_1,g_2\right)-\frac{C}{B}q,$$

where $\zeta\left(q;g_1,g_2\right)$ is the Weierstrass zeta function defined as $\zeta'\left(q;g_1,g_2\right)=-\wp(q;g_1,g_2)$ [30] and $A=\alpha-\beta$, $B=2k(\alpha-\beta)$, $C=h(\beta-\nu)(-\alpha+\beta-\nu+1)-k\nu$ and $q=x+(\beta-\alpha)y-\nu z+(\nu-\beta)t$.

2.2 Exact solutions of (3) using simplest equation method

In this subsecrion we use the simplest equation method [7, 8] to solve the ODE (10) and henceforth one obtains the exact solutions of the generalized second extended (3+1)-dimensional Jimbo-Miwa equation (3). We use the Bernoulli and Riccati equations as the simplest equations. The Bernoulli equation

$$H'(q) = cH(q) + dH^2(q), \tag{12}$$

where c and d are constants has solution

$$H(z) = c\left\{\frac{\cosh[c(q+C)] + \sinh[c(q+C)]}{1 - d\cosh[c(q+C)] - d\sinh[c(q+C)]}\right\},$$

with C being a constant of integration.

The Riccati equation

$$H'(q) = cH(q) + dH^2(q) + e, \tag{13}$$

where c, d and e are constants, has two solutions, namely

$$H(q) = -\frac{c}{2d} - \frac{\theta}{2d}\tanh\left[\frac{1}{2}\theta(q+C)\right]$$

and

$$H(q) = -\frac{c}{2d} - \frac{\theta}{2d}\tanh\left(\frac{1}{2}\theta q\right) + \frac{\operatorname{sech}\left(\frac{\theta q}{2}\right)}{C\cosh\left(\frac{\theta q}{2}\right) - \frac{2d}{\theta}\sinh\left(\frac{\theta q}{2}\right)},$$

with $\theta^2 = c^2 - 4de > 0$ and C a constant of integration.

The solutions of the ODE (10) are assumed to be of the form

$$F(q) = \sum_{i=0}^{M} A_i (H(q))^i, \tag{14}$$

where $H(z)$ solves the Bernoulli or Riccati equation, M is a positive integer which is determined by the balancing procedure and $A_i, (i = 0, 1, \cdots, M)$ are parameters to be determined.

Solutions of (3) using Bernoulli as the simplest equation

From equation (10) the balancing procedure yields $M = 1$, so the solutions of (10) can be written as

$$F(q) = A_0 + A_1 H(q). \tag{15}$$

Substituting (15) into (10) and invoking the Bernoulli equation (12) we obtain the algebraic equation

$$\begin{aligned}
&\alpha A_1 c^4 H(q) - A_1\beta c^4 H(q) + 15\alpha A_1 c^3 dH(q)^2 - 15A_1\beta c^3 dH(q)^2 + 2\alpha A_1^2 c^3 kH(q)^2\\
&- 2A_1^2\beta c^3 kH(q)^2 + 50\alpha A_1 c^2 d^2 H(q)^3 - 50A_1\beta c^2 d^2 H(q)^3 + 8\alpha A_1^2 c^2 dkH(q)^3\\
&- 8A_1^2\beta c^2 dkH(q)^3 - \alpha A_1\beta c^2 hH(q) + \alpha A_1 c^2 h\nu H(q) + A_1\beta^2 c^2 hH(q) - 2A_1\beta c^2 h\nu H(q)\\
&+ A_1\beta c^2 hH(q) + A_1 c^2 h\nu^2 H(q) - A_1 c^2 h\nu H(q) - A_1 c^2 k\nu H(q) + 60\alpha A_1 cd^3 H(q)^4\\
&- 60A_1\beta cd^3 H(q)^4 + 10\alpha A_1^2 cd^2 kH(q)^4 - 10A_1^2\beta cd^2 kH(q)^4 - 3\alpha A_1\beta cdhH(q)^2\\
&+ 3\alpha A_1 cdh\nu H(q)^2 + 3A_1\beta^2 cdhH(q)^2 - 6A_1\beta cdh\nu H(q)^2 + 3A_1\beta cdhH(q)^2\\
&+ 3A_1 cdh\nu^2 H(q)^2 - 3A_1 cdh\nu H(q)^2 - 3A_1 cdk\nu H(q)^2 + 24\alpha A_1 d^4 H(q)^5\\
&- 24A_1\beta d^4 H(q)^5 + 4\alpha A_1^2 d^3 kH(q)^5 - 4A_1^2\beta d^3 kH(q)^5 - 2\alpha A_1\beta d^2 hH(q)^3
\end{aligned}$$

$$
\begin{aligned}
&+2\alpha A_1 d^2 h\nu H(q)^3+2A_1\beta^2 d^2 hH(q)^3-4A_1\beta d^2 h\nu H(q)^3+2A_1\beta d^2 hH(q)^3\\
&+2A_1 d^2 h\nu^2 H(q)^3-2A_1 d^2 h\nu H(q)^3-2A_1 d^2 k\nu H(q)^3=0.
\end{aligned}
$$

Equating all coefficients of the function H^i to zero, we obtain the following algebraic system of equations in terms of A_0 and A_1:

$$
\begin{aligned}
&\alpha A_1 c^4-A_1\beta c^4-\alpha A_1\beta c^2 h+\alpha A_1 c^2 h\nu+A_1\beta^2 c^2 h-2A_1\beta c^2 h\nu+A_1\beta c^2 h+A_1 c^2 h\nu^2\\
&-A_1 c^2 h\nu-A_1 c^2 k\nu=0,\\
&15\alpha A_1 c^3 d-15A_1\beta c^3 d+2\alpha A_1^2 c^3 k-2A_1^2\beta c^3 k-3\alpha A_1\beta cdh+3\alpha A_1 cdh\nu+3A_1\beta^2 cdh\\
&-6A_1\beta cdh\nu+3A_1\beta cdh+3A_1 cdh\nu^2-3A_1 cdh\nu-3A_1 cdk\nu=0,\\
&50\alpha A_1 c^2 d^2-50A_1\beta c^2 d^2+8\alpha A_1^2 c^2 dk-8A_1^2\beta c^2 dk-2\alpha A_1\beta d^2 h+2\alpha A_1 d^2 h\nu\\
&+2A_1\beta^2 d^2 h-4A_1\beta d^2 h\nu+2A_1\beta d^2 h+2A_1 d^2 h\nu^2-2A_1 d^2 h\nu-2A_1 d^2 k\nu=0\\
&60\alpha A_1 cd^3-60A_1\beta cd^3+10\alpha A_1^2 cd^2 k-10A_1^2\beta cd^2 k=0\\
&24\alpha A_1 d^4-24A_1\beta d^4+4\alpha A_1^2 d^3 k-4A_1^2\beta d^3 k=0.
\end{aligned}
$$

Solving the above system with the aid of Mathematica, we obtain

$$
\alpha=\beta,\ k=\frac{h(\nu-1)(\nu-\beta)}{\nu},\ A_0=\text{arbitrary},\ A_1=-\frac{6d}{k}.
$$

Thus a solution of the generalized second extended (3+1)-dimensional Jimbo-Miwa equation (3) using the Bernoulli equation as the simplest equation is

$$
u(x,y,z,t)=A_0-\frac{6cd}{k}\left\{\frac{\cosh[c(q+C)]+\sinh[c(q+C)]}{1-d\cosh[c(q+C)]-d\sinh[c(q+C)]}\right\},
$$

where $q=x+(\beta-\alpha)y-\nu z+(\nu-\beta)t$ and C is an arbitrary constant.

