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Research Article

Solutions of the Difference Equation $x_{n+1} = x_n x_{n-1} - 1$

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Our goal in this paper is to investigate the long-term behavior of solutions of the following difference equation: $x_{n+1} = x_n x_{n-1} - 1$, $n = 0, 1, 2, \ldots$, where the initial conditions x_{-1} and x_0 are real numbers. We examine the boundedness of solutions, periodicity of solutions, and existence of unbounded solutions and how these behaviors depend on initial conditions.

1. Introduction and Preliminaries

We investigate the long-term behavior of solutions of the second-order difference equation

$$x_{n+1} = x_n x_{n-1} - 1, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the initial conditions x_{-1} and x_0 are real numbers. In particular, we examine the boundedness and periodic behaviors of solutions and the dependence of such behaviors on initial conditions.

Over the last decade, a number of rational second-order difference equations have been extensively studied due to their unique and diversified behavior of solutions. See, for example, [1–10] and the related references therein. Several of the rational difference equations have been used in mathematical biology by Beverton and Holt [11] and Pielou [12]. In addition, some rational difference equations exhibit a trichotomy behavior of solutions relative to the relationships between the parameters: convergence of solutions, periodicity of solutions, and existence of unbounded solutions. See [13] for one such example. The difference equation (1.1) belongs to the class of equations of the form

$$x_{n+1} = x_{n-k}x_{n-k-1} - 1, \quad \text{for } n = 0, 1, \ldots, \tag{1.2}$$

and a particular choice of k, where $k = 0, 1, \ldots$.

2. The Equilibria and Periodic Solutions of (1.1)

In this section, we show that (1.1) possesses exactly two equilibria and three periodic solutions with minimal period three.

After solving the equation $\overline{x}^2 - \overline{x} - 1 = 0$, we find that (1.1) has exactly two equilibria, one positive and one negative, which we denote by $\overline{x}_1$ and $\overline{x}_2$, respectively:

$$\overline{x}_1 =: \frac{1+\sqrt{5}}{2}, \qquad \overline{x}_2 =: \frac{1-\sqrt{5}}{2}. \tag{2.1}$$

Note that there are no solutions which are eventually constant. Indeed, if $x_N = x_{N+1} = \overline{x}$ for some $N \geq 0$, then from $x_{N+1} = x_N x_{N-1} - 1$, it follows that

$$x_{N-1} = \frac{x_{N+1}+1}{x_N} = \frac{\overline{x}+1}{\overline{x}} = \overline{x}. \tag{2.2}$$

Repeating this procedure, we obtain $x_n = \overline{x}$, for $-1 \leq n \leq N+1$, as claimed.

Also note that there are no solutions that are eventually periodic with minimal period two. Indeed, if $x_N = x_{N+2k}$ and $x_{N+1} = x_{N+2k+1}$, $k \geq 0$, then we have

$$x_{N+3} = x_{N+2}x_{N+1} - 1 = x_N x_{N+1} - 1 = x_{N+2} = x_N, \tag{2.3}$$

so that $x_n = x_N$, $n \geq N$, which is a contradiction.

The following result shows that there exist exactly three periodic solutions of (1.1) with minimal period three and gives a description of each.

Theorem 2.1. *There exist exactly three periodic solutions of* (1.1) *with minimal period three. They are given by the three pairs of initial conditions* $x_{-1} = -1$, $x_0 = -1$; $x_{-1} = -1$, $x_0 = 0$; *and* $x_{-1} = 0$, $x_0 = -1$.

Proof. We can write terms of a period-three solution of (1.1) as

$$\begin{gathered} x_{-1} = a, \qquad x_0 = b, \qquad x_1 = ab - 1, \\ x_2 = b(ab-1) - 1 = a \Longleftrightarrow ab^2 - b - 1 = a \Longleftrightarrow ab^2 = a + b + 1, \\ x_3 = a(ab-1) - 1 = b \Longleftrightarrow a^2 b - a - 1 = b \Longleftrightarrow a^2 b = a + b + 1. \end{gathered} \tag{2.4}$$

Therefore, this is indeed a solution of period three if the system below is satisfied.

$$\begin{gathered} ab^2 = a + b + 1, \\ a^2 b = a + b + 1. \end{gathered} \tag{2.5}$$

Hence we see that $ab^2 = a^2 b$. Thus, this is true if $a = 0$, or if $b = 0$, or if $a = b$ when $a \neq 0$ and $b \neq 0$. Now it suffices to consider the following three cases.

