Asymptotic Analysis -1 (2020) 1–12
DOI 10.3233/ASY-201598
IOS Press

Solutions for parametric double phase Robin problems

Nikolaos S. Papageorgiou [a], Calogero Vetro [b,*] and Francesca Vetro [c,d]
[a] *Department of Mathematics, Zografou campus, National Technical University, 15780, Athens, Greece*
E-mail: npapg@math.ntua.gr
[b] *Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy*
E-mail: calogero.vetro@unipa.it
[c] *Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[d] *Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
E-mail: francescavetro@tdtu.edu.vn

Abstract. We consider a parametric double phase problem with Robin boundary condition. We prove two existence theorems. In the first the reaction is $(p-1)$-superlinear and the solutions produced are asymptotically big as $\lambda \to 0^+$. In the second the conditions on the reaction are essentially local at zero and the solutions produced are asymptotically small as $\lambda \to 0^+$.

Keywords: Unbalanced growth, asymptotically big solutions, asymptotically small solutions, superlinear reaction, C-condition

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper we study the following parametric two phase Robin problem

$$\begin{cases} -\operatorname{div}(a(z)|\nabla u|^{p-2}\nabla u) - \Delta_q u + \xi(z)|u|^{p-2}u = \lambda f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_\vartheta} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega,\ 1 < q < p < +\infty. \end{cases} \tag{P_λ}$$

In this problem $a \in L^\infty(\Omega)$ with $a(z) > 0$ for a.a. $z \in \Omega$ and Δ_q denotes the q-Laplace differential operator defined by

$$\Delta_q u = \operatorname{div}\big(|\nabla u|^{q-2}\nabla u\big) \quad \text{for all } W^{1,q}(\Omega).$$

The differential operator in problem (P_λ) is related to the two-phase integral functional

$$u \to \int_\Omega \big[a(z)|\nabla u|^p + |\nabla u|^q\big]\,dz.$$

*Corresponding author. E-mail: calogero.vetro@unipa.it.

In the integral functional, the integrand is the function

$$\vartheta(z, y) = a(z)|y|^p + |y|^q \quad \text{for all } z \in \Omega, \text{ all } y \in \mathbb{R}^N.$$

Since we do not assume that the coefficient $a(\cdot)$ is bounded away from zero, this integrand exhibits unbalanced growth, namely we have

$$|y|^q \leqslant \vartheta(z, y) \leqslant c_0\big[1 + |y|^p\big] \quad \text{for some } c_0 > 0, \text{ all } z \in \Omega, \text{ all } y \in \mathbb{R}^N.$$

Such functionals were investigated first in the context of problems related to elasticity theory, by Marcellini [10] and Zhikov [20]. Recently the interest for such functional was revived with the remarkable works of Mingione and coworkers (see Baroni–Colombo–Mingione [1], Colombo–Mingione [3,4], De Filippis–Mingione [5]), who proved local regularity results for minimizers of such functionals. A global regularity theory is still elusive and so the tools and techniques used in the study of (p, q)-equations (see, for example, Papageorgiou–Vetro–Vetro [15]) are not applicable in two-phase problems. Even the ambient space changes and it is no longer the Sobolev space $W^{1,p}(\Omega)$, but the Musielak–Orlicz–Sobolev space $W^{1,\vartheta}(\Omega)$ (see Section 2). In the left hand side of (P_λ) we also have a potential term $x \to \xi(z)|x|^{p-2}x$ with $\xi \in L^\infty(\Omega)$, $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$. The reaction $\lambda f(z, x)$ is parametric, with $\lambda > 0$ being the parameter and $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous). We prove two existence theorems and provide information about the asymptotic behavior of the solutions as $\lambda \to 0^+$. In the first existence theorem we assume that $f(z, \cdot)$ exhibits $(p - 1)$-superlinear growth near $\pm\infty$. However, we do not employ the Ambrosetti–Rabinowitz condition (the AR-condition for short), which is common in the literature when dealing with superlinear problems. In this case we show that for the solution u_λ, we have $\|u_\lambda\| \to +\infty$ as $\lambda \to 0^+$. In the second, the hypotheses on $f(z, \cdot)$, aside from the "subcritical" growth condition, concern only its behavior near zero. In this case we show that $\|u_\lambda\| \to 0^+$ as $\lambda \to 0^+$. In the boundary condition $\frac{\partial u}{\partial n_\vartheta}$ denotes the conormal derivative of u with respect to the modular function ϑ. We interpret this derivative using the nonlinear Green's identity (see Papageorgiou–Rădulescu–Repovš [11], Corollary 1.5.16, p. 34). When $u \in C^1(\overline{\Omega})$, we have

$$\frac{\partial u}{\partial n_\vartheta} = \big[a(z)|\nabla u|^{p-2} + |\nabla u|^{q-2}\big]\frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