Solutions of (3) using Riccati as the simplest equation

Substituting (15) into (10) and using the Riccati equation (13) we obtain

$$
\begin{aligned}
&4d^3k\alpha A_1^2H(q)^5-4d^3k\beta A_1^2H(q)^5+24d^4\alpha A_1H(q)^5-24d^4\beta A_1H(q)^5\\
&+10cd^2k\alpha A_1^2H(q)^4-10cd^2k\beta A_1^2H(q)^4+60cd^3\alpha A_1H(q)^4-60cd^3\beta A_1H(q)^4\\
&+8c^2dk\alpha A_1^2H(q)^3+8d^2ek\alpha A_1^2H(q)^3-8c^2dk\beta A_1^2H(q)^3-8d^2ek\beta A_1^2H(q)^3\\
&+2d^2h\beta^2A_1H(q)^3+2d^2h\nu^2A_1H(q)^3+50c^2d^2\alpha A_1H(q)^3+40d^3e\alpha A_1H(q)^3\\
&-50c^2d^2\beta A_1H(q)^3-40d^3e\beta A_1H(q)^3+2d^2h\beta A_1H(q)^3-2d^2h\alpha\beta A_1H(q)^3\\
&-2d^2h\nu A_1H(q)^3-2d^2k\nu A_1H(q)^3+2d^2h\alpha\nu A_1H(q)^3-4d^2h\beta\nu A_1H(q)^3\\
&+2c^3k\alpha A_1^2H(q)^2+12cdek\alpha A_1^2H(q)^2-2c^3k\beta A_1^2H(q)^2-12cdek\beta A_1^2H(q)^2\\
&+3cdh\beta^2A_1H(q)^2+3cdh\nu^2A_1H(q)^2+15c^3d\alpha A_1H(q)^2+60cd^2e\alpha A_1H(q)^2\\
&-15c^3d\beta A_1H(q)^2-60cd^2e\beta A_1H(q)^2+3cdh\beta A_1H(q)^2-3cdh\alpha\beta A_1H(q)^2\\
&-3cdh\nu A_1H(q)^2-3cdk\nu A_1H(q)^2+3cdh\alpha\nu A_1H(q)^2-6cdh\beta\nu A_1H(q)^2\\
&+4de^2k\alpha A_1^2H(q)+4c^2ek\alpha A_1^2H(q)-4de^2k\beta A_1^2H(q)-4c^2ek\beta A_1^2H(q)\\
&+c^2h\beta^2A_1H(q)+2deh\beta^2A_1H(q)+c^2h\nu^2A_1H(q)+2deh\nu^2A_1H(q)+c^4\alpha A_1H(q)\\
&+16d^2e^2\alpha A_1H(q)+22c^2de\alpha A_1H(q)-c^4\beta A_1H(q)-16d^2e^2\beta A_1H(q)\\
&-22c^2de\beta A_1H(q)+c^2h\beta A_1H(q)+2deh\beta A_1H(q)-c^2h\alpha\beta A_1H(q)
\end{aligned}
$$

$$
\begin{aligned}
&-2deh\alpha\beta A_1H(q)-c^2h\nu A_1H(q)-2deh\nu A_1H(q)-c^2k\nu A_1H(q)-2dek\nu A_1H(q)\\
&+c^2h\alpha\nu A_1H(q)+2deh\alpha\nu A_1H(q)-2c^2h\beta\nu A_1H(q)-4deh\beta\nu A_1H(q)+2ce^2k\alpha A_1^2\\
&-2ce^2k\beta A_1^2+ceh\beta^2A_1+ceh\nu^2A_1+8cde^2\alpha A_1+c^3e\alpha A_1-8cde^2\beta A_1-c^3e\beta A_1\\
&+ceh\beta A_1-ceh\alpha\beta A_1-ceh\nu A_1-cek\nu A_1+ceh\alpha\nu A_1-2ceh\beta\nu A_1=0.
\end{aligned}
$$

As before, equating coefficients of H^i to zero, we obtain

$$
\begin{aligned}
&e\alpha A_1c^3-e\beta A_1c^3+2e^2k\alpha A_1^2c-2e^2k\beta A_1^2c+eh\beta^2A_1c+eh\nu^2A_1c+8de^2\alpha A_1c\\
&-8de^2\beta A_1c+eh\beta A_1c-eh\alpha\beta A_1c-eh\nu A_1c-ek\nu A_1c+eh\alpha\nu A_1c-2eh\beta\nu A_1c=0,\\
&\alpha A_1c^4-\beta A_1c^4+4ek\alpha A_1^2c^2-4ek\beta A_1^2c^2+h\beta^2A_1c^2+h\nu^2A_1c^2+22de\alpha A_1c^2\\
&-22de\beta A_1c^2+h\beta A_1c^2-h\alpha\beta A_1c^2-h\nu A_1c^2-k\nu A_1c^2+h\alpha\nu A_1c^2-2h\beta\nu A_1c^2\\
&+4de^2k\alpha A_1^2-4de^2k\beta A_1^2+2deh\beta^2A_1+2deh\nu^2A_1+16d^2e^2\alpha A_1-16d^2e^2\beta A_1\\
&+2deh\beta A_1-2deh\alpha\beta A_1-2deh\nu A_1-2dek\nu A_1+2deh\alpha\nu A_1-4deh\beta\nu A_1=0,\\
&2k\alpha A_1^2c^3-2k\beta A_1^2c^3+15d\alpha A_1c^3-15d\beta A_1c^3+12dek\alpha A_1^2c-12dek\beta A_1^2c\\
&+3dh\beta^2A_1c+3dh\nu^2A_1c+60d^2e\alpha A_1c-60d^2e\beta A_1c+3dh\beta A_1c-3dh\alpha\beta A_1c\\
&-3dh\nu A_1c-3dk\nu A_1c+3dh\alpha\nu A_1c-6dh\beta\nu A_1c=0,\\
&40e\alpha A_1d^3-40e\beta A_1d^3+8ek\alpha A_1^2d^2-8ek\beta A_1^2d^2+2h\beta^2A_1d^2+2h\nu^2A_1d^2\\
&+50c^2\alpha A_1d^2-50c^2\beta A_1d^2+2h\beta A_1d^2-2h\alpha\beta A_1d^2-2h\nu A_1d^2-2k\nu A_1d^2\\
&+2h\alpha\nu A_1d^2-4h\beta\nu A_1d^2+8c^2k\alpha A_1^2d-8c^2k\beta A_1^2d=0,\\
&60c\alpha A_1d^3-60c\beta A_1d^3+10ck\alpha A_1^2d^2-10ck\beta A_1^2d^2=0,\\
&24\alpha A_1d^4-24\beta A_1d^4+4k\alpha A_1^2d^3-4k\beta A_1^2d^3=0.
\end{aligned}
$$

Solving the above system of algebraic equations we obtain

$$
\alpha=\beta,\ \ k=\frac{h(\nu-1)(\nu-\beta)}{\nu},\ \ A_0=\text{arbitrary},\ \ A_1=-\frac{6d}{k}.
$$

Thus solutions of the generalized second extended (3+1)-dimensional Jimbo-Miwa equation (3) using the Riccati equation as the simplest equation are

$$
u(x,y,z,t)=A_0-\frac{6d}{k}\left\{-\frac{c}{2d}-\frac{\theta}{2d}\tanh\left[\frac{1}{2}\theta(q+C)\right]\right\}
$$

and

$$
u(t,x,y,z)=A_0-\frac{6d}{k}\left\{-\frac{c}{2d}-\frac{\theta}{2d}\tanh\left(\frac{1}{2}\theta q\right)+\frac{\operatorname{sech}\left(\frac{\theta q}{2}\right)}{C\cosh\left(\frac{\theta q}{2}\right)-\frac{2d}{\theta}\sinh\left(\frac{\theta q}{2}\right)}\right\},
$$

where $q=x+(\beta-\alpha)y-\nu z+(\nu-\beta)t$, $\theta^2=c^2-4de>0$ and C is an arbitrary constant.

3 Conservation laws of (3) using Ibragimov's theorem

In this section we derive the conservation laws of (3) by appealing to Ibragimov's new conservation theorem [31].

