Hindawi
Journal of Function Spaces
Volume 2018, Article ID 2193234, 10 pages
https://doi.org/10.1155/2018/2193234

Research Article

Solutions for Integral Boundary Value Problems of Nonlinear Hadamard Fractional Differential Equations

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Received 25 August 2018; Accepted 19 October 2018; Published 1 November 2018

Academic Editor: Xinguang Zhang

In this paper using fixed point methods we establish some existence theorems of positive (nontrivial) solutions for integral boundary value problems of nonlinear Hadamard fractional differential equations.

1. Introduction

In this work we study the following integral boundary value problems of nonlinear Hadamard fractional differential equations

$$\begin{aligned} D^{\beta}\left(\varphi_p\left(D^{\alpha}u(t)\right)\right) &= f(t,u(t)), \quad 1 < t < e, \\ u(1) = u'(1) &= u'(e) = 0, \\ D^{\alpha}u(1) &= 0, \\ \varphi_p\left(D^{\alpha}u(e)\right) &= \mu\int_1^e \varphi_p\left(D^{\alpha}u(t)\right)\frac{dt}{t}, \end{aligned} \tag{1}$$

where α, β, and μ are three positive real numbers with $\alpha \in (2,3]$, $\beta \in (1,2]$, and $\mu \in [0,\beta)$, $\varphi_p(s) = |s|^{p-2}s$ is the p-Laplacian for $p > 1, s \in \mathbb{R}$, and f is a continuous function on $[1,e] \times \mathbb{R}$. Moreover, let $\varphi_p^{-1} = \varphi_q$ with $1/p + 1/q = 1$. In what follows, we offer some related definitions and lemmas for Hadamard fractional calculus.

Definition 1 (see [1, Page 111]). The αth Hadamard fractional order derivative of a function $y : [1,+\infty) \longrightarrow \mathbb{R}$ is defined by

$$D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} y(s)\frac{ds}{s}, \tag{2}$$

where $\alpha > 0$, $n = [\alpha] + 1$, and $[\alpha]$ denotes the largest integer which is less than or equal to α. Moreover, we here also offer the αth Hadamard fractional order integral of $y : [1,+\infty) \longrightarrow \mathbb{R}$ which is defined by

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} y(s)\frac{ds}{s}, \tag{3}$$

where Γ is the gamma function.

Lemma 2 (see [1, Theorem 2.3]). *Let $\alpha > 0$, $n = [\alpha] + 1$. Then*

$$\begin{aligned} I^{\alpha}D^{\alpha}y(t) = y(t) &+ c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2} + \cdots \\ &+ c_n(\log t)^{\alpha-n}, \end{aligned} \tag{4}$$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$.

In recent years, there have been some significant developments in the study of boundary value problems for nonlinear fractional differential equations; we refer to [2–11] and the references therein. For more related works, see also [12–49]. For example, by using monotone iterative methods, Wang et al. [3] investigated a class of boundary value problems of Hadamard fractional differential equations involving nonlocal multipoint discrete and Hadamard integral boundary conditions and established monotone iterative sequences, which can converge to the unique positive solution of their problems. Similar methods are also applied in [4, 5, 12–15].

For differential equations with the p-Laplacian, see, for example, [6, 7, 15–20] and the references therein. In [6], Wang considered the nonlinear Hadamard fractional differential equation with integral boundary condition and p-Laplacian operator

$$\begin{gathered} D^{\beta}\varphi_p\left(D^{\alpha}u(t)\right) = f(t,u(t)), \quad t \in (1,T), \\ u(T) = \lambda I^{\sigma}u(\eta), \\ D^{\alpha}u(1) = 0, \\ u(1) = 0, \end{gathered} \tag{5}$$

where f grows $(p-1)$–sublinearly at $+\infty$, and by using the Schauder fixed point theorem, a solution existence result is obtained. In [7], Li and Lin used the Guo-Krasnosel'skii fixed point theorem to obtain the existence and uniqueness of positive solutions for (1) with $\mu = 0$.

However, we note that these are seldom considered Hadamard fractional differential equations with the p-Laplacian in the literature; in this paper we are devoted to this direction. We first utilize the Guo-Krasnosel'skii fixed point theorem to obtain two positive solutions existence theorems when f grows $(p-1)$–superlinearly and $(p-1)$–sublinearly with the p-Laplacian, and secondly by using the fixed point index, we obtain a nontrivial solution existence theorem without the p-Laplacian, but the nonlinearity can allow being sign-changing and unbounded from below. This improves and generalizes some semipositone problems [21–31].

2. Preliminaries

In this section, we first calculate Green's functions associated with (1) and then transform the boundary value problem into its integral form. For this, we give the following lemma.

Lemma 3. *Let α, β, μ, φ_p, and D^{α}, D^{β} be as in (1). Then (1) can take the integral form*

$$u(t) = \int_1^e G(t,s)\,\varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}, \quad \text{for } t \in [1,e], \tag{6}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \cdot \begin{cases} (\log t)^{\alpha-1}(1-\log s)^{\alpha-2} - (\log t - \log s)^{\alpha-1}, & 1 \le s \le t \le e, \\ (\log t)^{\alpha-1}(1-\log s)^{\alpha-2}, & 1 \le t \le s \le e, \end{cases} \tag{7}$$

and

$$\begin{gathered} H(t,\tau) = H_1(t,\tau) + \frac{\mu}{(\beta-\mu)\Gamma(\beta)}(\log t)^{\beta-1}\log\tau(1-\log\tau)^{\beta-1}, \quad \text{for } t,\tau \in [1,e], \\ H_1(t,\tau) = \frac{1}{\Gamma(\beta)}\begin{cases} (\log t)^{\beta-1}(1-\log\tau)^{\beta-1} - (\log t - \log\tau)^{\beta-1}, & 1 \le \tau \le t \le e, \\ (\log t)^{\beta-1}(1-\log\tau)^{\beta-1}, & 1 \le t \le \tau \le e. \end{cases} \end{gathered} \tag{8}$$

Proof. Use $y(t)$ to replace $f(t,u)$ in (1). Let $D^{\beta}(\varphi_p(D^{\alpha}u(t))) = y(t)$. Then from Lemma 2 we have

$$\varphi_p\left(D^{\alpha}u(t)\right) = I^{\beta}y(t) + c_1(\log t)^{\beta-1} + c_2(\log t)^{\beta-2}, \quad \text{for } c_i \in \mathbb{R},\ i = 1,2. \tag{9}$$