We mention that recently existence and multiplicity results for two phase problems were proved by Gasiński–Papageorgiou [6], Ge–Lv–Lu [7], Liu–Dai [9], Papageorgiou–Rădulescu–Repovš [12–14], Papageorgiou–Vetro–Vetro [16]. In the framework of double-phase problems with variable growth we refer to Cencelj–Rădulescu–Repovš [2], Ragusa–Tachikawa [18] and Zhang–Rădulescu [19].

2. Mathematical background – Hypotheses

As we already mentioned in the Introduction, the right function space framework for the analysis of problem (P_λ) is provided by the so-called Musielak–Orlicz–Sobolev spaces.

We consider the Carathéodory function

$$\vartheta(z, x) = a(z)x^p + x^q \quad \text{for all } z \in \Omega, \text{ all } x \geqslant 0.$$

Then the Musielak–Orlicz space $L^{\vartheta}(\Omega)$ is defined by

$$L^{\vartheta}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable and } \rho_{\vartheta}(u) = \int_{\Omega} \vartheta\big(z, |u|\big)\, dz < +\infty \right\}.$$

We furnish $L^{\vartheta}(\Omega)$ with the so-called "Luxemburg norm" defined by

$$\|u\|_{\vartheta} = \inf\left[\lambda > 0 : \rho_{\vartheta}\left(\frac{u}{\lambda}\right) \leqslant 1 \right].$$

Then $L^{\vartheta}(\Omega)$ becomes a separable, reflexive (in fact uniformly convex) Banach space. Also, we introduce the weighted Lebesgue space

$$L^p_a(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable and } \|u\|_{a,p} = \left[\int_{\Omega} a(z)|u|^p\, dz \right]^{1/p} < +\infty \right\}.$$

We know that

$$L^p(\Omega) \hookrightarrow L^{\vartheta}(\Omega) \hookrightarrow L^q(\Omega) \cap L^p_a(\Omega),$$

and $\min\{\|u\|^p_{\vartheta}, \|u\|^q_{\vartheta}\} \leqslant \|u\|^q_q + \|u\|^p_{a,p} \leqslant \max\{\|u\|^p_{\vartheta}, \|u\|^q_{\vartheta}\}$ for all $u \in L^{\vartheta}(\Omega)$.

Then, we can define the corresponding Sobolev-type space $W^{1,\vartheta}(\Omega)$ by setting

$$W^{1,\vartheta}(\Omega) = \big\{ u \in L^{\vartheta}(\Omega) : |\nabla u| \in L^{\vartheta}(\Omega) \big\}.$$

We furnish $W^{1,\vartheta}(\Omega)$ with the norm

$$\|u\| = \|u\|_{\vartheta} + \|\nabla u\|_{\vartheta} \quad \text{for all } u \in W^{1,\vartheta}(\Omega)$$

(here $\|\nabla u\|_{\vartheta} = \||\nabla u|\|_{\vartheta}$). Normed this way, the space $W^{1,\vartheta}(\Omega)$ is separable and reflexive (in fact uniformly convex). We know that

$$W^{1,\vartheta}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{compactly}$$

for every $r \in (1, q^*)$ with

$$q^* = \begin{cases} \frac{Nq}{N-q} & \text{if } q < N, \\ +\infty & \text{if } N \leqslant q \end{cases}$$

(the critical Sobolev exponent corresponding to q).

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff measure (surface measure) $\sigma(\cdot)$. Using this measure, we can define in the usual way the boundary Lebesgue spaces $L^s(\partial\Omega)$ $(1 \leqslant s \leqslant +\infty)$. We know that there exists a unique continuous linear map $\gamma_0 : W^{1,q}(\Omega) \to L^q(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,q}(\Omega) \cap C(\overline{\Omega}).$$

The trace map extends the notion of boundary values to all Sobolev functions. We know that

$$\operatorname{im}\gamma_0 = W^{\frac{1}{q'},q}(\partial\Omega) \quad \left(\frac{1}{q}+\frac{1}{q'}=1\right) \quad \text{and} \quad \ker\gamma_0 = W_0^{1,q}(\Omega).$$

Moreover, the trace map is compact into $L^s(\partial\Omega)$ for all $s \in [1, \frac{(N-1)q}{N-q})$ if $q < N$ and into $L^s(\partial\Omega)$ for all $s \geqslant 1$ if $q \geqslant N$. In the sequel, for the sake of notational simplicity, we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

If X is a Banach space and $\varphi \in C^1(X,\mathbb{R})$, then we say that $\varphi(\cdot)$ satisfies the "C-condition", if every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $(1+\|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence. Also by K_φ we denote the critical set of φ, that is, $K_\varphi = \{u \in X : \varphi'(u) = 0\}$.