We begin by determining the adjoint equation of (3) by utilizing

$$F^* \equiv \frac{\delta}{\delta u}\left(v(u_{xxxy}+k\left(u_y u_x\right)_x+h(u_{xt}+u_{yt}+u_{zt})-ku_{xz})\right)=0, \tag{16}$$

where $\delta/\delta u$ is the Euler-Lagrange operator defined by

$$\begin{aligned}\frac{\delta}{\delta u} = &-D_x\frac{\partial}{\partial u_x}-D_y\frac{\partial}{\partial u_y}+D_x^2\frac{\partial}{\partial u_{xx}}+D_xD_t\frac{\partial}{\partial u_{xt}}+D_yD_t\frac{\partial}{\partial u_{yt}}+D_zD_t\frac{\partial}{\partial u_{zt}}\\ &+D_xD_z\frac{\partial}{\partial u_{xz}}+D_xD_y\frac{\partial}{\partial u_{xy}}+D_x^3D_y\frac{\partial}{\partial u_{xxxy}}\end{aligned} \tag{17}$$

and the total differential operators D_t, D_x, D_y and D_z are given by

$$\begin{aligned}D_t &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{tx}\frac{\partial}{\partial u_x}+u_{ty}\frac{\partial}{\partial u_y}+u_{tz}\frac{\partial}{\partial u_z}+\cdots,\\ D_x &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+u_{xy}\frac{\partial}{\partial u_y}+u_{xz}\frac{\partial}{\partial u_z}+\cdots,\\ D_y &= \frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}+u_{yy}\frac{\partial}{\partial u_y}+u_{yt}\frac{\partial}{\partial u_t}+u_{yx}\frac{\partial}{\partial u_x}+u_{yz}\frac{\partial}{\partial u_z}+\cdots,\\ D_z &= \frac{\partial}{\partial z}+u_z\frac{\partial}{\partial u}+u_{zz}\frac{\partial}{\partial u_z}+u_{zt}\frac{\partial}{\partial u_t}+u_{zy}\frac{\partial}{\partial u_y}+u_{zx}\frac{\partial}{\partial u_x}+\cdots.\end{aligned} \tag{18}$$

Thus the adjoint equation (16) becomes

$$hv_{xt}+hv_{yt}+hv_{zt}+2kv_xu_{xy}+ku_xv_{xy}+ku_yv_{xx}-kv_{xz}+v_{xxxy}=0. \tag{19}$$

The Lagrangian of (3) and its adjoint equation (19) is

$$\mathscr{L} = v\left(h\left(u_{xt}+u_{yt}+u_{zt}\right)+ku_xu_{xy}+ku_{xx}u_y-ku_{xz}+u_{xxxy}\right) \tag{20}$$

and the extended symmetries [31] are

$$\begin{aligned}&Y_1=\frac{\partial}{\partial t},\ Y_2=\frac{\partial}{\partial x},\ Y_3=\frac{\partial}{\partial y},\ Y_4=\frac{\partial}{\partial z},\ Y_5=f_1(t)\frac{\partial}{\partial u},\ Y_6=f_2(z)\frac{\partial}{\partial u},\\ &Y_7=-3ht\frac{\partial}{\partial t}+(2kt+hz-hx)\frac{\partial}{\partial x}+(2hx+hy-4hz+hu)\frac{\partial}{\partial u},\\ &Y_8=ht\frac{\partial}{\partial t}-(kt+hz)\frac{\partial}{\partial x}-hz\frac{\partial}{\partial y}-hz\frac{\partial}{\partial z}+(kt+2hz)\frac{\partial}{\partial u},\\ &Y_9=ht\frac{\partial}{\partial t}-kt\frac{\partial}{\partial x}+(hy-hz)\frac{\partial}{\partial y}+(kt-hy+2hz)\frac{\partial}{\partial u}-hv\frac{\partial}{\partial v}.\end{aligned}$$

To obtain the conserved vectors corresponding to the Lie point symmetries (5) and the Lagrangian (20) we use [31]

$$T^i=\xi^i\mathscr{L}+W^\alpha\left[\frac{\partial\mathscr{L}}{\partial u_i^\alpha}-D_k\frac{\partial\mathscr{L}}{\partial u_{ik}^\alpha}+\cdots\right]+D_k(W^\alpha)\left[\frac{\partial\mathscr{L}}{\partial u_{ik}^\alpha}-D_k\frac{\partial\mathscr{L}}{\partial u_{ijk}^\alpha}+\cdots\right]+\cdots,$$

where W^α is the Lie characteristic function given by $W^\alpha=\eta^\alpha-\xi^j u_j^\alpha$, $\alpha=1,2$ and j runs from $1,\cdots,4$ in this particular case. Thus the conserved vectors corresponding to the nine Lie point symmetries are given by, respectively

$$T_x^1=\frac{1}{2}ku_{zt}v-\frac{1}{2}hu_{tt}v-\frac{1}{2}ku_xu_{yt}v-ku_yu_{xt}v+\frac{1}{2}ku_tu_{xy}v-\frac{3}{4}u_{xxyt}v+\frac{1}{2}hu_tv_t$$

$$
\begin{aligned}
&+\frac{1}{2}ku_tu_xv_y+ku_tu_yv_x-\frac{1}{2}ku_tv_z-\frac{1}{2}u_{xt}v_{xy}+\frac{1}{2}v_xu_{xyt}-\frac{1}{4}v_{xx}u_{yt}\\
&+\frac{1}{4}v_yu_{xxt}+\frac{3}{4}u_tv_{xxy},\\
T_y^1 =&-\frac{1}{2}hu_{tt}v-\frac{1}{2}ku_xu_{xt}v-\frac{1}{2}ku_tu_{xx}v-\frac{1}{4}u_{xxxt}v+\frac{1}{2}hu_tv_t+\frac{1}{2}ku_tu_xv_x\\
&-\frac{1}{4}v_{xx}u_{xt}+\frac{1}{4}v_xu_{xxt}+\frac{1}{4}u_tv_{xxx},\\
T_z^1 =&-\frac{1}{2}hu_{tt}v+\frac{1}{2}ku_{xt}v+\frac{1}{2}hu_tv_t-\frac{1}{2}ku_tv_x,\\
T_t^1 =&\frac{1}{2}hu_{zt}v+\frac{1}{2}hu_{yt}v+\frac{1}{2}hu_{xt}v-ku_{xz}v+ku_xu_{xy}v+ku_{xx}u_yv+u_{xxxy}v\\
&+\frac{1}{2}hu_tv_x+\frac{1}{2}hu_tv_y+\frac{1}{2}hu_tv_z;
\end{aligned}
$$

$$
\begin{aligned}
T_x^2 =&hu_{zt}v+hu_{yt}v+\frac{1}{2}hu_{xt}v+ku_xu_{xy}v-\frac{1}{2}ku_{xz}v+\frac{1}{4}u_{xxxy}v+\frac{1}{2}hv_tu_x\\
&+\frac{1}{2}ku_x^2v_y+ku_xu_yv_x-\frac{1}{2}ku_xv_z+\frac{3}{4}u_xv_{xxy}-\frac{1}{2}u_{xx}v_{xy}-\frac{1}{4}v_{xx}u_{xy}\\
&+\frac{1}{2}v_xu_{xxy}+\frac{1}{4}u_{xxx}v_y,\\
T_y^2 =&\frac{1}{2}hv_tu_x-\frac{1}{2}hu_{xt}v-ku_{xx}u_xv-\frac{1}{4}u_{xxxx}v+\frac{1}{2}ku_x^2v_x+\frac{1}{4}u_xv_{xxx}\\
&-\frac{1}{4}u_{xx}v_{xx}+\frac{1}{4}u_{xxx}v_x,\\
T_z^2 =&-\frac{1}{2}hu_{xt}v+\frac{1}{2}ku_{xx}v+\frac{1}{2}hv_tu_x-\frac{1}{2}ku_xv_x,\\
T_t^2 =&-\frac{1}{2}hu_{xz}v-\frac{1}{2}hu_{xy}v-\frac{1}{2}hu_{xx}v+\frac{1}{2}hu_xv_y+\frac{1}{2}hu_xv_z+\frac{1}{2}hu_xv_x;
\end{aligned}
$$