Case 1. Suppose that $a = 0$. Then $a^2 b = a + b + 1$ implies that $0 = b + 1$ and hence gives us $b = -1$. Therefore, a period-three solution exists if $a = 0$ and $b = -1$.

Case 2. Suppose that $b = 0$. Then $a^2 b = a + b + 1$ implies that $0 = a + 1$ and hence gives us $a = -1$. Therefore, a period-three solution exists if $a = -1$ and $b = 0$.

Case 3. Suppose that $a = b$ such that $a \neq 0$ and $b \neq 0$. Then $a^2 b = a+b+1$ implies that $a^3 = 2a+1$, which gives us $a = -1$. Therefore, a period-three solution exists if $a = -1$ and $b = -1$. (Note that $a = (1 \pm \sqrt{5})/2$ are also solutions of $a^3 = 2a + 1$, which are the two equilibria of (1.1).)

Hence, there exist exactly three periodic solutions with minimal period three of (1.1) given by

$$\begin{gathered} x_{-1} = -1, \quad x_0 = -1, \quad x_1 = 0, \ldots; \quad x_{-1} = -1, \quad x_0 = 0, \quad x_1 = -1, \ldots; \\ x_{-1} = 0, \quad x_0 = -1, \quad x_1 = -1, \ldots, \end{gathered} \tag{2.6}$$

as claimed. □

In the sequel, we will refer to any one of these three periodic solutions of (1.1) as

$$\ldots, 0, -1, -1, 0, -1, -1, \ldots. \tag{2.7}$$

The following theorem demonstrates, in a similar way as in [14, 15], that (1.1) has solutions that are eventually periodic with minimal period three.

Theorem 2.2. *All eventually periodic solutions with minimal period three are of the form*

$$(x_{-1}, x_0, \dots, x_N, x_{N+1}, 0, -1, -1, 0, -1, -1, 0, -1, -1, \dots), \tag{2.8}$$

Where $N \geq -1$, $x_{N+1} = a \in R \setminus \{0\}$, $x_N = 1/a$, *and, if* $N \neq -1$, $x_{n-1} = (x_{n+1} + 1)/x_n$ *for* $0 \leq n \leq N$.

Proof. If $\{x_n\}_{n=-1}^{\infty}$ is an eventually periodic solution with minimal period three, then by Theorem 2.1 there is an $N \geq -1$ such that $x_{N+2} = 0$ and $x_{N+3} = -1$. Hence $0 = x_{N+2} = x_{N+1}x_N - 1$ and consequently $x_N \neq 0 \neq x_{N+1}$ and $x_{N+1} = 1/x_N$. Let $x_{N+1} = a$ (which implies that $x_N = 1/a$). From (1.1), if $N \neq -1$, we get $x_{n-1} = (x_{n+1} + 1)/x_n$, for $0 \leq n \leq N$, as desired. □

Remark 2.3. If, in Theorem 2.2, $a = -1$, then, among some others, we obtain the solutions in Theorem 2.1.

The next theorem shows that no periodic or eventually periodic solution of (1.1) converges to a minimal period-three solution.

Theorem 2.4. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of* (1.1) *that is neither periodic nor eventually periodic with minimal period three. Then* $\{x_n\}_{n=-1}^{\infty}$ *does not converge to the minimal period-three solution*

$$\dots, 0, -1, -1, 0, -1, -1, \dots. \tag{2.9}$$

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of (1.1) that is neither periodic nor eventually periodic with minimal period three. For the sake of contradiction, assume that $\{x_n\}_{n=-1}^{\infty}$ converges to the minimal period-three solution

$$\dots, 0, -1, -1, 0, -1, -1, \dots. \tag{2.10}$$

Then we can choose $N \geq 0$ such that there exist

$$-\frac{1}{2} < \delta_1, \delta_2, \delta_3, \dots < \frac{1}{2}, \qquad -1 < \epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \dots < 1, \tag{2.11}$$