Note that $D^{\alpha}u(1) = 0$ implies $\varphi_p(D^{\alpha}u(1)) = 0$, and then $c_2 = 0$. Therefore, we obtain

$$\varphi_p\left(D^{\alpha}u(t)\right) = I^{\beta}y(t) + c_1(\log t)^{\beta-1}. \tag{10}$$

Next, we calculate $\varphi_p(D^{\alpha}u(e))$ and $\mu\int_1^e \varphi_p(D^{\alpha}u(t))(dt/t)$:

$$\begin{aligned} \varphi_p\left(D^{\alpha}u(e)\right) &= I^{\beta}y(e) + c_1 \\ &= c_1 + \frac{1}{\Gamma(\beta)}\int_1^e (1-\log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\mu\int_1^e \varphi_p\left(D^{\alpha}u(t)\right)\frac{dt}{t} \\ &= \mu\int_1^e I^{\beta}y(t)\frac{dt}{t} + \mu c_1\int_1^e(\log t)^{\beta-1}\frac{dt}{t} \\ &= \frac{\mu c_1}{\beta} + \frac{\mu}{\Gamma(\beta)}\int_1^e\int_1^t(\log t - \log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau}\frac{dt}{t}. \end{aligned} \tag{12}$$

The condition $\varphi_p(D^{\alpha}u(e)) = \mu\int_1^e \varphi_p(D^{\alpha}u(t))(dt/t)$ enables us to obtain

$$\begin{aligned} c_1 = {} & \frac{\mu\beta}{(\beta-\mu)\Gamma(\beta)}\int_1^e\int_1^t(\log t-\log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau}\frac{dt}{t} \\ & - \frac{\beta}{(\beta-\mu)\Gamma(\beta)}\int_1^e(1-\log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau}. \end{aligned} \tag{13}$$

Substituting c_1 into (10) gives

$$\begin{aligned} \varphi_p\left(D^{\alpha}u(t)\right) = {} & \frac{1}{\Gamma(\beta)}\int_1^t(\log t-\log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau} \\ & - \frac{\beta(\log t)^{\beta-1}}{(\beta-\mu)\Gamma(\beta)}\int_1^e(1-\log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau} \\ & + \frac{\mu\beta(\log t)^{\beta-1}}{(\beta-\mu)\Gamma(\beta)}\int_1^e\int_1^t(\log t-\log\tau)^{\beta-1}y(\tau)\frac{d\tau}{\tau}\frac{dt}{t} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\beta)} \int_1^t (\log t - \log\tau)^{\beta-1} y(\tau) \frac{d\tau}{\tau} - \frac{(\log t)^{\beta-1}}{\Gamma(\beta)} \\
&\cdot \int_1^e (1-\log\tau)^{\beta-1} y(\tau) \frac{d\tau}{\tau} + \frac{(\log t)^{\beta-1}}{\Gamma(\beta)} \\
&\cdot \int_1^e (1-\log\tau)^{\beta-1} y(\tau) \frac{d\tau}{\tau} - \frac{\beta(\log t)^{\beta-1}}{(\beta-\mu)\Gamma(\beta)} \\
&\cdot \int_1^e (1-\log\tau)^{\beta-1} y(\tau) \frac{d\tau}{\tau} + \frac{\mu\beta(\log t)^{\beta-1}}{(\beta-\mu)\Gamma(\beta)} \\
&\cdot \int_1^e \int_1^t (\log t - \log\tau)^{\beta-1} y(\tau) \frac{d\tau}{\tau}\frac{dt}{t} = -\frac{1}{\Gamma(\beta)} \\
&\cdot \int_1^t \left[(\log t)^{\beta-1}(1-\log\tau)^{\beta-1} - (\log t - \log\tau)^{\beta-1}\right] \\
&\cdot y(\tau)\frac{d\tau}{\tau} - \frac{1}{\Gamma(\beta)} \int_t^e (\log t)^{\beta-1}(1-\log\tau)^{\beta-1} \\
&\cdot y(\tau)\frac{d\tau}{\tau} + \frac{\mu\beta(\log t)^{\beta-1}}{(\beta-\mu)\Gamma(\beta)} \\
&\cdot \int_1^e \int_1^t (\log t - \log\tau)^{\beta-1} y(\tau) \frac{d\tau}{\tau}\frac{dt}{t} \\
&- \frac{\mu(\log t)^{\beta-1}}{(\beta-\mu)\Gamma(\beta)} \int_1^e (1-\log\tau)^{\beta-1} y(\tau)\frac{d\tau}{\tau} \\
&= -\int_1^e H_1(t,\tau) y(\tau)\frac{d\tau}{\tau} + \frac{\mu\beta(\log t)^{\beta-1}}{(\beta-\mu)\Gamma(\beta)} \\
&\cdot \int_1^e \int_\tau^e (\log t - \log\tau)^{\beta-1} y(\tau)\frac{dt}{t}\frac{d\tau}{\tau} \\
&- \frac{\mu(\log t)^{\beta-1}}{(\beta-\mu)\Gamma(\beta)} \int_1^e (1-\log\tau)^{\beta-1} y(\tau)\frac{d\tau}{\tau} \\
&= -\int_1^e H_1(t,\tau) y(\tau)\frac{d\tau}{\tau} + \frac{\mu(\log t)^{\beta-1}}{(\beta-\mu)\Gamma(\beta)} \\
&\cdot \int_1^e (1-\log\tau)^{\beta} y(\tau)\frac{d\tau}{\tau} - \frac{\mu(\log t)^{\beta-1}}{(\beta-\mu)\Gamma(\beta)} \\
&\cdot \int_1^e (1-\log\tau)^{\beta-1} y(\tau)\frac{d\tau}{\tau} = -\int_1^e H_1(t,\tau) \\
&\cdot y(\tau)\frac{d\tau}{\tau} - \int_1^e \frac{\mu}{(\beta-\mu)\Gamma(\beta)} (\log t)^{\beta-1} \\
&\cdot \log\tau(1-\log\tau)^{\beta-1} y(\tau)\frac{d\tau}{\tau} = -\int_1^e H(t,\tau) \\
&\cdot y(\tau)\frac{d\tau}{\tau}.
\end{aligned} \tag{14}
$$

Note that $-\varphi_p(D^\alpha u(t)) = \varphi_p(-D^\alpha u(t))$, and hence we obtain

$$
-D^\alpha u(t) = \varphi_q\left(\int_1^e H(t,\tau) y(\tau)\frac{d\tau}{\tau}\right), \quad \text{for } \alpha \in (2,3],\ t\in[1,e]. \tag{15}
$$