$$
\begin{aligned}
T_x^3 =&-\frac{1}{2}hu_{yt}v-\frac{1}{2}ku_yu_{xy}v+\frac{1}{2}ku_{yz}v-\frac{1}{2}ku_xu_{yy}v-\frac{3}{4}u_{xxyy}v+\frac{1}{2}hv_tu_y\\
&+ku_y^2v_x+\frac{1}{2}ku_xu_yv_y-\frac{1}{2}ku_yv_z+\frac{3}{4}u_yv_{xxy}-\frac{1}{2}u_{xy}v_{xy}+\frac{1}{2}v_xu_{xyy}\\
&-\frac{1}{4}u_{yy}v_{xx}+\frac{1}{4}v_yu_{xxy},\\
T_y^3 =&hu_{zt}v+\frac{1}{2}hu_{yt}v+hu_{xt}v-ku_{xz}v+\frac{1}{2}ku_xu_{xy}v+\frac{1}{2}ku_{xx}u_yv+\frac{3}{4}u_{xxxy}v\\
&+\frac{1}{2}hv_tu_y+\frac{1}{2}ku_xu_yv_x-\frac{1}{4}v_{xx}u_{xy}+\frac{1}{4}v_xu_{xxy}+\frac{1}{4}u_yv_{xxx},\\
T_z^3 =&-\frac{1}{2}hu_{yt}v+\frac{1}{2}ku_{xy}v+\frac{1}{2}hv_tu_y-\frac{1}{2}ku_yv_x,\\
T_t^3 =&-\frac{1}{2}hu_{yz}v-\frac{1}{2}hu_{yy}v-\frac{1}{2}hu_{xy}v+\frac{1}{2}hu_yv_x+\frac{1}{2}hu_yv_z+\frac{1}{2}hu_yv_y;
\end{aligned}
$$

$$
\begin{aligned}
T_x^4 =&kv_xu_y^2+\frac{1}{2}hv_tu_y-\frac{1}{2}kv_zu_y+\frac{1}{2}kv_yu_xu_y+ku_zv_xu_y+ku_xv_xu_y-kvu_{xz}u_y\\
&-\frac{1}{2}kvu_{xy}u_y+\frac{3}{4}v_{xxy}u_y+\frac{1}{2}kv_yu_x^2+\frac{1}{2}hv_tu_z-\frac{1}{2}ku_zv_z+\frac{1}{2}hvu_{zt}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}kvu_{zz}+\frac{1}{2}hvu_{yt}+\frac{1}{2}kvu_{yz}+\frac{1}{2}hv_tu_x-\frac{1}{2}kv_zu_x+\frac{1}{2}ku_zv_yu_x\\
&-\frac{1}{2}kvu_{yz}u_x-\frac{1}{2}kvu_{yy}u_x+\frac{1}{2}hvu_{xt}-\frac{1}{2}kvu_{xz}+\frac{1}{2}kvu_zu_{xy}+kvu_xu_{xy}\\
&-\frac{1}{2}u_{xz}v_{xy}-\frac{1}{2}u_{xy}v_{xy}+\frac{1}{2}v_xu_{xyz}+\frac{1}{2}v_xu_{xyy}-\frac{1}{2}v_{xy}u_{xx}-\frac{1}{4}u_{yz}v_{xx}\\
&-\frac{1}{4}u_{yy}v_{xx}-\frac{1}{4}u_{xy}v_{xx}+\frac{1}{4}v_yu_{xxz}+\frac{1}{4}v_yu_{xxy}+\frac{1}{2}v_xu_{xxy}+\frac{3}{4}u_zv_{xxy}\\
&+\frac{3}{4}u_xv_{xxy}-\frac{3}{4}vu_{xxyz}-\frac{3}{4}vu_{xxyy}+\frac{1}{4}v_yu_{xxx}+\frac{1}{4}vu_{xxxy},
\end{aligned}
$$

$$
\begin{aligned}
T_y^4=&\frac{1}{2}hu_{zt}v+\frac{1}{2}hu_{yt}v+\frac{1}{2}hu_{xt}v-\frac{1}{2}ku_xu_{xz}v+\frac{1}{2}ku_xu_{xy}v-ku_{xx}u_xv\\
&-ku_{xz}v-\frac{1}{2}ku_{xx}u_zv+\frac{1}{2}ku_{xx}u_yv-\frac{1}{4}u_{xxxz}v+\frac{3}{4}u_{xxxy}v-\frac{1}{4}u_{xxxx}v\\
&+\frac{1}{2}hv_tu_x+\frac{1}{2}hv_tu_y+\frac{1}{2}hv_tu_z+\frac{1}{2}ku_xu_yv_x+\frac{1}{2}ku_xu_zv_x+\frac{1}{2}ku_x^2v_x\\
&-\frac{1}{4}v_{xx}u_{xy}+\frac{1}{4}v_xu_{xxy}+\frac{1}{4}u_yv_{xxx}-\frac{1}{4}v_{xx}u_{xz}+\frac{1}{4}v_xu_{xxz}+\frac{1}{4}u_zv_{xxx}\\
&+\frac{1}{4}u_xv_{xxx}-\frac{1}{4}u_{xx}v_{xx}+\frac{1}{4}u_{xxx}v_x,
\end{aligned}
$$

$$
\begin{aligned}
T_z^4=&\frac{1}{2}hu_{zt}v+\frac{1}{2}hu_{yt}v+\frac{1}{2}hu_{xt}v-\frac{1}{2}ku_{xz}v+\frac{1}{2}ku_{xy}v+ku_xu_{xy}v+\frac{1}{2}ku_{xx}v\\
&+ku_{xx}u_yv+u_{xxxy}v+\frac{1}{2}hv_tu_x+\frac{1}{2}hv_tu_y+\frac{1}{2}hv_tu_z-\frac{1}{2}ku_yv_x\\
&-\frac{1}{2}ku_zv_x-\frac{1}{2}ku_xv_x,
\end{aligned}
$$

$$
\begin{aligned}
T_t^4=&\frac{1}{2}hu_yv_x-\frac{1}{2}hu_{zz}v-hu_{yz}v-\frac{1}{2}hu_{yy}v-hu_{xz}v-hu_{xy}v-\frac{1}{2}hu_{xx}v+\frac{1}{2}hu_xv_y\\
&+\frac{1}{2}hu_xv_z+\frac{1}{2}hu_zv_x+\frac{1}{2}hu_xv_x+\frac{1}{2}hu_yv_z+\frac{1}{2}hu_zv_y+\frac{1}{2}hu_yv_y+\frac{1}{2}hu_zv_z;
\end{aligned}
$$

$$
\begin{aligned}
T_x^5&=\frac{1}{2}hf_1'(t)v+f_1(t)\left(\frac{1}{2}kv_z-\frac{1}{2}hv_t-\frac{1}{2}ku_xv_y-ku_yv_x-\frac{3}{4}v_{xxy}-\frac{1}{2}ku_{xy}v\right),\\
T_y^5&=\frac{1}{2}kf_1(t)u_{xx}v+\frac{1}{2}hf_1'(t)v-\frac{1}{2}hf(t)v_t-\frac{1}{2}kf(t)u_xv_x-\frac{1}{4}f(t)v_{xxx},\\
T_z^5&=\frac{1}{2}hf_1'(t)v-\frac{1}{2}hf_1(t)v_t+\frac{1}{2}kf_1(t)v_x,\\
T_t^5&=-\frac{1}{2}hf_1(t)v_x-\frac{1}{2}hf_1(t)v_y-\frac{1}{2}hf_1(t)v_z;
\end{aligned}
$$