with

$$\begin{gathered} x_{N-2} = -1 + \epsilon_0, \quad x_{N-1} = \delta_1, \quad x_N = -1 + \epsilon_1, \quad x_{N+1} = -1 + \epsilon_2, \quad x_{N+2} = \delta_2, \\ x_{N+3} = -1 + \epsilon_3, \quad x_{N+4} = -1 + \epsilon_4, \quad x_{N+5} = \delta_3, \quad x_{N+6} = -1 + \epsilon_5, \dots, \end{gathered} \tag{2.12}$$

that is, for $n = 0, 1, \dots$,

$$x_{N+(3n-2)} = -1 + \epsilon_{2n}, \qquad x_{N+(3n-1)} = \delta_{n+1}, \qquad x_{N+3n} = -1 + \epsilon_{2n+1}. \tag{2.13}$$

Now, first observe that

$$\begin{aligned} x_{N-2} &= -1 + \epsilon_0, \\ x_{N-1} &= \delta_1, \\ x_N &= -1 + \epsilon_1 = -1 + \delta_1(-1 + \epsilon_0). \end{aligned} \tag{2.14}$$

We claim that $\delta_1 \neq 0$. Otherwise, $x_{N-1} = \delta_1 = 0$. Then, by (1.1), $x_N = x_{N+1} = -1$, which gives us a solution that is eventually periodic with minimal period three, a contradiction. This leaves us with $\delta_1 > 0$ or $\delta_1 < 0$. If $\delta_1 > 0$, then since $x_{N-2} = -1 + \epsilon_0 < 0$ by (2.11), we have $x_N < -1$ and so $\epsilon_1 < 0$. If $\delta_1 < 0$, then since $x_{N-2} = -1 + \epsilon_0 < 0$ and $x_N = -1 + \epsilon_1 < 0$ by (2.11), we have $x_N \in (-1, 0)$ and so $\epsilon_1 > 0$. Therefore, we need only consider the following two cases.

Case 1. If $\delta_1 > 0$, $\epsilon_1 < 0$, then we obtain the following inequalities from (2.11) and (2.12):

$$\begin{aligned} x_{N-1} &= \delta_1 > 0, \\ x_N &= -1 + \epsilon_1 < -1, \\ x_{N+1} &= -1 + \epsilon_2 = -1 + \delta_1(-1 + \epsilon_1) < -1, \\ x_{N+2} &= \delta_2 = (-1 + \epsilon_1)[-1 + \delta_1(-1 + \epsilon_1)] - 1 > 0, \\ x_{N+3} &= -1 + \epsilon_3 = \delta_2[-1 + \delta_1(-1 + \epsilon_1)] - 1 < -1. \end{aligned} \tag{2.15}$$

We next compute the following:

$$\begin{aligned} & x_{N+2} = \delta_2 = (-1 + \epsilon_1)[-1 + \delta_1(-1 + \epsilon_1)] - 1 > \delta_1 \\ & \Longleftrightarrow (-1 + \epsilon_1)[-1 + \delta_1(-1 + \epsilon_1)] > \delta_1 + 1 \\ & \Longleftrightarrow 1 + \delta_1 - \delta_1\epsilon_1 - \epsilon_1 - \delta_1\epsilon_1 + \delta_1\epsilon_1^2 > \delta_1 + 1 \\ & \Longleftrightarrow -2\delta_1\epsilon_1 - \epsilon_1 + \delta_1\epsilon_1^2 > 0 \\ & \Longleftrightarrow \delta_1\epsilon_1^2 > \epsilon_1(2\delta_1 + 1) \\ & \Longleftrightarrow \delta_1\epsilon_1 < 2\delta_1 + 1, \end{aligned} \tag{2.16}$$

which is true. Hence, $\delta_2 > \delta_1 > 0$ and $\epsilon_3 < 0$. Therefore, this case applies to δ_2, ϵ_3, and by induction and (2.11) and (2.12), we obtain

$$0 < \delta_1 < \delta_2 < \cdots < \delta_n < \cdots . \tag{2.17}$$

Hence, δ_n cannot converge to 0 as $n \to \infty$ in this case.