Then, if we let $x(t) = \varphi_q(\int_1^e H(t,\tau)y(\tau)(d\tau/\tau))$, $t\in[1,e]$, from Lemma 2 we obtain

$$
\begin{aligned}
u(t) &= -I^\alpha x(t) + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2} \\
&\quad + c_3(\log t)^{\alpha-3}, \quad \text{for } c_i\in\mathbb{R},\ i=1,2,3.
\end{aligned} \tag{16}
$$

The condition $u(1) = u'(1) = 0$ implies that $c_2 = c_3 = 0$. Then we substitute e into the first derivative of u, and we calculate c_1 as follows:

$$
c_1 = \frac{1}{\Gamma(\alpha)} \int_1^e (1-\log s)^{\alpha-2} x(s)\frac{ds}{s}. \tag{17}
$$

As a result, from (16) we have

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_1^t (\log t - \log s)^{\alpha-1} x(s)\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_1^e (\log t)^{\alpha-1}(1-\log s)^{\alpha-2} x(s)\frac{ds}{s} \\
&= \int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau)y(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}, \\
&\qquad \text{for } t\in[1,e].
\end{aligned} \tag{18}
$$

This completes the proof. □

Lemma 4. *Green's functions G, H defined by (7) and (8) have the following properties:*

(i) G, H are continuous, nonnegative functions on $[1,e]\times[1,e]$,

(ii) $(\log t)^{\alpha-1}[(1-\log s)^{\alpha-2} - (1-\log s)^{\alpha-1}] \le \Gamma(\alpha)G(t,s) \le (1-\log s)^{\alpha-2} - (1-\log s)^{\alpha-1}$, for $t,s\in[1,e]$.

From [7, Lemma 7] and [8, Lemma 2.2] we easily obtain this lemma, so we omit its proof.

Let

$$
\begin{aligned}
G_1(t,s) &= \int_1^e G(t,\tau)H(\tau,s)\frac{d\tau}{\tau}, \\
\phi(s) &= \frac{1}{\Gamma(\alpha)} \\
&\quad\cdot \int_1^e \left[(1-\log t)^{\alpha-2} - (1-\log t)^{\alpha-1}\right] H(t,s)\frac{dt}{t}, \\
&\qquad \text{for } t,s\in[1,e].
\end{aligned} \tag{19}
$$

Then we obtain the following lemma.

Lemma 5. *There exist $\kappa_1 = \int_1^e (\log t)^{\alpha-1}\phi(t)(dt/t)$, $\kappa_2 = \int_1^e \phi(t)(dt/t)$ such that*

$$
\kappa_1\phi(s) \le \int_1^e G_1(t,s)\phi(t)\frac{dt}{t} \le \kappa_2\phi(s), \quad \text{for } s\in[1,e]. \tag{20}
$$

Proof. We only prove the left inequality above. From Lemma 4(ii) we have

$$\int_1^e G_1(t,s)\phi(t)\frac{\mathrm{d}t}{t} = \int_1^e\int_1^e G(t,\tau) \cdot H(\tau,s)\frac{\mathrm{d}\tau}{\tau}\phi(t)\frac{\mathrm{d}t}{t} \geq \frac{1}{\Gamma(\alpha)} \cdot \int_1^e\int_1^e (\log t)^{\alpha-1}\left[(1-\log\tau)^{\alpha-2}-(1-\log\tau)^{\alpha-1}\right] \cdot H(\tau,s)\frac{\mathrm{d}\tau}{\tau}\phi(t)\frac{\mathrm{d}t}{t} = \kappa_1\phi(s). \quad (21)$$

This completes the proof. □

Let $\mathscr{E} = C[1,e]$ be the Banach space equipped with the norm $\|u\| = \max_{t\in[1,e]}|u(t)|$. Then we define two sets on $\mathscr{E}$ as follows:

$$P = \{u \in \mathscr{E} : u(t) \geq 0,\ \forall t \in [1,e]\},$$
$$P_0 = \left\{u \in \mathscr{E} : u(t) \geq (\log t)^{\alpha-1}\|u\|,\ \forall t \in [1,e]\right\}. \quad (22)$$

Consequently, P, P_0 are cones on $\mathscr{E}$. From Lemma 3 we can define an operator A on $\mathscr{E}$ as follows:

$$(Au)(t) = \int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}, \quad \text{for } u \in \mathscr{E},\ t \in [1,e]. \quad (23)$$

The continuity of G, H, f implies that $A : \mathscr{E} \longrightarrow \mathscr{E}$ is a completely continuous operator and the existence of solutions for (1) if and only if the existence of fixed points for A.

Lemma 6 (see [50]). *Let $\mathscr{E}$ be a Banach space and Ω a bounded open set in $\mathscr{E}$. Suppose that $A : \Omega \longrightarrow \mathscr{E}$ is a continuous compact operator. If there exists $u_0 \in E \setminus \{0\}$ such that*

$$u - Au \neq \mu u_0, \quad \forall u \in \partial\Omega,\ \mu \geq 0, \quad (24)$$

then the topological degree $\deg(I - A, \Omega, 0) = 0$.

Lemma 7 (see [50]). *Let $\mathscr{E}$ be a Banach space and Ω a bounded open set in $\mathscr{E}$ with $0 \in \Omega$. Suppose that $A : \Omega \longrightarrow \mathscr{E}$ is a continuous compact operator. If*

$$Au \neq \mu u, \quad \forall u \in \partial\Omega,\ \mu \geq 1, \quad (25)$$

then the topological degree $\deg(I - A, \Omega, 0) = 1$.

Lemma 8 (see [50]). *Let $\mathscr{E}$ be a Banach space and $P \subset \mathscr{E}$ a cone in $\mathscr{E}$. Assume that Ω_1, Ω_2 are open subsets of $\mathscr{E}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow P$ be a completely continuous operator such that either*

(G1) $\|Au\| \leq \|u\|$, $u \in \partial\Omega_1 \cap P$, and $\|Au\| \geq \|u\|$, $u \in \partial\Omega_2 \cap P$,

or

(G2) $\|Au\| \geq \|u\|$, $u \in \partial\Omega_1 \cap P$, and $\|Au\| \leq \|u\|$, $u \in \partial\Omega_2 \cap P$.

Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Positive Solutions for (1)

Let $B_\varrho := \{u \in \mathscr{E} : \|u\| < \varrho\}$ for $\varrho > 0$. Now, we first list our assumptions on f:

(H1) $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$,

(H2) there exist $\delta_1 \in (1,e)$, $t_0 \in (1,e)$ such that $\liminf_{u\longrightarrow+\infty}(f(t,u)/\varphi_p(u)) \geq \varphi_p(N_1)$, $\liminf_{u\longrightarrow0^+}(f(t,u)/\varphi_p(u)) \geq \varphi_p(N_2)$, uniformly on $t \in [\delta_1, e]$, where $2N_1^{-1}$, $N_2^{-1} \in (0, (\log\delta_1)^{\alpha-1}\int_1^e G(t_0,s)\varphi_q(\int_{\delta_1}^e H(s,\tau)(\mathrm{d}\tau/\tau))(\mathrm{d}s/s))$,

(H3) there exists $\rho_1 > 0$ such that $f(t,u) \leq \varphi_p(N_3\rho_1)$, $\forall u \in [0,\rho_1]$, $t \in [1,e]$, where $N_3^{-1} > \int_1^e G(e,s)\varphi_q(\int_1^e H(s,\tau)(\mathrm{d}\tau/\tau))(\mathrm{d}s/s)$,

(H4) $\limsup_{u\longrightarrow+\infty}(f(t,u)/\varphi_p(u)) \leq \varphi_p(M_1)$, $\limsup_{u\longrightarrow0^+}(f(t,u)/\varphi_p(u)) \leq \varphi_p(M_2)$, uniformly on $t \in [1,e]$, where $(2M_1)^{-1}, M_2^{-1} > \int_1^e G(e,s)\varphi_q(\int_1^e H(s,\tau)(\mathrm{d}\tau/\tau))(\mathrm{d}s/s)$,

(H5) there exist $\rho_2 > 0$, $\delta_1 \in (1,e)$, $t_0 \in (1,e)$ such that $f(t,u) \geq \varphi_p(M_3\rho_2)$, $\forall u \in [(\log\delta_1)^{\alpha-1}\rho_2, \rho_2]$, $t \in [\delta_1, e]$, where

$$M_3^{-1} \in \left(0, \int_1^e G(t_0,s)\varphi_q\left(\int_{\delta_1}^e H(s,\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\right). \quad (26)$$

Lemma 9. *Suppose that (H1) holds. Then $A(P) \subset P_0$.*

Proof. If $u \in P$, from Lemma 4 we have

$$(Au)(t) \leq \frac{1}{\Gamma(\alpha)}\int_1^e\left[(1-\log s)^{\alpha-2}-(1-\log s)^{\alpha-1}\right] \cdot \varphi_q\left(\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}, \quad \forall t \in [1,e]. \quad (27)$$

On the other hand,

$$(Au)(t) \geq (\log t)^{\alpha-1}\cdot\frac{1}{\Gamma(\alpha)} \cdot \int_1^e\left[(1-\log s)^{\alpha-2}-(1-\log s)^{\alpha-1}\right] \cdot \varphi_q\left(\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \geq (\log t)^{\alpha-1}\|Au\|, \quad \forall t \in [1,e]. \quad (28)$$

This completes the proof. □

Remark 10. Our aim is to find operator equation $u = Au$ has fixed points in P, and from Lemma 9, these fixed points must belong to the cone P_0. Therefore, our work space can be chosen P_0 rather than P.

In what follows, we discuss the existence of positive solutions for (1) in P_0.

Theorem 11. *Suppose that (H1)-(H3) hold. Then (1) has at least two positive solutions.*

Proof. From (H3), when $u \in \partial B_{\rho_1} \cap P_0$, we have

$$\begin{aligned}(Au)(t) &\le \max_{t\in[1,e]} \int_1^e G(t,s)\varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\le \int_1^e G(e,s)\varphi_q\left(\int_1^e H(s,\tau)\varphi_p(N_3\rho_1)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &= N_3\rho_1 \int_1^e G(e,s)\varphi_q\left(\int_1^e H(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} < \rho_1, \\ &\qquad \forall t \in [1,e].\end{aligned} \tag{29}$$

Hence, we obtain

$$\|Au\| < \|u\|, \quad \text{for } u \in \partial B_{\rho_1} \cap P_0. \tag{30}$$

On the other hand, by the second limit inequality in (H2), there exists $r_1 \in (0,\rho_1)$ such that

$$f(t,u) \ge \varphi_p(N_2 u), \quad \forall u \in [0,r_1],\ t \in [\delta_1, e]. \tag{31}$$

Note that if $u \in \partial B_{r_1} \cap P_0,\ t \in [\delta_1,e]$, from the definition of P_0 we have

$$u(t) \ge (\log\delta_1)^{\alpha-1}\|u\|. \tag{32}$$

This, together with (31), implies that

$$\begin{aligned}\|Au\| &= \max_{t\in[1,e]} (Au)(t) \ge (Au)(t_0) = \int_1^e G(t_0,s) \\ &\quad\cdot \varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\ge \int_1^e G(t_0,s)\varphi_q\left(\int_{\delta_1}^e H(s,\tau)\right. \\ &\quad\left.\cdot \varphi_p\left(N_2(\log\delta_1)^{\alpha-1}\|u\|\right)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &= N_2(\log\delta_1)^{\alpha-1}\|u\| \int_1^e G(t_0,s) \\ &\quad\cdot \varphi_q\left(\int_{\delta_1}^e H(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} > \|u\|, \\ &\qquad \text{for } u \in \partial B_{r_1} \cap P_0.\end{aligned} \tag{33}$$

By the first limit inequality in (H2), there exist $R_1 > \rho_1$ and $C_1 > 0$ such that

$$f(t,u) \ge \varphi_p(N_1 u) - C_1, \quad \forall u \in \mathbb{R}^+,\ t \in [\delta_1, e]. \tag{34}$$

Note that R_1 can be chosen large enough, and if $u \in \partial B_{R_1} \cap P_0$, together with (32), there exists $C_2 > 0$ such that

$$f(t,u) \ge \varphi_p\left(N_1(\log\delta_1)^{\alpha-1}R_1 - C_2\right), \quad \forall t \in [\delta_1, e]. \tag{35}$$