$$
\begin{aligned}
T_x^6&=-\frac{1}{2}kf_2'(z)v+f_2(z)\left(\frac{1}{2}kv_z-\frac{1}{2}hv_t-\frac{1}{2}ku_xv_y-ku_yv_x-\frac{3}{4}v_{xxy}-\frac{1}{2}ku_{xy}v\right),\\
T_y^6&=\frac{1}{2}kf_2(z)u_{xx}v-\frac{1}{2}hf_2(z)v_t-\frac{1}{2}kf_2(z)u_xv_x-\frac{1}{4}f_2(z)v_{xxx},\\
T_z^6&=\frac{1}{2}kf_2(z)v_x-\frac{1}{2}hf_2(z)v_t,\\
T_t^6&=\frac{1}{2}hf_2'(z)v-\frac{1}{2}hf_2(z)v_x-\frac{1}{2}hf_2(z)v_y-\frac{1}{2}hf_2(z)v_z;
\end{aligned}
$$

$$
\begin{aligned}
T_x^7 =& 2vu_th^2 - xv_th^2 - \frac{1}{2}yv_th^2 + 2zv_th^2 - \frac{1}{2}uv_th^2 - \frac{3}{2}tu_tv_th^2 + \frac{3}{2}tvu_{tt}h^2 \\
& - xvu_{zt}h^2 + zvu_{zt}h^2 - xvu_{yt}h^2 + zvu_{yt}h^2 - \frac{1}{2}xv_tu_xh^2 + \frac{1}{2}zv_tu_xh^2 \\
& - \frac{1}{2}xvu_{xt}h^2 + \frac{1}{2}zvu_{xt}h^2 - \frac{1}{2}kxv_yu_x^2h + \frac{1}{2}kzv_yu_x^2h + 2kvh \\
& - \frac{1}{2}kvu_zh + kxv_zh + \frac{1}{2}kyv_zh - 2kzv_zh + \frac{1}{2}kuv_zh + \frac{3}{2}ktu_tv_zh \\
& + \frac{1}{2}ktvu_{zt}h + 2kvu_yh + 2ktvu_{yt}h + ktv_tu_xh + \frac{1}{2}kxv_zu_xh \\
& - \frac{1}{2}kzv_zu_xh + \frac{5}{2}kvu_yu_xh - kxv_yu_xh - \frac{1}{2}kyv_yu_xh + 2kzv_yu_xh \\
& - \frac{1}{2}kuv_yu_xh - \frac{3}{2}ktu_tv_yu_xh + \frac{3}{2}ktvu_{yt}u_xh - 2kxu_yv_xh - kyu_yv_xh \\
& + 4kzu_yv_xh - kuu_yv_xh - 3ktu_tu_yv_xh - kxu_yu_xv_xh + kzu_yu_xv_xh \\
& + ktvu_{xt}h + 3ktvu_yu_{xt}h + \frac{1}{2}kxvu_{xz}h - \frac{1}{2}kzvu_{xz}h - kxvu_{xy}h \\
& - \frac{1}{2}kyvu_{xy}h + 2kzvu_{xy}h - \frac{1}{2}kuvu_{xy}h - \frac{3}{2}ktvu_tu_{xy}h - kxvu_xu_{xy}h \\
& + kzvu_xu_{xy}h - v_xu_{xy}h + u_xv_{xy}h + \frac{3}{2}tu_{xt}v_{xy}h + v_{xy}h - \frac{3}{2}tv_xu_{xyt}h \\
& - \frac{3}{4}v_yu_{xx}h + \frac{1}{2}xv_{xy}u_{xx}h - \frac{1}{2}zv_{xy}u_{xx}h + \frac{1}{4}u_yv_{xx}h + \frac{3}{4}tu_{yt}v_{xx}h \\
& + \frac{1}{4}xu_{xy}v_{xx}h - \frac{1}{4}zu_{xy}v_{xx}h + \frac{1}{4}v_{xx}h - \frac{3}{4}tv_yu_{xxt}h + \frac{9}{4}vu_{xxy}h \\
& - \frac{1}{2}xv_xu_{xxy}h + \frac{1}{2}zv_xu_{xxy}h - \frac{3}{2}xv_{xxy}h - \frac{3}{4}yv_{xxy}h + 3zv_{xxy}h \\
& - \frac{3}{4}uv_{xxy}h - \frac{9}{4}tu_tv_{xxy}h - \frac{3}{4}xu_xv_{xxy}h + \frac{3}{4}zu_xv_{xxy}h + \frac{9}{4}tvu_{xxyt}h \\
& - \frac{1}{4}xv_yu_{xxx}h + \frac{1}{4}zv_yu_{xxx}h - \frac{1}{4}xvu_{xxxy}h + \frac{1}{4}zvu_{xxxy}h + k^2tv_yu_x^2 \\
& - k^2tv_zu_x + 2k^2tu_yu_xv_x - k^2tvu_{xz} + 2k^2tvu_xu_{xy} - ktv_{xy}u_{xx} \\
& - \frac{1}{2}ktu_{xy}v_{xx} + ktv_xu_{xxy} + \frac{3}{2}ktu_xv_{xxy} + \frac{1}{2}ktv_yu_{xxx} + \frac{1}{2}ktvu_{xxxy}, \\
T_y^7 =& 2vu_th^2 - xv_th^2 - \frac{1}{2}yv_th^2 + 2zv_th^2 - \frac{1}{2}uv_th^2 - \frac{3}{2}tu_tv_th^2 + \frac{3}{2}tvu_{tt}h^2 \\
& - \frac{1}{2}xv_tu_xh^2 + \frac{1}{2}zv_tu_xh^2 + \frac{1}{2}xvu_{xt}h^2 - \frac{1}{2}zvu_{xt}h^2 + kvu_x^2h + ktv_tu_xh \\
& - \frac{1}{2}kxu_x^2v_xh + \frac{1}{2}kzu_x^2v_xh - kxu_xv_xh - \frac{1}{2}kyu_xv_xh + 2kzu_xv_xh \\
& - \frac{1}{2}kuu_xv_xh - \frac{3}{2}ktu_tu_xv_xh - ktvu_{xt}h + \frac{3}{2}ktvu_xu_{xt}h + kxvu_{xx}h \\
& + \frac{1}{2}kyvu_{xx}h - 2kzvu_{xx}h + \frac{1}{2}kuvu_{xx}h + \frac{3}{2}ktvu_tu_{xx}h + kxvu_xu_{xx}h \\
& - kzvu_xu_{xx}h - \frac{3}{4}v_xu_{xx}h + \frac{1}{2}u_xv_{xx}h + \frac{3}{4}tu_{xt}v_{xx}h + \frac{1}{4}xu_{xx}v_{xx}h \\
& - \frac{1}{4}zu_{xx}v_{xx}h + \frac{1}{2}v_{xx}h - \frac{3}{4}tv_xu_{xxt}h + vu_{xxx}h - \frac{1}{4}xv_xu_{xxx}h + \frac{1}{4}zv_xu_{xxx}h \\
& - \frac{1}{2}xv_{xxx}h - \frac{1}{4}yv_{xxx}h + zv_{xxx}h - \frac{1}{4}uv_{xxx}h - \frac{3}{4}tu_tv_{xxx}h - \frac{1}{4}xu_xv_{xxx}h
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}zu_xv_{xxx}h+\frac{3}{4}tvu_{xxxt}h+\frac{1}{4}xvu_{xxxx}h-\frac{1}{4}zvu_{xxxx}h+k^2tu_x^2v_x\\
&-2k^2tvu_xu_{xx}-\frac{1}{2}ktu_{xx}v_{xx}+\frac{1}{2}ktv_xu_{xxx}+\frac{1}{2}ktu_xv_{xxx}-\frac{1}{2}ktvu_{xxxx},\\
T_z^7=&\,2h^2u_tv-\frac{1}{2}h^2v_tu+\frac{3}{2}h^2tu_{tt}v+\frac{1}{2}h^2xu_{xt}v-\frac{1}{2}h^2zu_{xt}v-2hku_xv\\
&+\frac{1}{2}hkv_xu-\frac{5}{2}hktu_{xt}v-\frac{1}{2}hkxu_{xx}v+\frac{1}{2}hkzu_{xx}v+k^2tu_{xx}v-hkv\\
&+\frac{1}{2}h^2zv_tu_x-\frac{1}{2}h^2xv_tu_x-\frac{3}{2}h^2tu_tv_t-h^2xv_t-\frac{1}{2}h^2yv_t+2h^2zv_t+hktv_tu_x\\
&+\frac{3}{2}hktu_tv_x-\frac{1}{2}hkzu_xv_x+\frac{1}{2}hkxu_xv_x+\frac{1}{2}hkyv_x-2hkzv_x+hkxv_x-k^2tu_xv_x,\\
T_t^7=&-\frac{1}{2}vh^2+\frac{1}{2}vu_zh^2-xv_zh^2-\frac{1}{2}yv_zh^2+2zv_zh^2-\frac{1}{2}uv_zh^2-\frac{3}{2}tu_tv_zh^2\\
&-\frac{3}{2}tvu_{zt}h^2+\frac{1}{2}vu_yh^2-xv_yh^2-\frac{1}{2}yv_yh^2+2zv_yh^2-\frac{1}{2}uv_yh^2-\frac{3}{2}tu_tv_yh^2\\
&-\frac{3}{2}tvu_{yt}h^2+\frac{1}{2}vu_xh^2-\frac{1}{2}xv_zu_xh^2+\frac{1}{2}zv_zu_xh^2-\frac{1}{2}xv_yu_xh^2+\frac{1}{2}zv_yu_xh^2\\
&-xv_xh^2-\frac{1}{2}yv_xh^2+2zv_xh^2-\frac{1}{2}uv_xh^2-\frac{3}{2}tu_tv_xh^2-\frac{1}{2}xu_xv_xh^2\\
&+\frac{1}{2}zu_xv_xh^2-\frac{3}{2}tvu_{xt}h^2+\frac{1}{2}xvu_{xz}h^2-\frac{1}{2}zvu_{xz}h^2+\frac{1}{2}xvu_{xy}h^2\\
&-\frac{1}{2}zvu_{xy}h^2+\frac{1}{2}xvu_{xx}h^2-\frac{1}{2}zvu_{xx}h^2+ktv_zu_xh+ktv_yu_xh+ktu_xv_xh\\
&+kht\left(2vu_{xz}-vu_{xy}-3vu_xu_{xy}-vu_{xx}-3vu_yu_{xx}\right)-3htvu_{xxxy};
\end{aligned}
$$