Case 2. If $\delta_1 < 0$, $\epsilon_1 > 0$, then we obtain the following:

$$\begin{gathered}
x_{N-1} = \delta_1 \in (-1,0), \\
x_N = -1 + \epsilon_1 \in (-1,0), \\
x_{N+1} = -1 + \epsilon_2 = -1 + \delta_1(-1+\epsilon_1) \in (-1,0), \\
x_{N+2} = \delta_2 = (-1+\epsilon_1)[-1+\delta_1(-1+\epsilon_1)] - 1 \in (-1,0), \\
x_{N+3} = -1 + \epsilon_3 = \delta_2[-1+\delta_1(-1+\epsilon_1)] - 1 \in (-1,0).
\end{gathered} \tag{2.18}$$

We next compute the following:

$$\begin{aligned}
& x_{N+2} = \delta_2 = (-1+\epsilon_1)[-1+\delta_1(-1+\epsilon_1)] - 1 < \delta_1 \\
& \Longleftrightarrow (-1+\epsilon_1)[-1+\delta_1(-1+\epsilon_1)] < \delta_1 + 1 \\
& \Longleftrightarrow 1 + \delta_1 - \delta_1\epsilon_1 - \epsilon_1 - \delta_1\epsilon_1 + \delta_1\epsilon_1^2 < \delta_1 + 1 \\
& \Longleftrightarrow -2\delta_1\epsilon_1 - \epsilon_1 + \delta_1\epsilon_1^2 < 0 \\
& \Longleftrightarrow \delta_1\epsilon_1^2 < \epsilon_1(2\delta_1 + 1) \\
& \Longleftrightarrow \delta_1\epsilon_1 < 2\delta_1 + 1,
\end{aligned} \tag{2.19}$$

which is true by (2.11) and (2.12). Hence, $\delta_2 < \delta_1 < 0$ and $\epsilon_3 > 0$. Therefore, this case applies to ϵ_3, δ_2, and by induction and (2.11) and (2.12), we obtain the following inequalities:

$$0 > \delta_1 > \delta_2 > \cdots > \delta_n > \cdots. \tag{2.20}$$

Hence, δ_n cannot converge to 0 as $n \to \infty$ in this case, and we are done with our proof by contradiction. □

Finally we show that the interval $(-1,0)$ is invariant.

Theorem 2.5. *If* $-1 < x_{-1}, x_0 < 0$, *then* $-1 < x_n < 0$ *for all* $n \geq -1$.

Proof. If $-1 < x_{-1}, x_0 < 0$, then $-1 < x_1 = x_0x_{-1} - 1 < 0$. From (1.1) and by induction, we then have that $-1 < x_n < 0$ for all $n \geq -1$. □

Remark 2.6. Notice, if the solution is not periodic or eventually periodic with minimal period three, then as we see from Theorem 2.5 and as we shall see from Theorems 4.1–4.5, either the solution is bounded, while inside the invariant interval $(-1,0)$, or the solution becomes unbounded.

3. Stability and Convergence of Solutions of (1.1)

In this section, we determine the stability nature of the two equilibria of (1.1) and leave open for the reader the possibility of convergence of solutions to the negative equilibrium.

Lemma 3.1. *The negative equilibrium solution of* (1.1), $\overline{x}_2$, *is locally asymptotically stable. The positive equilibrium solution of* (1.1), $\overline{x}_1$, *is unstable.*

Proof. The characteristic equation of the equilibria of (1.1) is the following:

$$\lambda^2 - \overline{x}_{1,2}\lambda - \overline{x}_{1,2} = 0, \tag{3.1}$$

with eigenvalues

$$\lambda_{\pm} = \frac{\overline{x}_{1,2} \pm \sqrt{\overline{x}_{1,2}^2 + 4\overline{x}_{1,2}}}{2}. \tag{3.2}$$

Therefore, it is easy to show that the eigenvalues for $\overline{x}_2$ are complex with $|\lambda_{\pm}| < 1$, and for $\overline{x}_1$, $|\lambda_+| > 1$ and $|\lambda_-| < 1$. □

Open Problem. If $-1 < x_{-1}, x_0 < 0$ show whether or not every solution $\{x_n\}_{n=-1}^{\infty}$ of (1.1) converges to $\overline{x}_2$ (see Theorem 2.5 as well as Lemma 3.1).

4. Unbounded Solutions of (1.1)

In this section, we find sets of initial conditions of (1.1) for which unbounded solutions exist.