Combining this and (33), we find

$$\begin{aligned}\|Au\| &\ge \int_1^e G(t_0,s)\varphi_q\left(\int_{\delta_1}^e H(s,\tau)\right. \\ &\quad\left.\cdot \varphi_p\left(N_1(\log\delta_1)^{\alpha-1}R_1 - C_2\right)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &= \left(N_1(\log\delta_1)^{\alpha-1}R_1 - C_2\right)\int_1^e G(t_0,s) \\ &\quad\cdot \varphi_q\left(\int_{\delta_1}^e H(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} \ge 2R_1 - C_3,\end{aligned} \tag{36}$$

where $C_3 = C_2\int_1^e G(t_0,s)\varphi_q(\int_{\delta_1}^e H(s,\tau)(d\tau/\tau))(ds/s)$. Consequently, we have

$$\|Au\| > \|u\|, \quad \text{for } \partial B_{R_1} \cap P_0, \text{ if } \|u\| \longrightarrow \infty. \tag{37}$$

In summary, from (30), (33), and (37) with $R_1 > \rho_1 > r_1$, Lemma 8 enables us to obtain that (1) has at least two positive solutions in $(\overline{B}_{R_1} \setminus B_{\rho_1}) \cap P_0$ and $(\overline{B}_{\rho_1} \setminus B_{r_1}) \cap P_0$. This completes the proof. □

Theorem 12. *Suppose that (H1), (H4)-(H5) hold. Then (1) has at least two positive solutions.*

Proof. If $u \in \partial B_{\rho_2} \cap P_0$, we have $\|u\| = \rho_2$, and $u \in [(\log\delta_1)^{\alpha-1}\rho_2, \rho_2]$, for $u \in P_0, t \in [\delta_1, e]$. Hence, from (H5) we obtain

$$\begin{aligned}\|Au\| &\ge \int_1^e G(t_0,s)\varphi_q\left(\int_1^e H(s,\tau) f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\ge \int_1^e G(t_0,s)\varphi_q\left(\int_{\delta_1}^e H(s,\tau)\varphi_p(M_3\rho_2)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\ge M_3\rho_2 \int_1^e G(t_0,s)\varphi_q\left(\int_{\delta_1}^e H(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} > \rho_2.\end{aligned} \tag{38}$$

This indicates that

$$\|Au\| > \|u\|, \quad \text{for } u \in \partial B_{\rho_2} \cap P_0. \tag{39}$$

On the other hand, by the second limit inequality in (H4), there exists $r_2 \in (0,\rho_2)$ such that

$$f(t,u) \le \varphi_p(M_2 u), \quad \forall u \in [0,r_2],\ t \in [1,e]. \tag{40}$$

This, if $u \in \partial B_{r_2} \cap P_0$, implies that

$$\begin{aligned}\|Au\| &\le \int_1^e G(e,s)\varphi_q\left(\int_1^e H(s,\tau)\varphi_p(M_2 u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\le \int_1^e G(e,s)\varphi_q\left(\int_1^e H(s,\tau)\varphi_p(M_2 r_2)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &= M_2 r_2 \int_1^e G(e,s)\varphi_q\left(\int_1^e H(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} < r_2.\end{aligned} \tag{41}$$

This gives

$$\|Au\| < \|u\|, \quad \text{for } u \in \partial B_{r_2} \cap P_0. \tag{42}$$

By the first limit inequality in (H4), there exist $R_2 > \rho_2$ and $C_4 > 0$ such that

$$f(t,u) \le \varphi_p(M_1 u + C_4), \quad \forall u \in \mathbb{R}^+,\ t \in [1,e]. \tag{43}$$

Consequently, if $u \in \partial B_{R_2} \cap P_0$ with R_2 large enough, we obtain

$$\begin{aligned}\|Au\| &\le \int_1^e G(e,s) \\ &\quad \cdot \varphi_q\left(\int_1^e H(s,\tau)\varphi_p(M_1R_2 + C_4)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &= (M_1R_2 + C_4)\int_1^e G(e,s)\varphi_q\left(\int_1^e H(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\le \frac{1}{2}R_2 + C_5,\end{aligned} \tag{44}$$

where $C_5 = C_4\int_1^e G(e,s)\varphi_q(\int_1^e H(s,\tau)(d\tau/\tau))(ds/s)$. Hence, we have

$$\|Au\| < \|u\|, \quad \text{for } u \in \partial B_{R_2} \cap P_0, \text{ if } \|u\| \longrightarrow \infty. \tag{45}$$

In a word, from (39), (42), and (45) with $R_2 > \rho_2 > r_2$, Lemma 8 enables us to obtain that (1) has at least two positive solutions in $(\overline{B}_{R_2} \setminus B_{\rho_2}) \cap P_0$ and $(\overline{B}_{\rho_2} \setminus B_{r_2}) \cap P_0$. This completes the proof. □

Example 13. Let

$$f(t,u) = \begin{cases} \rho_1^{p-1-\gamma_1} N_3^{p-1} u^{\gamma_1}, & u \in (\rho_1, +\infty),\ t \in [1,e], \\ \rho_1^{p-1-\gamma_2} N_3^{p-1} u^{\gamma_2}, & u \in [0, \rho_1],\ t \in [1,e], \end{cases} \tag{46}$$

where $\gamma_1 \in (p-1, +\infty)$, $\gamma_2 \in (0, p-1)$, and N_3, ρ_1 are defined by (H3). Then

$$\begin{aligned}\liminf_{u\longrightarrow+\infty}\frac{f(t,u)}{\varphi_p(u)} &= \liminf_{u\longrightarrow+\infty}\frac{\rho_1^{p-1-\gamma_1}N_3^{p-1}u^{\gamma_1}}{u^{p-1}} = +\infty \\ &\ge \varphi_p(N_1), \\ \liminf_{u\longrightarrow0^+}\frac{f(t,u)}{\varphi_p(u)} &= \liminf_{u\longrightarrow0^+}\frac{\rho_1^{p-1-\gamma_2}N_3^{p-1}u^{\gamma_2}}{u^{p-1}} = +\infty \\ &\ge \varphi_p(N_2).\end{aligned} \tag{47}$$

Moreover, for $u \in [0, \rho_1]$, $t \in [1,e]$ we have

$$f(t,u) \le \rho_1^{p-1-\gamma_2} N_3^{p-1} \rho_1^{\gamma_2} = (N_3\rho_1)^{p-1}. \tag{48}$$

Therefore, (H1)-(H3) hold.