Let $A : W^{1,\vartheta}(\Omega) \to W^{1,\vartheta}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega \big[a(z)|\nabla u|^{p-2} + |\nabla u|^{q-2}\big](\nabla u, \nabla h)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,\vartheta}(\Omega).$$

This map has the following properties (see Liu–Dai [9], Proposition 3.1).

Proposition 1. *If $a \in L^\infty(\Omega)$ and $a(z) > 0$ for a.a. $z \in \Omega$, then $A(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W^{1,\vartheta}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$ in $W^{1,\vartheta}(\Omega)$).*

The hypotheses on the data of (P_λ) are the following:

H_0: $a \in L^\infty(\Omega)$ with $a(z) \geqslant 0$ for a.a. $z \in \Omega$, $\xi \in L^\infty(\Omega)$ with $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$, $\beta \in L^\infty(\partial\Omega)$ with $\beta(z) \geqslant 0$ for σ-a.a. $z \in \partial\Omega$, $\xi \not\equiv 0$ or $\beta \not\equiv 0$ and $\frac{Np}{N+p-1} < q$.

Remark 1. The last condition in hypotheses H_0, which relates the two exponents p and q, implies that $W^{1,\vartheta}(\Omega) \hookrightarrow L^p(\partial\Omega)$ compactly via the trace map $\gamma_0(\cdot)$.

H_1: $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leqslant \widehat{a}(z)[1+|x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\widehat{a} \in L^\infty(\Omega)$, $p < r < q^*$;
(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to\pm\infty}\frac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
(iii) there exists $\tau \in ((r-q)\max\{1, \frac{N}{q}\}, q^*)$ with $\tau > q$ such that

$$0 < \widehat{\eta} \leqslant \liminf_{x\to\pm\infty}\frac{f(z,x)x - pF(z,x)}{|x|^\tau} \quad \text{uniformly for a.a. } z \in \Omega;$$

(iv) there exist $1 < \mu < q$ and $c_1 > 0$ such that

$$-c_1 \leqslant \liminf_{x\to 0}\frac{F(z,x)}{|x|^\mu} \leqslant \limsup_{x\to 0}\frac{F(z,x)}{|x|^\mu} \leqslant c_1 \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark 2. From hypotheses H_1(ii), (iii), we have that

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

So the reaction $f(z,\cdot)$ is $(p-1)$-superlinear. However, this superlinear growth of $f(z,\cdot)$ is not expressed using the AR-condition. Recall that the AR-condition says that there exist $\eta > p$ and $M > 0$ such that

$$0 < \eta F(z,x) \leqslant f(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geqslant M, \tag{1a}$$

$$0 < \operatorname*{essinf}_{\Omega} F(\cdot, \pm M). \tag{1b}$$

Integrating (1a) and using (1b), we obtain the following weaker condition

$$\begin{aligned} &c_2|x|^\eta \leqslant F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geqslant M, \text{ some } c_2 > 0 \\ &\quad\Rightarrow \quad c_2|x|^\eta \leqslant f(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geqslant M. \end{aligned}$$

In this paper instead of the AR-condition, we employ hypothesis H_1(iii) which is less restrictive and incorporates in our framework superlinear nonlinearities which fail to satisfy the AR-condition. For example consider the following function (for the sake of simplicity we drop the z-dependence)

$$f(x) = \begin{cases} |x|^{\mu-2}x & \text{if } |x| \leqslant 1, \\ |x|^{p-2}x \ln|x| + |x|^{s-2}x & \text{if } 1 < |x|, \end{cases}$$

with $1 < \mu < q$ and $1 < s < p$. The function satisfies hypothesis H_1, but fails to satisfy the AR-condition.