$$
\begin{aligned}
T_x^8=&\,\frac{1}{2}tu_tv_th^2-\frac{1}{2}vu_th^2-\frac{1}{2}tvu_{tt}h^2-\frac{1}{2}zv_tu_zh^2-\frac{1}{2}zvu_{zt}h^2-\frac{1}{2}zv_tu_yh^2\\
&-\frac{1}{2}zvu_{yt}h^2-\frac{1}{2}zv_tu_xh^2-\frac{1}{2}zvu_{xt}h^2-\frac{1}{2}kzv_yu_x^2h-\frac{1}{2}kvh-\frac{1}{2}ktv_th\\
&-\frac{1}{2}kvu_zh+kzv_zh-\frac{1}{2}ktu_tv_zh+\frac{1}{2}kzu_zv_zh-\frac{1}{2}ktvu_{zt}h-\frac{1}{2}kzvu_{zz}h\\
&-\frac{1}{2}kvu_yh+\frac{1}{2}kzv_zu_yh-ktvu_{yt}h-\frac{1}{2}kzvu_{yz}h-\frac{1}{2}ktv_tu_xh+\frac{1}{2}kzv_zu_xh\\
&-kzv_yu_xh+\frac{1}{2}ktu_tv_yu_xh-\frac{1}{2}kzu_zv_yu_xh-\frac{1}{2}kzu_yv_yu_xh-\frac{1}{2}ktvu_{yt}u_xh\\
&+\frac{1}{2}kzvu_{yz}u_xh+\frac{1}{2}kzvu_{yy}u_xh-kzu_y^2v_xh-2kzu_yv_xh+ktu_tu_yv_xh\\
&-kzu_zu_yv_xh-kzu_yu_xv_xh-\frac{1}{2}ktvu_{xt}h-ktvu_yu_{xt}h+\frac{1}{2}kzvu_{xz}h\\
&+kzvu_yu_{xz}h-kzvu_{xy}h+\frac{1}{2}ktvu_tu_{xy}h-\frac{1}{2}kzvu_zu_{xy}h+\frac{1}{2}kzvu_yu_{xy}h\\
&-kzvu_xu_{xy}h-\frac{1}{2}tu_{xt}v_{xy}h+\frac{1}{2}zu_{xz}v_{xy}h+\frac{1}{2}zu_{xy}v_{xy}h+\frac{1}{2}tv_xu_{xyt}h\\
&-\frac{1}{2}zv_xu_{xyz}h-\frac{1}{2}zv_xu_{xyy}h+\frac{1}{2}zv_{xy}u_{xx}h-\frac{1}{4}tu_{yt}v_{xx}h+\frac{1}{4}zu_{yz}v_{xx}h\\
&+\frac{1}{4}zu_{yy}v_{xx}h+\frac{1}{4}zu_{xy}v_{xx}h+\frac{1}{4}tv_yu_{xxt}h-\frac{1}{4}zv_yu_{xxz}h-\frac{1}{4}zv_yu_{xxy}h\\
&-\frac{1}{2}zv_xu_{xxy}h-\frac{3}{2}zv_{xxy}h+\frac{3}{4}tu_tv_{xxy}h-\frac{3}{4}zu_zv_{xxy}h-\frac{3}{4}zu_yv_{xxy}h
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{4}zu_xv_{xxy}h-\frac{3}{4}tvu_{xxyt}h+\frac{3}{4}zvu_{xxyz}h+\frac{3}{4}zvu_{xxyy}h-\frac{1}{4}zv_yu_{xxx}h\\
&-\frac{1}{4}zvu_{xxxy}h-\frac{1}{2}k^2tv_yu_x^2+\frac{1}{2}k^2tv_z+\frac{1}{2}k^2tv_zu_x-\frac{1}{2}k^2tv_yu_x-k^2tu_yv_x\\
&-k^2tu_yu_xv_x+\frac{1}{2}k^2tvu_{xz}-\frac{1}{2}k^2tvu_{xy}-k^2tvu_xu_{xy}+\frac{1}{2}ktv_{xy}u_{xx}+\frac{1}{4}ktu_{xy}v_{xx}\\
&-\frac{1}{2}ktv_xu_{xxy}-\frac{3}{4}ktv_{xxy}-\frac{3}{4}ktu_xv_{xxy}-\frac{1}{4}ktv_yu_{xxx}-\frac{1}{4}ktvu_{xxxy}-zv_th^2,\\
T_y^8=&-\frac{1}{2}vu_th^2-zv_th^2+\frac{1}{2}tu_tv_th^2-\frac{1}{2}tvu_{tt}h^2-\frac{1}{2}zv_tu_zh^2-\frac{1}{2}zvu_{zt}h^2-\frac{1}{2}zv_tu_yh^2\\
&-\frac{1}{2}zvu_{yt}h^2-\frac{1}{2}zv_tu_xh^2-\frac{1}{2}zvu_{xt}h^2+\frac{1}{2}kvh-\frac{1}{2}ktv_th+\frac{1}{2}kvu_xh-\frac{1}{2}ktv_tu_xh\\
&-\frac{1}{2}kzu_x^2v_xh-kzu_xv_xh+\frac{1}{2}ktu_tu_xv_xh-\frac{1}{2}kzu_zu_xv_xh-\frac{1}{2}kzu_yu_xv_xh\\
&+\frac{1}{2}ktvu_{xt}h-\frac{1}{2}ktvu_xu_{xt}h+kzvu_{xz}h+\frac{1}{2}kzvu_xu_{xz}h-\frac{1}{2}kzvu_xu_{xy}h+kzvu_{xx}h\\
&-\frac{1}{2}ktvu_tu_{xx}h+\frac{1}{2}kzvu_zu_{xx}h-\frac{1}{2}kzvu_yu_{xx}h+kzvu_xu_{xx}h-\frac{1}{4}tu_{xt}v_{xx}h\\
&+\frac{1}{4}zu_{xz}v_{xx}h+\frac{1}{4}zu_{xy}v_{xx}h+\frac{1}{4}zu_{xx}v_{xx}h+\frac{1}{4}tv_xu_{xxt}h-\frac{1}{4}zv_xu_{xxz}h-\frac{1}{4}zv_xu_{xxy}h\\
&-\frac{1}{4}zv_xu_{xxx}h-\frac{1}{2}zv_{xxx}h+\frac{1}{4}tu_tv_{xxx}h-\frac{1}{4}zu_zv_{xxx}h-\frac{1}{4}zu_yv_{xxx}h-\frac{1}{4}zu_xv_{xxx}h\\
&-\frac{1}{4}tvu_{xxxt}h+\frac{1}{4}zvu_{xxxz}h-\frac{3}{4}zvu_{xxxy}h+\frac{1}{4}zvu_{xxxx}h-\frac{1}{2}k^2tu_x^2v_x-\frac{1}{2}k^2tu_xv_x\\
&+\frac{1}{2}k^2tvu_{xx}+k^2tvu_xu_{xx}+\frac{1}{4}kt\left(u_{xx}v_{xx}-v_xu_{xxx}-v_{xxx}-u_xv_{xxx}+vu_{xxxx}\right),\\
T_z^8=&-\frac{1}{2}h^2u_tv-\frac{1}{2}h^2tu_{tt}v-\frac{1}{2}h^2zu_{zt}v-\frac{1}{2}h^2zu_{yt}v-\frac{1}{2}h^2zu_{xt}v+\frac{1}{2}hku_xv+hktu_{xt}v\\
&+\frac{1}{2}hkzu_{xz}v-\frac{1}{2}hkzu_{xy}v-hkzu_xu_{xy}v-\frac{1}{2}hkzu_{xx}v-hkzu_{xx}u_yv-hzu_{xxxy}v\\
&-\frac{1}{2}k^2tu_{xx}v+\frac{1}{2}hkv-\frac{1}{2}h^2zv_tu_x-\frac{1}{2}h^2zv_tu_y-\frac{1}{2}h^2zv_tu_z+\frac{1}{2}h^2tu_tv_t-h^2zv_t\\
&-\frac{1}{2}hktv_tu_x-\frac{1}{2}hktu_tv_x-\frac{1}{2}hktv_t+\frac{1}{2}hkzu_yv_x+\frac{1}{2}hkzu_zv_x+\frac{1}{2}hkzu_xv_x\\
&+hkzv_x+\frac{1}{2}k^2tu_xv_x+\frac{1}{2}k^2tv_x,\\
T_t^8=&\,vh^2+\frac{1}{2}vu_zh^2-zv_zh^2+\frac{1}{2}tu_tv_zh^2-\frac{1}{2}zu_zv_zh^2+\frac{1}{2}tvu_{zt}h^2+\frac{1}{2}zvu_{zz}h^2+\frac{1}{2}vu_yh^2\\
&-\frac{1}{2}zv_zu_yh^2-zv_yh^2+\frac{1}{2}tu_tv_yh^2-\frac{1}{2}zu_zv_yh^2-\frac{1}{2}zu_yv_yh^2+\frac{1}{2}tvu_{yt}h^2+zvu_{yz}h^2\\
&+\frac{1}{2}zvu_{yy}h^2+\frac{1}{2}vu_xh^2-\frac{1}{2}zv_zu_xh^2-\frac{1}{2}zv_yu_xh^2-zv_xh^2+\frac{1}{2}tu_tv_xh^2-\frac{1}{2}zu_zv_xh^2\\
&-\frac{1}{2}zu_yv_xh^2-\frac{1}{2}zu_xv_xh^2+\frac{1}{2}tvu_{xt}h^2+zvu_{xz}h^2+zvu_{xy}h^2+\frac{1}{2}zvu_{xx}h^2-\frac{1}{2}ktv_zh\\
&-\frac{1}{2}ktv_yh-\frac{1}{2}ktv_zu_xh-\frac{1}{2}ktv_yu_xh-\frac{1}{2}ktv_xh-\frac{1}{2}ktu_xv_xh-\frac{1}{2}ktvu_{xz}h\\
&+\frac{1}{2}ktvu_{xy}h+ktvu_xu_{xy}h+\frac{1}{2}ktvu_{xx}h+ktvu_yu_{xx}h+tvu_{xxxy}h;
\end{aligned}
$$