Example 14. Let

$$f(t,u) = \begin{cases} (\log\delta_1)^{-(\alpha-1)\gamma_3}\rho_2^{p-1-\gamma_3}M_3^{p-1}u^{\gamma_3}, & u \in \left[(\log\delta_1)^{\alpha-1}\rho_2, +\infty\right),\ t \in [1,e], \\ (\log\delta_1)^{-(\alpha-1)\gamma_4}\rho_2^{p-1-\gamma_4}M_3^{p-1}u^{\gamma_4}, & u \in \left[0, (\log\delta_1)^{\alpha-1}\rho_2\right),\ t \in [1,e], \end{cases} \tag{49}$$

where $\gamma_3 \in (0, p-1)$, $\gamma_4 \in (p-1, +\infty)$, and M_3, ρ_2 are defined by (H5). Then

$$\begin{aligned}&\limsup_{u\longrightarrow+\infty}\frac{f(t,u)}{\varphi_p(u)} \\ &\quad = \limsup_{u\longrightarrow+\infty}\frac{(\log\delta_1)^{-(\alpha-1)\gamma_3}\rho_2^{p-1-\gamma_3}M_3^{p-1}u^{\gamma_3}}{u^{p-1}} = 0 \\ &\quad \le \varphi_p(M_1), \\ &\limsup_{u\longrightarrow0^+}\frac{f(t,u)}{\varphi_p(u)} \\ &\quad = \limsup_{u\longrightarrow0^+}\frac{(\log\delta_1)^{-(\alpha-1)\gamma_4}\rho_2^{p-1-\gamma_4}M_3^{p-1}u^{\gamma_4}}{u^{p-1}} = 0 \\ &\quad \le \varphi_p(M_2).\end{aligned} \tag{50}$$

Moreover, for $u \in [(\log\delta_1)^{\alpha-1}\rho_2, \rho_2]$, $t \in [\delta_1, e]$ we have

$$\begin{aligned}f(t,u) &\ge (\log\delta_1)^{-(\alpha-1)\gamma_3}\rho_2^{p-1-\gamma_3}M_3^{p-1}u^{\gamma_3} \\ &= (M_3\rho_2)^{p-1}.\end{aligned} \tag{51}$$

Therefore, (H1), (H4)-(H5) hold.

4. Nontrivial Solutions for (1)

In this section we consider the boundary value problem (1) without the p-Laplacian, i.e., $p = 2$. In this case, (1) can be transformed into its integral form as follows:

$$\begin{aligned}u(t) &= \int_1^e G(t,s)\int_1^e H(s,\tau) f(\tau, u(\tau))\frac{d\tau}{\tau}\frac{ds}{s} \\ &= \int_1^e G_1(t,s) f(s, u(s))\frac{ds}{s}, \quad \text{for } t \in [1,e].\end{aligned} \tag{52}$$

As said in Section 3, we define an operator, still denoted by A, as follows:

$$(Au)(t) = \int_1^e G_1(t,s) f(s,u(s))\frac{ds}{s}, \quad \text{for } u \in \mathscr{E},\ t \in [1,e]. \tag{53}$$

In what follows, we aim to find the existence of fixed points of A. For this, we list our assumptions on f:

(H6) $f \in C([1,e]\times\mathbb{R},\mathbb{R})$,

(H7) There exist nonnegative functions $a(t), b(t) \in \mathscr{E}$ with $b \not\equiv 0$ and $K(u)\in C(\mathbb{R},\mathbb{R}^+)$ such that

$$f(t,u) \geq -a(t) - b(t)K(u), \quad \forall u\in\mathbb{R},\ t\in[1,e]. \tag{54}$$

Moreover,

$$\lim_{|u|\longrightarrow\infty}\frac{K(u)}{|u|} = 0, \tag{55}$$

(H8) $\liminf_{|u|\longrightarrow\infty}(f(t,u)/|u|) > \kappa_1^{-1}$, uniformly in $t\in[1,e]$,

(H9) $\liminf_{|u|\longrightarrow 0}(|f(t,u)|/|u|) < \kappa_2^{-1}$, uniformly in $t\in[1,e]$.

Theorem 15. *Suppose that (H6)-(H9) hold. Then (1) has at least one nontrivial solution.*

Proof. From (H9) there exist $\varepsilon_3\in(0,\kappa_2^{-1})$ and $r_3>0$ such that

$$|f(t,u)| \leq \left(\kappa_2^{-1}-\varepsilon_3\right)|u|, \quad \forall t\in[1,e],\ |u|\in[0,r_3). \tag{56}$$

For this r_3, we show that

$$Au \neq \mu u, \quad u\in\partial B_{r_3},\ \mu\geq 1. \tag{57}$$

If otherwise, there exist $u_1\in\partial B_{r_3}$, $\mu_1\geq 1$ such that

$$Au_1 = \mu_1 u_1, \tag{58}$$

and hence, we obtain

$$\begin{aligned} |u_1(t)| &= \frac{1}{\mu_1}\left|(Au_1)(t)\right| \leq \left|(Au_1)(t)\right| \\ &\leq \int_1^e G_1(t,s)\left|f(s,u_1(s))\right|\frac{ds}{s} \\ &\leq \left(\kappa_2^{-1}-\varepsilon_3\right)\int_1^e G_1(t,s)\left|u_1(s)\right|\frac{ds}{s}. \end{aligned} \tag{59}$$

Multiply both sides of the above inequality by $\phi(t)$ and integrate from 1 to e and together with Lemma 5 we obtain

$$\begin{aligned} &\int_1^e |u_1(t)|\phi(t)\frac{dt}{t} \\ &\quad\leq \left(\kappa_2^{-1}-\varepsilon_3\right)\int_1^e\int_1^e G_1(t,s)\left|u_1(s)\right|\frac{ds}{s}\phi(t)\frac{dt}{t} \\ &\quad\leq \left(\kappa_2^{-1}-\varepsilon_3\right)\kappa_2\int_1^e |u_1(t)|\phi(t)\frac{dt}{t}. \end{aligned} \tag{60}$$

This implies that $\int_1^e |u_1(t)|\phi(t)(dt/t) = 0$, and $u_1\equiv 0$ for the fact that $\phi(t)\not\equiv 0$, for $t\in[1,e]$, which contradicts $u_1\in\partial B_{r_3}$. Therefore, (57) is true, and from Lemma 7 we obtain

$$\deg\left(I-A, B_{r_3}, 0\right) = 1. \tag{61}$$

On the other hand, by (H8), there exist $\varepsilon_4>0$ and $X_0>0$ such that

$$f(t,u) \geq \left(\kappa_1^{-1}+\varepsilon_4\right)|u|, \quad \forall t\in[1,e],\ |u|>X_0. \tag{62}$$