First observe that when the initial conditions $x_{-1}, x_0 > \overline{x}_1$ or $x_{-1}, x_0 < -1$, then existence of unbounded solutions appears. Specifically, the following two theorems will show existence of unbounded solutions relative to the set of these initial conditions. The theorem below is included to make this paper self-contained. One can see that the theorem below overlaps with [1, Theorem 9.1].

Theorem 4.1. *If* $x_{-1}, x_0 > \overline{x}_1 = (1+\sqrt{5})/2$, *then the solution* $\{x_n\}_{n=0}^{\infty}$ *is eventually strictly increasing and tends to* $+\infty$.

Proof. Since $x_0 > (1+\sqrt{5})/2$, we have $1/x_0 < 2/(1+\sqrt{5}) = (\sqrt{5}-1)/2$. Thus,

$$1 + \frac{1}{x_0} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-1}. \tag{4.1}$$

Therefore, $x_{-1} > 1 + 1/x_0$. Thus $x_{-1}x_0 > x_0 + 1$. Rewriting, $x_0 x_{-1} - 1 > x_0$. Hence, $x_1 > x_0$. It follows by induction that the solution $\{x_n\}_{n=0}^{\infty}$ is eventually strictly increasing.

To prove that the sequence tends to $+\infty$, we assume otherwise. Since the sequence is increasing, it must be bounded and thus must converge. But, the equation has only two equilibria, and they are both less than x_0. We have a contradiction. The proof is complete. □

Recently there has been considerable interest in showing the existence of monotonic solutions of nonlinear difference equations. Various methods for this can be found, for example, in [6, 16–23].

Theorem 4.2. *Suppose* $x_{-1}, x_0 < -1$. *Then*

$$0 < x_1 < x_4 < x_7 < \cdots,$$
$$\cdots < x_8 < x_6 < x_5 < x_3 < x_2 < x_0 < -1, \tag{4.2}$$

and the subsequences $\{x_{3n}\}_{n=0}^{\infty}$ *and* $\{x_{3n+2}\}_{n=0}^{\infty}$ *tend to* $-\infty$ *and* $\{x_{3n+1}\}_{n=0}^{\infty}$ *tends to* $+\infty$.

Proof. We first have

$$x_1 = x_0 x_{-1} - 1 > 0. \tag{4.3}$$

Then obviously

$$x_2 = x_1 x_0 - 1 < -1. \tag{4.4}$$

But we will prove that $x_2 < x_0 < -1$.

Since $(x_0+1)^2 > 0$, we have that $x_0^2 + 2x_0 + 1 > 0$. Thus, $-1 < 2/x_0 + 1/x_0^2$. Since $x_{-1} < -1$, we must have $x_{-1} < 2/x_0 + 1/x_0^2$. Thus $x_{-1}x_0 > 2 + 1/x_0$. Therefore, $x_{-1}x_0 - 1 > 1 + 1/x_0$. Hence $x_1 > 1 + 1/x_0$. From here we have $x_1 x_0 - 1 < x_0$, and the result that $x_2 < x_0$ follows.

Next, we show that x_3 is not only less than -1, but it is less than x_2. To this end, since $x_2 < x_0$, and $x_1 = x_0 x_{-1} - 1 > 0$, we have $x_1 x_2 < x_1 x_0$. This gives $x_2 x_1 - 1 < x_1 x_0 - 1$, and hence, $x_3 < x_2$.

To show that $x_4 > x_1$, we start by observing that $x_3 < -1$, $x_1 > 0$, and so $x_3 < x_1$. Thus, since $x_3 = x_2 x_1 - 1 < x_1$, we get, with $x_2 + 1 < 0$,

$$\begin{aligned}
& x_1 x_2 - x_1 - 1 < 0 \\
& \Longrightarrow x_1(x_2 - 1) < 1 \\
& \Longrightarrow x_1(x_2 - 1)(x_2 + 1) > x_2 + 1 \\
& \Longrightarrow x_1 x_2^2 - x_1 > x_2 + 1 \\
& \Longrightarrow x_1 x_2^2 - x_2 - 1 > x_1 \\
& \Longrightarrow x_2(x_2 x_1 - 1) - 1 > x_1 \\
& \Longrightarrow x_3 x_2 - 1 > x_1 \\
& \Longrightarrow x_4 > x_1.
\end{aligned} \tag{4.5}$$

By induction, it can be proved that

$$0 < x_1 < x_4 < x_7 < \cdots,$$
$$\cdots < x_8 < x_6 < x_5 < x_3 < x_2 < x_0 < -1, \tag{4.6}$$

that is, $\{x_{3n+1}\}_{n=0}^{\infty}$ is a positive increasing subsequence and $\{x_{3n}\}_{n=0}^{\infty}$ and $\{x_{3n+2}\}_{n=0}^{\infty}$ are negative decreasing subsequences.