$$
\begin{aligned}
T_x^9 = &-\frac{1}{2}vu_th^2+\frac{1}{2}yv_th^2-zv_th^2+\frac{1}{2}tu_tv_th^2-\frac{1}{2}tvu_{tt}h^2+\frac{1}{2}yv_tu_yh^2-\frac{1}{2}zv_tu_yh^2\\
&-\frac{1}{2}yvu_{yt}h^2+\frac{1}{2}zvu_{yt}h^2-\frac{1}{2}kvh-\frac{1}{2}ktv_th-\frac{1}{2}kyv_zh+kzv_zh-\frac{1}{2}ktu_tv_zh\\
&-\frac{1}{2}ktvu_{zt}h-\frac{1}{2}kvu_yh-\frac{1}{2}kyv_zu_yh+\frac{1}{2}kzv_zu_yh-ktvu_{yt}h+\frac{1}{2}kyvu_{yz}h\\
&-\frac{1}{2}kzvu_{yz}h-\frac{1}{2}ktv_tu_xh-\frac{1}{2}kvu_yu_xh+\frac{1}{2}kyv_yu_xh-kzv_yu_xh+\frac{1}{2}ktu_tv_yu_xh\\
&+\frac{1}{2}kyu_yv_yu_xh-\frac{1}{2}kzu_yv_yu_xh-\frac{1}{2}ktvu_{yt}u_xh-\frac{1}{2}kyvu_{yy}u_xh+\frac{1}{2}kzvu_{yy}u_xh\\
&+kyu_y^2v_xh-kzu_y^2v_xh+kyu_yv_xh-2kzu_yv_xh+ktu_tu_yv_xh-\frac{1}{2}ktvu_{xt}h\\
&-ktvu_yu_{xt}h+\frac{1}{2}kyvu_{xy}h-kzvu_{xy}h+\frac{1}{2}ktvu_tu_{xy}h-\frac{1}{2}kyvu_yu_{xy}h\\
&+\frac{1}{2}kzvu_yu_{xy}h+\frac{1}{2}v_xu_{xy}h-\frac{1}{2}tu_{xt}v_{xy}h-\frac{1}{2}yu_{xy}v_{xy}h+\frac{1}{2}zu_{xy}v_{xy}h+\frac{1}{2}tv_xu_{xyt}h\\
&+\frac{1}{2}yv_xu_{xyy}h-\frac{1}{2}zv_xu_{xyy}h-\frac{1}{4}u_yv_{xx}h-\frac{1}{4}tu_{yt}v_{xx}h-\frac{1}{4}yu_{yy}v_{xx}h+\frac{1}{4}zu_{yy}v_{xx}h\\
&-\frac{1}{4}v_{xx}h+\frac{1}{4}tv_yu_{xxt}h-\frac{3}{4}vu_{xxy}h+\frac{1}{4}yv_yu_{xxy}h-\frac{1}{4}zv_yu_{xxy}h+\frac{3}{4}yv_{xxy}h\\
&-\frac{3}{2}zv_{xxy}h+\frac{3}{4}tu_tv_{xxy}h+\frac{3}{4}yu_yv_{xxy}h-\frac{3}{4}zu_yv_{xxy}h-\frac{3}{4}tvu_{xxyt}h-\frac{3}{4}yvu_{xxyy}h\\
&+\frac{3}{4}zvu_{xxyy}h-\frac{1}{2}k^2tv_yu_x^2+\frac{1}{2}k^2tv_z+\frac{1}{2}k^2tv_zu_x-\frac{1}{2}k^2tv_yu_x-k^2tu_yv_x\\
&-k^2tu_yu_xv_x+\frac{1}{2}k^2tvu_{xz}-\frac{1}{2}k^2tvu_{xy}-k^2tvu_xu_{xy}+\frac{1}{2}ktv_{xy}u_{xx}+\frac{1}{4}ktu_{xy}v_{xx}\\
&-\frac{1}{2}ktv_xu_{xxy}-\frac{3}{4}ktv_{xxy}-\frac{3}{4}ktu_xv_{xxy}-\frac{1}{4}ktv_yu_{xxx}-\frac{1}{4}ktvu_{xxxy},\\
T_y^9 = &-\frac{1}{2}vu_th^2+\frac{1}{2}yv_th^2-zv_th^2+\frac{1}{2}tu_tv_th^2-\frac{1}{2}tvu_{tt}h^2+yvu_{zt}h^2-zvu_{zt}h^2\\
&+\frac{1}{2}yv_tu_yh^2-\frac{1}{2}zv_tu_yh^2+\frac{1}{2}yvu_{yt}h^2-\frac{1}{2}zvu_{yt}h^2+yvu_{xt}h^2-zvu_{xt}h^2\\
&+\frac{1}{2}kvh-\frac{1}{2}ktv_th+\frac{1}{2}kvu_xh-\frac{1}{2}ktv_tu_xh+\frac{1}{2}kyu_xv_xh-kzu_xv_xh\\
&+\frac{1}{2}ktu_tu_xv_xh+\frac{1}{2}kyu_yu_xv_xh-\frac{1}{2}kzu_yu_xv_xh+\frac{1}{2}ktvu_{xt}h-\frac{1}{2}ktvu_xu_{xt}h\\
&-kyvu_{xz}h+kzvu_{xz}h+\frac{1}{2}kyvu_xu_{xy}h-\frac{1}{2}kzvu_xu_{xy}h-\frac{1}{2}kyvu_{xx}h+kzvu_{xx}h\\
&-\frac{1}{2}ktvu_tu_{xx}h+\frac{1}{2}kyvu_yu_{xx}h-\frac{1}{2}kzvu_yu_{xx}h-\frac{1}{4}tu_{xt}v_{xx}h-\frac{1}{4}yu_{xy}v_{xx}h\\
&+\frac{1}{4}zu_{xy}v_{xx}h+\frac{1}{4}tv_xu_{xxt}h+\frac{1}{4}yv_xu_{xxy}h-\frac{1}{4}zv_xu_{xxy}h+\frac{1}{4}yv_{xxx}h-\frac{1}{2}zv_{xxx}h\\
&+\frac{1}{4}tu_tv_{xxx}h+\frac{1}{4}yu_yv_{xxx}h-\frac{1}{4}zu_yv_{xxx}h-\frac{1}{4}tvu_{xxxt}h+\frac{3}{4}yvu_{xxxy}h\\
&-\frac{3}{4}zvu_{xxxy}h-\frac{1}{2}k^2tu_x^2v_x-\frac{1}{2}k^2tu_xv_x+\frac{1}{2}k^2tvu_{xx}+k^2tvu_xu_{xx}+\frac{1}{4}ktu_{xx}v_{xx}\\
&-\frac{1}{4}ktv_xu_{xxx}-\frac{1}{4}ktv_{xxx}-\frac{1}{4}ktu_xv_{xxx}+\frac{1}{4}ktvu_{xxxx},\\
T_z^9 = &-\frac{1}{2}h^2u_tv-\frac{1}{2}h^2tu_{tt}v-\frac{1}{2}h^2yu_{yt}v+\frac{1}{2}h^2zu_{yt}v+\frac{1}{2}hku_xv+hktu_{xt}v
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}hkyu_{xy}v-\frac{1}{2}hkzu_{xy}v-\frac{1}{2}k^2tu_{xx}v+\frac{1}{2}hkv-\frac{1}{2}h^2zv_tu_y+\frac{1}{2}h^2yv_tu_y\\
&+\frac{1}{2}h^2tu_tv_t+\frac{1}{2}h^2yv_t-h^2zv_t-\frac{1}{2}hktv_tu_x-\frac{1}{2}hktu_tv_x-\frac{1}{2}hktv_t+\frac{1}{2}hkzu_yv_x\\
&-\frac{1}{2}hkyu_yv_x-\frac{1}{2}hkyv_x+hkzv_x+\frac{1}{2}k^2tu_xv_x+\frac{1}{2}k^2tv_x,
\end{aligned}
$$