For every fixed ϵ with $\|b\|\epsilon\in(0,\varepsilon_4)$, $\|b\| = \max_{t\in[1,e]}|b(t)|$, and from (H7), there exists $X_1>X_0$ such that

$$K(u) \leq \epsilon|u|, \quad \forall |u|>X_1. \tag{63}$$

Combining the two inequalities above, (H7) enables us to find

$$\begin{aligned} f(t,u) &\geq \left(\kappa_1^{-1}+\varepsilon_4\right)|u| - a(t) - b(t)K(u) \\ &\geq \left(\kappa_1^{-1}+\varepsilon_4\right)|u| - a(t) - \epsilon b(t)|u| \\ &\geq \left(\kappa_1^{-1}+\varepsilon_4-\|b\|\epsilon\right)|u| - a(t), \\ &\qquad\qquad \forall |u|>X_1,\ t\in[1,e]. \end{aligned} \tag{64}$$

If we take $C_6 = (\kappa_1^{-1}+\varepsilon_4-\|b\|\epsilon)X_1 + \max_{t\in[1,e],|u|\leq X_1}|f(t,u)|$, $K^* = \max_{|u|\leq X_1}K(u)$. Then we easily have

$$\begin{aligned} f(t,u) &\geq \left(\kappa_1^{-1}+\varepsilon_4-\|b\|\epsilon\right)|u| - a(t) - C_6, \\ &\qquad\qquad \forall u\in\mathbb{R},\ t\in[1,e]. \end{aligned} \tag{65}$$

Note that ϵ can be chosen arbitrarily small, and we let

$$R_3 \geq \max\left\{\frac{\left(\kappa_1^{-1}+2\left(\varepsilon_4-\|b\|\epsilon\right)\right)\int_1^e W(s)\left(a(s)+\|b\|K^*+C_6\right)(ds/s)}{\left(\varepsilon_4-\|b\|\epsilon\right)\Gamma(\alpha) - \|b\|\epsilon\left(\kappa_1^{-1}+2\left(\varepsilon_4-\|b\|\epsilon\right)\right)\int_1^e W(s)(ds/s)}, \frac{\int_1^e W(s)\left(a(s)+\|b\|K^*+C_6\right)(ds/s)}{\Gamma(\alpha)-\|b\|\epsilon\int_1^e W(s)(ds/s)}\right\}, \tag{66}$$

where $W(s) = \int_1^e (1-\log\tau)^{\alpha-2}H(\tau,s)(d\tau/\tau)$, for $s\in[1,e]$. Now we prove that

$$u - Au \neq \mu\phi, \quad \forall u\in\partial B_{R_3},\ \mu\geq 0, \tag{67}$$

where ϕ is defined by (19). Indeed, if (67) is not true, then there exists $u_2\in\partial B_{R_3}$ and $\mu_0>0$ such that

$$u_2 - Au_2 = \mu_0\phi. \tag{68}$$

Let $\widetilde{u}(t) = \int_1^e G_1(t,s)[a(s)+b(s)K(u_2(s))+C_6](ds/s)$. Then $\widetilde{u}\in P_0$ and

$$\begin{aligned} \widetilde{u}(t) &= \int_1^e G_1(t,s)\left[a(s)+b(s)K(u_2(s))+C_6\right]\frac{ds}{s} \\ &\leq \int_1^e\int_1^e \frac{1}{\Gamma(\alpha)}(\log t)^{\alpha-1}(1-\log\tau)^{\alpha-2}H(\tau,s)\frac{d\tau}{\tau} \end{aligned}$$

$$\begin{aligned}&\cdot\left[a(s)+b(s)K(u_2(s))+C_6\right]\frac{ds}{s}\\&\leq\frac{1}{\Gamma(\alpha)}(\log t)^{\alpha-1}\int_1^e\int_1^e(1-\log\tau)^{\alpha-2}H(\tau,s)\frac{d\tau}{\tau}\\&\cdot\left[a(s)+b(s)K(u_2(s))+C_6\right]\frac{ds}{s}\\&=\frac{1}{\Gamma(\alpha)}(\log t)^{\alpha-1}\int_1^e W(s)\\&\cdot\left[a(s)+b(s)K(u_2(s))+C_6\right]\frac{ds}{s}.\end{aligned}\tag{69}$$

Consequently, we have

$$\begin{aligned}\|\tilde{u}\|&\leq\frac{1}{\Gamma(\alpha)}\int_1^e W(s)\left[a(s)+b(s)K(u_2(s))+C_6\right]\frac{ds}{s}\\&\leq\frac{1}{\Gamma(\alpha)}\int_1^e W(s)(a(s)+C_6)\frac{ds}{s}\\&+\frac{\|b\|}{\Gamma(\alpha)}\left(\int_{|u_2|\leq X_1}W(s)K(u_2(s))\frac{ds}{s}\right.\\&\left.+\int_{|u_2|>X_1}W(s)K(u_2(s))\frac{ds}{s}\right)\leq\frac{1}{\Gamma(\alpha)}\\&\cdot\int_1^e W(s)\left(a(s)+\|b\|K^*+C_6\right)\frac{ds}{s}+\frac{\|b\|\epsilon}{\Gamma(\alpha)}\\&\cdot\int_{|u_2|>X_1}W(s)|u_2(s)|\frac{ds}{s}\leq\frac{1}{\Gamma(\alpha)}\\&\cdot\int_1^e W(s)\left(a(s)+\|b\|K^*+\|b\|\epsilon R_3+C_6\right)\frac{ds}{s}.\end{aligned}\tag{70}$$

Plus $\tilde{u}$ into (68) gives

$$\begin{aligned}u_2(t)+\tilde{u}(t)&=(Au_2)(t)+\tilde{u}(t)+\mu_0\phi(t)\\&=\int_1^e G_1(t,s)\left[f(s,u_2(s))+a(s)+b(s)K(u_2(s))\right.\\&\left.+C_6\right]\frac{ds}{s}+\mu_0\phi(t).\end{aligned}\tag{71}$$