Let $\widehat{\gamma}_p : W^{1,\vartheta}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\widehat{\gamma}_p(u) = \int_\Omega a(z)|\nabla u|^p\,dz + \int_\Omega \xi(z)|u|^p\,dz + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma \quad \text{for all } u \in W^{1,\vartheta}(\Omega).$$

Proposition 2. *If hypotheses H_0 hold, then $c_3\|u\|^p \leqslant \widehat{\gamma}_p(u)$ for some $c_3 > 0$, all $u \in W^{1,\vartheta}(\Omega)$.*

Proof. We argue by contradiction. So, suppose that the result of the proposition is not true. Then on account of the p-homogeneity of $\widehat{\gamma}_p(\cdot)$, we can find $\{u_n\}_{n\geqslant 1} \subseteq W^{1,\vartheta}(\Omega)$ such that

$$\|u_n\| = 1 \quad \text{and} \quad \widehat{\gamma}_p(u_n) < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

We may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,\vartheta}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3}$$

From (2) and (3) it follows that

$$\begin{aligned}
&\int_\Omega a(z)|\nabla u|^p\,dz = 0 \\
&\quad\Rightarrow\quad \big|\nabla u(z)\big| = 0 \quad \text{for a.a. } z\in\Omega \\
&\quad\Rightarrow\quad u \equiv c \in \mathbb{R}.
\end{aligned}$$

Then from (2) in the limit as $n \to +\infty$ we have

$$\begin{aligned}
&|c|^p\left[\int_\Omega \xi(z)\,dz + \int_{\partial\Omega} \beta(z)\,d\sigma\right] = 0 \\
&\quad\Rightarrow\quad c = 0 \quad (\text{see hypotheses } H_0) \\
&\quad\Rightarrow\quad u_n \to 0 \quad \text{in } W^{1,\vartheta}(\Omega),
\end{aligned}$$

which contradicts (2). □

For every $\lambda > 0$, let $\varphi_\lambda : W^{1,\vartheta}(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\widehat{\gamma}_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \lambda\int_\Omega F(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently $\varphi_\lambda \in C^1(W^{1,\vartheta}(\Omega),\mathbb{R})$.

3. Asymptotically big solutions

In this section we show that for all $\lambda > 0$ small problem (P_λ) has a solution $u_\lambda \in W^{1,\vartheta}(\Omega)$ such that $\|u_\lambda\| \to +\infty$ as $\lambda \to 0^+$.

Proposition 3. *If hypotheses H_0, H_1 hold and $\lambda > 0$, then the functional $\varphi_\lambda(\cdot)$ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,\vartheta}(\Omega)$ such that

$$\big|\varphi_\lambda(u_n)\big| \leqslant c_4 \quad \text{for some } c_4 > 0, \text{ all } n \in \mathbb{N}, \tag{4}$$

$$\big(1 + \|u_n\|\big)\varphi_\lambda'(u_n) \to 0 \quad \text{in } W^{1,\vartheta}(\Omega)^* \text{ as } n \to +\infty. \tag{5}$$

From (5) we have

$$\begin{aligned}
&\left|\big\langle A(u_n), h\big\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n h\,dz + \int_{\partial\Omega}\beta(z)|u_n|^{p-2}u_n h\,d\sigma - \lambda\int_\Omega f(z,u_n)h\,dz\right| \\
&\quad\leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \text{for all } h \in W^{1,\vartheta}(\Omega), \text{ with } \varepsilon_n \to 0^+.
\end{aligned} \tag{6}$$

In (6) we choose $h = u_n \in W^{1,\vartheta}(\Omega)$ and obtain

$$-\widehat{\gamma}_p(u_n) - \|\nabla u_n\|_q^q + \lambda \int_\Omega f(z,u_n)u_n\,dz \leqslant \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Also from (4) we have

$$\widehat{\gamma}_p(u_n) + \frac{p}{q}\|\nabla u_n\|_q^q - \lambda \int_\Omega pF(z,u_n)\,dz \leqslant pc_4 \quad \text{for all } n \in \mathbb{N}. \tag{8}$$

We add (7) and (8) and recall that $q < p$. Then

$$\lambda \int_\Omega \big[f(z,u_n)u_n - pF(z,u_n)\big]\,dz \leqslant c_5 \quad \text{for some } c_5 > 0, \text{ all } n \in \mathbb{N}. \tag{9}$$

Hypotheses H_1(i), (iii) imply that

$$c_6|x|^\tau - c_7 \leqslant f(z,x)x - pF(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_6, c_7 > 0. \tag{10}$$

We use (10) in (9) and obtain

$$\begin{aligned} &\|u_n\|_\tau^\tau \leqslant c_8 \quad \text{for some } c_8 > 0, \text{ all } n \in \mathbb{N} \\ &\quad \Rightarrow \quad \{u_n\}_{n\geqslant 1} \subseteq L^\tau(\Omega) \quad \text{is bounded.} \end{aligned} \tag{11}$$