$$
\begin{aligned}
T_t^9=&\frac{1}{2}vh^2+\frac{1}{2}yv_zh^2-zv_zh^2+\frac{1}{2}tu_tv_zh^2+\frac{1}{2}tvu_{zt}h^2+\frac{1}{2}yv_zu_yh^2-\frac{1}{2}zv_zu_yh^2\\
&+\frac{1}{2}yv_yh^2-zv_yh^2+\frac{1}{2}tu_tv_yh^2+\frac{1}{2}yu_yv_yh^2-\frac{1}{2}zu_yv_yh^2+\frac{1}{2}tvu_{yt}h^2-\frac{1}{2}yvu_{yz}h^2\\
&+\frac{1}{2}zvu_{yz}h^2-\frac{1}{2}yvu_{yy}h^2+\frac{1}{2}zvu_{yy}h^2+\frac{1}{2}yv_xh^2-zv_xh^2+\frac{1}{2}tu_tv_xh^2\\
&+\frac{1}{2}yu_yv_xh^2-\frac{1}{2}zu_yv_xh^2+\frac{1}{2}tvu_{xt}h^2-\frac{1}{2}yvu_{xy}h^2+\frac{1}{2}zvu_{xy}h^2-\frac{1}{2}ktv_zh\\
&-\frac{1}{2}ktv_yh-\frac{1}{2}ktv_zu_xh-\frac{1}{2}ktv_yu_xh-\frac{1}{2}ktv_xh-\frac{1}{2}ktu_xv_xh-\frac{1}{2}ktvu_{xz}h\\
&+\frac{1}{2}ktvu_{xy}h+ktvu_xu_{xy}h+\frac{1}{2}ktvu_{xx}h+ktvu_yu_{xx}h+tvu_{xxxy}h.
\end{aligned}
$$

Remark. It should be noted that the above conservation laws include the energy conservation law, which corresponds to the time translation and three momentum conservation laws, which correspond to the three space translations.

4 Conclusions

In this paper we studied the generalized second extended (3+1)-dimensional Jimbo-Miwa equation (3). Symmetry reductions of this equation were performed several times until it was reduced to a nonlinear fourth-order ordinary differential equation. The general solution of this ordinary differential equation was obtained in terms of the Weierstrass zeta function. Travelling wave solutions of (3) were also derived using the simplest equation method. Finally, the conservation laws of (3) were computed by invoking the conservation theorem due to Ibragimov. These conservation laws included an energy conservation law, which corresponded to the time translation and three momentum conservation laws that corresponded to the three space translations.

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