Next we verify that these subsequences are unbounded and thus that our solution is unbounded. Assume that the two decreasing sequences, $\{x_{3n}\}_{n=0}^{\infty}$ and $\{x_{3n+2}\}_{n=0}^{\infty}$, are bounded from below. Then they each must converge to a finite limit (which is the same finite limit and less than $\overline{x}_2$). But by (1.1), the third increasing subsequence, $\{x_{3n+1}\}_{n=0}^{\infty}$, must also converge to a finite limit (which is positive), where

$$x_{3n+1} = x_{3n}x_{3n-1} - 1 = |x_{3n}| \cdot \left|x_{3(n-1)+2}\right| - 1. \tag{4.7}$$

This is impossible because there are no periodic solutions with minimal period two (see Section 2). So $\{x_{3n}\}_{n=0}^{\infty}$ and $\{x_{3n+2}\}_{n=0}^{\infty}$ are unbounded (they tend to $-\infty$) and thus $\{x_{3n+1}\}_{n=0}^{\infty}$ is also unbounded (it tends to $+\infty$) by (1.1) again. □

Theorem 4.3. *Suppose that $x_{-1} < 0$ and $x_0 > 0$. Then the solution of* (1.1) *has three subsequences $\{x_{3n}\}_{n=1}^{\infty}$, $\{x_{3n+1}\}_{n=1}^{\infty}$, and $\{x_{3n+2}\}_{n=1}^{\infty}$, where two of them tend to $-\infty$ and one tends to $+\infty$.*

Proof. Note that $x_1 = x_{-1}x_0 - 1 < -1$ and $x_2 = x_0x_1 - 1 < -1$. Then the result follows from Theorem 4.2. □

It might be interesting to note that if the order of the initial conditions in the above theorem is changed, the behavior of the solution can be drastically different. For example, if we let $x_{-1} = -0.58$ and $x_0 = 0.618$ then the solution does what the above theorem guarantees. On the other hand, if $x_{-1} = 0.618$ and $x_0 = -0.58$, then the solution enters the interval $(-1, 0)$, remains there, and thus is bounded. Furthermore, it should be pointed out that if the initial conditions are such that $x_{-1} > 0$ and $x_0 < 0$, then the solution in certain cases is bounded and in other cases is unbounded.

In addition, observe that if $0 < x_{-1}, x_0 < \overline{x}_1$, then the solutions of (1.1) exhibit somewhat chaotic behavior relative to the initial conditions. A little change in the initial conditions can cause a drastic difference in the long-term behavior of the solutions. For instance, if $x_{-1} = 1.5$ and $x_0 = 1.6$, then the solution enters and then remains in the interval $(-1, 0)$, and hence is bounded. Whereas if $x_{-1} = 1.5$ and $x_0 = 1.61$, then the solution has three unbounded subsequences: one tending to $+\infty$ and the other two tending to $-\infty$.

Theorem 4.4. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of* (1.1). *Suppose that*

(i) $0 < x_{-1}, x_0 < 1$;

(ii) $x_0^2x_{-1}^2 - 2x_0x_{-1} + 1 - x_{-1} > 0$.

Then $\{x_{3n}\}_{n=2}^{\infty}$ is a positive increasing subsequence; $\{x_{3n+2}\}_{n=1}^{\infty}$ is a negative decreasing subsequence; $\{x_{3n+1}\}_{n=2}^{\infty}$ is a negative decreasing subsequence. Consequently, there does not exist an $N \geq -1$ such that $-1 < x_n < 0$ for all $n \geq N$. Furthermore, all three subsequences are unbounded and so the solution is unbounded.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of (1.1). Suppose that

(i) $0 < x_{-1}, x_0 < 1$;

(ii) $x_0^2x_{-1}^2 - 2x_0x_{-1} + 1 - x_{-1} > 0$.