First assume that $q < N$. From hypothesis H_1(iii) it is clear that we may assume that $\tau < r < q^*$. Let $t \in (0,1)$ be such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{q^*}. \tag{12}$$

Using the interpolation inequality (see Papageorgiou–Winkert [17], Proposition 2.3.17, p. 116), we have

$$\begin{aligned} &\|u_n\|_r \leqslant \|u_n\|_\tau^{1-t}\|u_n\|_{q^*}^t \\ &\quad \Rightarrow \quad \|u_n\|_r^r \leqslant c_9\|u_n\|^{tr} \quad \text{for some } c_9 > 0, \text{ all } n \in \mathbb{N} \\ &\qquad\qquad \big(\text{see (11) and recall that } W^{1,\vartheta}(\Omega) \hookrightarrow L^{q^*}(\Omega)\big). \end{aligned} \tag{13}$$

From (6) with $h = u_n \in W^{1,\vartheta}(\Omega)$ we obtain

$$\begin{aligned} &\widehat{\gamma}_p(u_n) + \|\nabla u_n\|_q^q - \lambda \int_\Omega f(z,u_n)u_n\,dz \leqslant \varepsilon_n \quad \text{for all } n \in \mathbb{N} \\ &\quad \Rightarrow \quad c_3\|u_n\|^p \leqslant \lambda \int_\Omega f(z,u_n)u_n\,dz + \varepsilon_n \quad (\text{see Proposition 2}) \\ &\qquad\qquad \leqslant \lambda c_{10}\big[1 + \|u_n\|^{tr}\big] + \varepsilon_n \quad \text{for some } c_{10} > 0, \text{ all } n \in \mathbb{N} \\ &\qquad\qquad\quad (\text{see hypothesis } H_1(\text{i}) \text{ and (13)}). \end{aligned} \tag{14}$$

From (12) we have

$$t = \frac{q^*(r-\tau)}{r(q^*-\tau)}$$
$$\Rightarrow \quad tr = \frac{q^*(r-\tau)}{q^*-\tau}. \tag{15}$$

On account of hypothesis H_1(iii) we have

$$(r-q)\frac{N}{q} < \tau \quad \text{(recall that we have assumed that } q < N)$$
$$\Rightarrow \quad N(r-q) < \tau q$$
$$\Rightarrow \quad Nr - N\tau < Nq - N\tau + \tau q$$
$$\Rightarrow \quad \frac{Nq(r-\tau)}{Nq - N\tau + \tau q} < q$$
$$\Rightarrow \quad \frac{q^*(r-\tau)}{q^*-\tau} < q$$
$$\Rightarrow \quad tr < q \quad \text{(see (15))}.$$

Then from (14) and since $q < p$, we infer that

$$\{u_n\}_{n\geqslant 1} \subseteq W^{1,\vartheta}(\Omega) \quad \text{is bounded.} \tag{16}$$

Next suppose that $q \geqslant N$. In this case we know that $q^* = +\infty$, while from the Sobolev embedding theorem, we have

$$W^{1,\vartheta}(\Omega) \hookrightarrow W^{1,q}(\Omega) \hookrightarrow L^s(\Omega) \quad \text{(for all } 1 \leqslant s < +\infty).$$

So, in the previous argument we need to replace q^* by $l > r$.
Then again from (12) we have

$$tr = \frac{l(r-\tau)}{l-\tau} \to r - \tau < q \quad \text{as } l \to +\infty \text{ (see hypothesis } H_1\text{(iii))}.$$

So, by choosing $l > r$ big, we will have

$$tr < q < p,$$

hence (16) holds again.
From (16) it follows that we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,\vartheta}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{17}$$

In (6) we choose $h = u_n - u \in W^{1,\vartheta}(\Omega)$, pass to the limit as $n \to +\infty$ and use (17). Then

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u \rangle = 0$$
$$\Rightarrow \quad u_n \to u \quad \text{in } W^{1,\vartheta}(\Omega) \text{ (see Proposition 1).}$$

We conclude that for every $\lambda > 0$ the functional $\varphi_\lambda(\cdot)$ satisfies the C-condition. □

Proposition 4. *If hypotheses H_0, H_1 hold, then we can find $\lambda^* > 0$ such that $0 < m_\lambda \leqslant \varphi_\lambda(u)$ for all $\|u\| = \rho_\lambda$, all $\lambda \in (0, \lambda^*)$.*