Note that $f(s,u_2(s))+a(s)+b(s)K(u_2(s))+C_6\in P, s\in[1,e]$ and $\phi\in P_0$. Lemma 9 enables us to know that $u_2+\tilde{u}\in P_0$. From (65) we have

$$\begin{aligned}(Au_2)(t)+\tilde{u}(t)&=\int_1^e G_1(t,s)\left[f(s,u_2(s))+a(s)\right.\\&\left.+b(s)K(u_2(s))+C_6\right]\frac{ds}{s}\geq\int_1^e G_1(t,s)\\&\cdot\left[f(s,u_2(s))+a(s)+C_6\right]\frac{ds}{s}\geq\int_1^e G_1(t,s)\\&\cdot\left(\kappa_1^{-1}+\varepsilon_4-\|b\|\epsilon\right)|u_2(s)|\frac{ds}{s}\geq\int_1^e G_1(t,s)\left(\kappa_1^{-1}\right.\\&\left.+\varepsilon_4-\|b\|\epsilon\right)u_2(s)\frac{ds}{s}.\end{aligned}\tag{72}$$

On the other hand, we have

$$\begin{aligned}&\kappa_1^{-1}\int_1^e G_1(t,s)\left[u_2(s)+\tilde{u}(s)\right]\frac{ds}{s}\\&+(\varepsilon_4-\|b\|\epsilon)\int_1^e G_1(t,s)u_2(s)\frac{ds}{s}\\&-\kappa_1^{-1}\int_1^e G_1(t,s)\tilde{u}(s)\frac{ds}{s}\\&\geq\kappa_1^{-1}\int_1^e G_1(t,s)\left[u_2(s)+\tilde{u}(s)\right]\frac{ds}{s}.\end{aligned}\tag{73}$$

This inequality holds if

$$\begin{aligned}&(\varepsilon_4-\|b\|\epsilon)\int_1^e G_1(t,s)u_2(s)\frac{ds}{s}\\&-\kappa_1^{-1}\int_1^e G_1(t,s)\tilde{u}(s)\frac{ds}{s}\geq 0.\end{aligned}\tag{74}$$

Indeed, $u_2+\tilde{u}\in P_0$ implies that $u_2(t)+\tilde{u}(t)\geq(\log t)^{\alpha-1}\|u_2+\tilde{u}\|\geq(\log t)^{\alpha-1}(\|u_2\|-\|\tilde{u}\|)$, for $t\in[1,e]$. Consequently,

$$\begin{aligned}&(\varepsilon_4-\|b\|\epsilon)\int_1^e G_1(t,s)\left[u_2(s)+\tilde{u}(s)\right]\frac{ds}{s}\\&-\left(\kappa_1^{-1}+\varepsilon_4-\|b\|\epsilon\right)\int_1^e G_1(t,s)\tilde{u}(s)\frac{ds}{s}\\&\geq(\varepsilon_4-\|b\|\epsilon)(R_3-\|\tilde{u}\|)\int_1^e G_1(t,s)(\log s)^{\alpha-1}\frac{ds}{s}\\&-\frac{\kappa_1^{-1}+\varepsilon_4-\|b\|\epsilon}{\Gamma(\alpha)}\\&\cdot\int_1^e W(s)\left(a(s)+\|b\|K^*+\|b\|\epsilon R_3+C_6\right)\frac{ds}{s}\\&\cdot\int_1^e G_1(t,s)(\log s)^{\alpha-1}\frac{ds}{s}\geq 0.\end{aligned}\tag{75}$$

As a result, we have

$$\begin{aligned}(Au_2)(t)+\tilde{u}(t)&\geq\kappa_1^{-1}\int_1^e G_1(t,s)\left[u_2(s)+\tilde{u}(s)\right]\frac{ds}{s}\\&:=\kappa_1^{-1}T(u_2+\tilde{u})(t),\quad\forall t\in[1,e],\end{aligned}\tag{76}$$

where $(Tu)(t)=\int_1^e G_1(t,s)u(s)(ds/s)$, for $u\in\mathscr{E}, t\in[1,e]$. Using (68) we obtain

$$\begin{aligned}u_2+\tilde{u}&=Au_2+\tilde{u}+\mu_0\phi\geq\kappa_1^{-1}T(u_2+\tilde{u})+\mu_0\phi\\&\geq\mu_0\phi.\end{aligned}\tag{77}$$

Define

$$\mu^*=\sup\{\mu>0:u_2+\tilde{u}\geq\mu\phi\}.\tag{78}$$

Note that $\mu_0\in\{\mu>0:u_2+\tilde{u}\geq\mu\phi\}$, and then $\mu^*\geq\mu_0$, $u_2+\tilde{u}\geq\mu^*\phi$. From Lemma 5 we have

$$\kappa_1^{-1}T(u_2+\tilde{u})\geq\mu^*\kappa_1^{-1}T\phi\geq\mu^*\phi,\tag{79}$$

and hence

$$u_2 + \widetilde{u} \geq \kappa_1^{-1} T\left(u_2 + \widetilde{u}\right) + \mu_0 \phi \geq \left(\mu_0 + \mu^*\right) \phi, \quad (80)$$

which contradicts the definition of μ^*. Therefore, (67) holds, and from Lemma 6 we obtain

$$\deg\left(I - A, B_{R_3}, 0\right) = 0. \quad (81)$$

This, together with (61), implies that

$$\begin{aligned} &\deg\left(I - A, B_{R_3} \setminus \overline{B}_{r_3}, 0\right) \\ &\quad = \deg\left(I - A, B_{R_3}, 0\right) - \deg\left(I - A, B_{r_3}, 0\right) = -1. \end{aligned} \quad (82)$$

Therefore the operator A has at least one fixed point in $B_{R_3} \setminus \overline{B}_{r_3}$, and (1) has at least one nontrivial solution. This completes the proof. □

Example 16. Let $f(t,u) = a|u| - bk(u), k(u) = \ln(|u|+1), u \in \mathbb{R}, t \in [1,e]$, where $a \in (\kappa_1^{-1}, +\infty)$ and $b \in (a, a+\kappa_2^{-1})$. Then $\lim_{|u| \longrightarrow +\infty}(k(u)/|u|) = 0$, and $\lim_{|u| \longrightarrow +\infty}((a|u| - b\ln(|u|+1))/|u|) = a > \kappa_1^{-1}$, $\lim_{|u| \longrightarrow 0}(|a|u| - b\ln(|u|+1)|/|u|) = |a - b| < \kappa_2^{-1}$. Therefore, (H6)-(H9) hold.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Acknowledgments

This work is supported by Natural Science Foundation of Shandong Province (ZR2018MA011, ZR2018MA009, and ZR2015AM014).

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