Since $x_0x_{-1} < 1$ by Hypothesis (i), then we have

$$x_1 = x_0x_{-1} - 1 < 0. \tag{4.8}$$

Then, since $x_1x_0 < 0$, we have

$$x_2 = x_1x_0 - 1 < -1. \tag{4.9}$$

We compute the sign of $x_3 = x_2x_1 - 1$:

$$\begin{aligned}
&x_3 > 0 \\
&\Longleftrightarrow x_2x_1 - 1 > 0 \\
&\Longleftrightarrow (x_1x_0 - 1)x_1 - 1 > 0 \\
&\Longleftrightarrow x_1^2x_0 - x_1 - 1 > 0 \\
&\Longleftrightarrow (x_0x_{-1} - 1)^2x_0 - (x_0x_{-1} - 1) - 1 > 0 \\
&\Longleftrightarrow x_0^3x_{-1}^2 - 2x_0^2x_{-1} + x_0 - x_0x_{-1} + 1 - 1 > 0 \\
&\Longleftrightarrow x_0^2x_{-1}^2 - 2x_0x_{-1} + 1 - x_{-1} > 0,
\end{aligned} \tag{4.10}$$

which is true by Hypotheses (i) and (ii). We next have

$$x_4 = x_3x_2 - 1 < -1 \tag{4.11}$$

since $x_3 > 0$, $x_2 < 0$. Then, also,

$$x_5 = x_4x_3 - 1 < -1 \tag{4.12}$$

since $x_4 < 0$, $x_3 > 0$.

With $x_4, x_5 < -1$ replacing $x_{-1}, x_0 < -1$ in Theorem 4.2, we have that

$$\begin{gathered}
0 < x_6 < x_9 < x_{12} < \cdots, \\
\cdots < x_{13} < x_{11} < x_{10} < x_8 < x_7 < x_5 < -1.
\end{gathered} \tag{4.13}$$

Hence, $\{x_{3n}\}_{n=2}^{\infty}$ is a positive increasing subsequence, $\{x_{3n+1}\}_{n=2}^{\infty}$ is a negative decreasing subsequence, and $\{x_{3n+2}\}_{n=1}^{\infty}$ is a negative decreasing subsequence, and so there does not exist an $N \geq -1$ such that $-1 < x_n < 0$ for all $n \geq N$. Furthermore, by Theorem 4.2, $\{x_{3n}\}_{n=2}^{\infty}$ tends to $+\infty$ and $\{x_{3n+1}\}_{n=2}^{\infty}$, and $\{x_{3n+2}\}_{n=1}^{\infty}$ both tend to $-\infty$. □

In conclusion, we show that, for a certain range of initial conditions, the solution of (1.1) is either unbounded or eventually enters the interval $(-1, 0)$ and remains there.

Theorem 4.5. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of* (1.1). *Suppose that $1 < x_{-1}, x_0 < \overline{x}_1$. Then one of the following occurs.*

(i) *The solution is unbounded.*

(ii) *There exists $n_0 \geq 1$ such that $x_n \in (-1, 0)$ for all $n \geq n_0$.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of (1.1). Assume that

$$1 < x_{-1}, x_0 < \overline{x}_1 = \frac{1+\sqrt{5}}{2}. \tag{4.14}$$

We will show that the solution is decreasing while the terms are positive. Since $1 < x_0 < (1+\sqrt{5})/2$, we have

$$\frac{1}{x_0} > \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}. \tag{4.15}$$

Thus

$$1 + \frac{1}{x_0} > 1 + \frac{2}{1+\sqrt{5}} = 1 + \frac{\sqrt{5}-1}{2} > x_{-1}, \tag{4.16}$$

which, since $x_0 > 0$, implies that

$$x_0 > x_0 x_{-1} - 1 = x_1. \tag{4.17}$$

Observe then that we have a decreasing sequence of terms as long as the terms are positive. Since our decreasing sequence is bounded above by $\overline{x}_1$ and since $\overline{x}_2$ is less than zero, this sequence is not bounded below by zero and thus crosses over to negative values.