Proof. On account of hypotheses H_1(i), (iv), we have

$$\left|F(z,x)\right| \leqslant c_{11}\left[|x|^\mu + |x|^r\right] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{11} > 0. \tag{18}$$

Then for every $u \in W^{1,\vartheta}(\Omega)$ we have

$$\varphi_\lambda(u) \geqslant \frac{c_3}{p}\|u\|^p - \lambda c_{12}\left[\|u\|^\mu + \|u\|^r\right] \quad \text{for some } c_{12} > 0$$
$$\text{(see Proposition 2 and (18)).} \tag{19}$$

Consider $u \in W^{1,\vartheta}(\Omega)$ with $\|u\| = \rho_\lambda = \lambda^{-\delta}$ where $0 < \delta < \frac{1}{r-p}$. Then from (19) we have

$$\begin{aligned}\varphi_\lambda(u) &\geqslant \frac{c_3}{p}\lambda^{-\delta p} - c_{12}\left[\lambda^{1-\delta\mu} + \lambda^{1-\delta r}\right] \\ &= \left[\frac{c_3}{p} - c_{12}\left(\lambda^{1-\delta(\mu-p)} + \lambda^{1-\delta(r-p)}\right)\right]\lambda^{-\delta p} = m_\lambda.\end{aligned} \tag{20}$$

Note that

$$0 < 1 - \delta(r-p) < 1 - \delta(\mu - p).$$

Then we can find $\lambda^* > 0$ such that

$$\lambda^{1-\delta(\mu-p)} + \lambda^{1-\delta(r-p)} < \frac{c_3}{c_{12}p} \quad \text{for all } \lambda \in \left(0, \lambda^*\right).$$

From (20) we infer that

$$\varphi_\lambda(u) \geqslant m_\lambda > 0 \quad \text{for all } u \in W^{1,\vartheta}(\Omega) \text{ with } \|u\| = \rho_\lambda, \text{ all } 0 < \lambda < \lambda^*.$$ □

Remark 3. From the above proof we see that $m_\lambda \to +\infty$ as $\lambda \to 0^+$ (see (20)).

Now we can produce solutions of (P_λ) which asymptotically as $\lambda \to 0^+$ become arbitrarily big in the $W^{1,\vartheta}(\Omega)$-norm.

Theorem 1. *If hypotheses H_0, H_1 hold, then we can find $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has a nontrivial solution $u_\lambda \in W^{1,\vartheta}(\Omega)$ and $\|u_\lambda\| \to +\infty$ as $\lambda \to 0^+$.*

Proof. Let $u \in W^{1,\vartheta}(\Omega)$ with $u(z) > 0$ for a.a. $z \in \Omega$. Then on account of hypothesis $H_1(ii)$ we have

$$\varphi_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{21}$$

Then (21) together with Propositions 3 and 4, permit the use of the mountain pass theorem. So, we can find $u_\lambda \in W^{1,\vartheta}(\Omega)$ such that

$$u_\lambda \in K_{\varphi_\lambda} \quad \text{and} \quad \varphi_\lambda(0) = 0 < m_\lambda \leqslant \varphi_\lambda(u_\lambda). \tag{22}$$

So, u_λ is a nontrivial solution of (P_λ) $(\lambda \in (0, \lambda^*))$. Using (18), we have

$$\begin{aligned}
&\varphi_\lambda(u_\lambda) \leqslant c_{13}\big[\|u_\lambda\|^p + \|u_\lambda\|^\mu + \|u_\lambda\|^r\big] \quad \text{for some } c_{13} > 0 \\
&\quad \Rightarrow \quad m_\lambda \leqslant c_{14}\big[1 + \|u_\lambda\|^r\big] \quad \text{for some } c_{14} > 0 \text{ (see (22) and recall that } 1 < \mu < p < r) \\
&\quad \Rightarrow \quad \|u_\lambda\| \to +\infty \quad \text{as } \lambda \to 0^+ \text{ (recall that } m_\lambda \to +\infty \text{ as } \lambda \to 0^+).
\end{aligned}$$

□

4. Asymptotically small solutions

In this section, we provide conditions on $f(z, x)$ which guarantee that for all $\lambda > 0$ small problem (P_λ) has a solution $\widehat{u}_\lambda \in W^{1,\vartheta}(\Omega)$ such that $\|\widehat{u}_\lambda\| \to 0^+$ as $\lambda \to 0^+$.

The new conditions on the function $f(z, x)$ in the reaction are the following:

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leqslant \widehat{a}(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\widehat{a} \in L^\infty(\Omega)$, $p < r < q^*$;

(ii) there exists $\tau \in (1, q)$ and $\delta, \widehat{c}, \widetilde{c}$ such that

$$\widehat{c}|x|^\tau \leqslant F(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leqslant \delta,$$

$$\limsup_{x \to 0} \frac{F(z, x)}{|x|^\tau} \leqslant \widetilde{c} \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark 4. The hypotheses on $f(z, \cdot)$ are minimal. We stress that no asymptotic condition as $x \to \pm\infty$ is imposed on $f(z, \cdot)$. Only the subcritical growth condition H_2(i), which guarantees that the energy functional of the problem is C^1. It is an interesting open question whether we can drop hypothesis H_2(i) and use cut-off techniques like those in Leonardi-Papageorgiou [8]. The lack of global regularity results for double phase problems, make such an approach problematic.

Theorem 2. *If hypotheses H_0, H_2 hold, then we can find $\widehat{\lambda}^* > 0$ such that for all $\lambda \in (0, \widehat{\lambda}^*)$ problem (P_λ) has a nontrivial solution $\widehat{u}_\lambda \in W^{1,\vartheta}(\Omega)$ and $\|\widehat{u}_\lambda\| \to 0^+$ as $\lambda \to 0^+$.*

Proof. As before $\varphi_\lambda : W^{1,\vartheta}(\Omega) \to \mathbb{R}$ is the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\widehat{\gamma}_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \lambda \int_\Omega F(z, u)\, dz \quad \text{for all } u \in W^{1,\vartheta}(\Omega).$$

We know that $\varphi_\lambda \in C^1(W^{1,\vartheta}(\Omega), \mathbb{R})$. Hypotheses H_2 imply that

$$\left|F(z,x)\right| \leqslant c_{15}\left[|x|^\tau + |x|^r\right] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{15} > 0. \tag{23}$$

Let $0 < \delta < \frac{1}{p}$. Then for $u \in W^{1,\vartheta}(\Omega)$ with $\|u\| = \lambda^\delta$, we have

$$\begin{aligned} \varphi_\lambda(u) &\geqslant \frac{c_3}{p}\lambda^{\delta p} - c_{16}\left[\lambda^{\delta\tau} + \lambda^{\delta r}\right] \quad \text{for some } c_{15} > 0 \text{ (see Proposition 1 and (23))} \\ &= \left[\frac{c_3}{p}\lambda^{\delta p - 1} - c_{16}\left(\lambda^{\delta\tau} + \lambda^{\delta r}\right)\right]\lambda. \end{aligned}$$

Note that $\delta p - 1 < 0$ and so we see that we can find $\widehat{\lambda}^* > 0$ such that for all $\lambda \in (0, \widehat{\lambda}^*)$ we have

$$\varphi_\lambda(u) > 0 \quad \text{for all } u \in W^{1,\vartheta}(\Omega) \text{ with } \|u\| = \lambda^\delta. \tag{24}$$

Let $B_\lambda = \{u \in W^{1,\vartheta}(\Omega) : \|u\| < \lambda^\delta\}$. The reflexivity of $W^{1,\vartheta}(\Omega)$ and the Eberlein–Smulian theorem imply that $\overline{B}_\lambda$ is sequentially weakly compact. The functional $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous (recall that $W^{1,\vartheta}(\Omega) \hookrightarrow L^p(\Omega)$ compactly). So, by the Weierstrass–Tonelli theorem, we can find $\widehat{u}_\lambda \in W^{1,\vartheta}(\Omega)$ such that

$$\varphi_\lambda(\widehat{u}_\lambda) = \min\left[\varphi_\lambda(u) : u \in \overline{B}_\lambda\right]. \tag{25}$$

Let $u \in C^1(\overline{\Omega}) \subseteq W^{1,\vartheta}(\Omega)$ with $u(z) > 0$ for all $z \in \overline{\Omega}$. Then we can find $t \in (0,1)$ small such that $0 < tu(z) \leqslant \delta$ for all $z \in \overline{\Omega}$, where $\delta > 0$ is as postulated by hypothesis H_2(ii). We have

$$\varphi_\lambda(tu) \leqslant \frac{t^p}{p}\widehat{\gamma}_p(u) + \frac{t^q}{q}\|\nabla u\|_q^q - \widehat{c}t^\tau\|u\|_\tau^\tau \quad \text{(see hypothesis } H_2\text{(ii))}.$$