Let x_N be the last positive term and x_{N+1} the first negative term of our solution. We now determine ranges of values of terms of our solution beginning with x_N and x_{N+1}, using "n" in place of "N" for convenience.

(1) x_n and x_{n+1}: observe that

$$x_n x_{n-1} - 1 > -1 \Longleftrightarrow x_n x_{n-1} > 0, \tag{4.18}$$

which is true. Hence, $-1 < x_{n+1} < 0$. Note that this, in turn, implies that $0 < x_n x_{n-1} < 1$. Then at least $0 < x_n < 1$ since $x_n < x_{n-1}$.

(2) x_{n+2}: by Statement (1), $-1 < x_{n+1}x_n < 0$ and so $-2 < x_{n+2} < -1$.

(3) $x_{n+3}, x_{n+4}, \ldots$.

Case 1 ($x_{n+3} > 0$). Then from Statement (2), $x_{n+4} = x_{n+3}x_{n+2} - 1 < -1$ and thus $x_{n+5} = x_{n+4}x_{n+3} - 1 < -1$. By Theorem 4.2, the solution is unbounded.

Case 2 ($x_{n+3} < 0$). We wish to show that $x_{n+3} > -1$:

$$x_{n+3} = x_{n+2}x_{n+1} - 1 > -1 \Longleftrightarrow x_{n+2}x_{n+1} > 0, \tag{4.19}$$

which is true since, by Statements (1) and (2), $x_{n+1}, x_{n+2} < 0$.

We next show that $-1 < x_{n+4} < 0$. First we show that $x_{n+4} < 0$:

$$\begin{aligned}
&x_{n+4} = x_{n+3}x_{n+2} - 1 < 0 \\
&\Longleftrightarrow (x_{n+2}x_{n+1} - 1)x_{n+2} - 1 < 0 \\
&\Longleftrightarrow \left(x_{n+1}^2 x_n - x_{n+1} - 1\right)(x_{n+1}x_n - 1) - 1 < 0 \\
&\Longleftrightarrow x_{n+1}^3 x_n^2 - 2x_{n+1}^2 x_n + x_{n+1} - x_{n+1}x_n + 1 - 1 < 0 \\
&\Longleftrightarrow x_{n+1}^3 x_n^2 - 2x_{n+1}^2 x_n + x_{n+1} - x_{n+1}x_n < 0 \\
&\Longleftrightarrow x_{n+1}^2 x_n^2 - 2x_{n+1}x_n + 1 - x_n > 0 \\
&\Longleftrightarrow x_{n+1}x_n(x_{n+1}x_n - 2) - (x_n - 1) > 0 \\
&\Longleftrightarrow \left(x_{n+2}^2 - 1\right) - (x_n - 1) > 0 \\
&\Longleftrightarrow x_{n+2}^2 > x_n,
\end{aligned} \tag{4.20}$$

which is true since, by Statements (1) and (2), $x_{n+2}^2 > 1$ and $0 < x_n < 1$.

Second, we show that $x_{n+4} > -1$: notice that

$$x_{n+4} = x_{n+3}x_{n+2} - 1 > -1 \Longleftrightarrow x_{n+3}x_{n+2} > 0, \tag{4.21}$$

which is true since $x_{n+3} < 0$ and $x_{n+2} < 0$ by Statement (2). Because $-1 < x_{n+3}, x_{n+4} < 0$, it follows from Theorem 2.5 that there exists $n_0 \geq 1$ such that $x_n \in (-1, 0)$ for all $n \geq n_0$. □

5. Conclusions and Future Work

It is of interest to continue the investigation of the monotonicity, periodicity, and boundedness nature of solutions of (1.1). It is of further interest to extend our study of solutions of (1.1) to an equation with an arbitrary constant parameter or a nonautonomous parameter, or to an equation with arbitrary delays.

(i) $x_{n+1} = x_n x_{n-1} - c$, $n = 0, 1, \ldots$, where c is any real.

(ii) $x_{n+1} = x_n x_{n-1} - c_n$, $n = 0, 1, \ldots$, where $\{c_n\}_{n=0}^{\infty}$ is a periodic sequence of real numbers.

(iii) $x_{n+1} = x_{n-\ell}x_{n-k} - 1$, $n = 0, 1, \ldots$, where $\ell, k \in \{0, 1, \ldots\}$.

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