Since $1 < \tau < q < p$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} &\varphi_\lambda(tu) < 0 \\ &\quad \Rightarrow \quad \varphi_\lambda(\widehat{u}_\lambda) < 0 = \varphi_\lambda(0) \quad \text{(see (25))} \\ &\quad \Rightarrow \quad \widehat{u}_\lambda \neq 0. \end{aligned} \tag{26}$$

Also from (24) and (26) it follows that

$$\|\widehat{u}_\lambda\| < \lambda^\delta. \tag{27}$$

Therefore $\widehat{u}_\lambda \in B_\lambda \setminus \{0\}$. On account of (25) we have

$$\begin{aligned} &\widehat{u}_\lambda \in K_{\varphi_\lambda} \\ &\quad \Rightarrow \quad \widehat{u}_\lambda \quad \text{is a nontrivial solution of } (P_\lambda),\ \lambda \in \left(0, \widehat{\lambda}^*\right). \end{aligned}$$

From (27) we see that $\|u_\lambda\| \to 0^+$ as $\lambda \to 0^+$. $\square$

References

[1] P. Baroni, M. Colombo and G. Mingione, Harnack inequalities for double phase functionals, *Nonlinear Anal.* **121** (2015), 206–222. doi:10.1016/j.na.2014.11.001.

[2] M. Cencelj, V.D. Rădulescu and D.D. Repovš, Double phase problems with variable growth, *Nonlinear Anal.* **177** (2018), part A, 270–287. doi:10.1016/j.na.2018.03.016.

[3] M. Colombo and G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Ration. Mech. Anal.* **218** (2015), 219–273. doi:10.1007/s00205-015-0859-9.

[4] M. Colombo and G. Mingione, Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.* **215** (2015), 443–496. doi:10.1007/s00205-014-0785-2.

[5] C. De Filippis and G. Mingione, On the regularity of minima for non-autonomous functionals, *J. Geom. Anal.* doi:10.1007/s12220-019-00225-z.

[6] L. Gasiński and N.S. Papageorgiou, Constant sign and nodal solutions for superlinear double phase problems, *Adv. Calc. Var.* doi:10.1515/acv-2019-0040.

[7] B. Ge, D.-J. Lv and J.F. Lu, Multiple solutions for a class of double phase problem without the Ambrosetti–Rabinowitz condition, *Nonlinear Anal.* **188** (2019), 294–315. doi:10.1016/j.na.2019.06.007.

[8] S. Leonardi and N.S. Papageorgiou, On a class of critical Robin problems, *Forum Math.* **32** (2020), 95–110. doi:10.1515/forum-2019-0160.

[9] W. Liu and G. Dai, Existence and multiplicity results for double phase problems, *J. Differential Equations* **265** (2018), 4311–4334. doi:10.1016/j.jde.2018.06.006.

[10] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Differential Equations* **90** (1991), 1–30. doi:10.1016/0022-0396(91)90158-6.

[11] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, *Nonlinear Analysis – Theory and Methods*, Springer Nature, Switzerland, 2019.

[12] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, Double-phase problems and a discontinuity property of the spectrum, *Proc. Amer. Math. Soc.* **147**(7) (2019), 2899–2910. doi:10.1090/proc/14466.

[13] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, Positive solutions for nonlinear parametric singular Dirichlet problems, *Bull. Math. Sci.* **9**(3) (2019), 1950011. doi:10.1142/S1664360719500115.

[14] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, Ground state and nodal solutions for a class of double phase problems, *Z. Angew. Math. Phys.* **71**(1) (2020), Paper No. 15. doi:10.1007/s00033-019-1239-3.

[15] N.S. Papageorgiou, C. Vetro and F. Vetro, Multiple solutions with sign information for a $(p, 2)$-equation with combined nonlinearities, *Nonlinear Anal.* **192** (2020), 111716. doi:10.1016/j.na.2019.111716.

[16] N.S. Papageorgiou, C. Vetro and F. Vetro, Multiple solutions for parametric double phase Dirichlet problems, *Commun. Contemp. Math.*, to appear.

[17] N.S. Papageorgiou and P. Winkert, *Applied Nonlinear Functional Analysis*, De Gruyter, Berlin, 2018.

[18] M.A. Ragusa and A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, *Adv. Nonlinear Anal.* **9**(1) (2020), 710–728. doi:10.1515/anona-2020-0022.

[19] Q. Zhang and V.D. Rădulescu, Double phase anisotropic variational problems and combined effects of reaction and absorption terms, *J. Math. Pures Appl.* **118** (2018), 159–203. doi:10.1016/j.matpur.2018.06.015.

[20] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.* **29** (1987), 33–66. doi:10.1070/IM1987v029n01ABEH000958.