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Solutions of the Einstein-Maxwell Equations with Many Black Holes

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Abstract. In Newtonian gravitational theory a system of point charged particles can be arranged in static equilibrium under their mutual gravitational and electrostatic forces provided that for each particle the charge, e, is related to the mass, m, by $e = G^{\frac{1}{2}}m$. Corresponding static solutions of the coupled source free Einstein-Maxwell equations have been given by Majumdar and Papapetrou. We show that these solutions can be analytically extended and interpreted as a system of charged black holes in equilibrium under their gravitational and electrical forces.

We also analyse some of stationary solutions of the Einstein-Maxwell equations discovered by Israel and Wilson. If space is asymptotically Euclidean we find that all of these solutions have naked singularities.

I. Introduction

In Newtonian theory a system of point charged particles can remain in static equilibrium if the charges e_i are all of the same sign and related to the masses m_i by

$$|e_i| = G^{\frac{1}{2}} m_i \,. \tag{1.1}$$

No matter how the particles are arranged, if this condition is satisfied, the electrostatic repulsions exactly balance the gravitational attractions. In 1947, Majumdar [1] and Papapetrou [2] independently discovered a class of static solutions to the source free Einstein-Maxwell equations which correspond to this Newtonian situation. The source free solutions given by Majumdar and Papapetrou are not geodesically complete. One way of completing them is to match the solutions to static interior solutions of dust whose charge density equals its mass density [3]. It is even more interesting, however, to study the analytic extension of the source free solutions themselves in view of the fruitful studies already carried out on the analytic extensions of the Schwarzschild [4], Reissner-Nordstrom [5] and Kerr [6–7] geometries. These latter solutions are all found to be asymptotically Euclidean. They each contain (for certain

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values of their parameters) event horizons which represent the boundary of the set of all events which can be reached by observers whose paths begin at infinity and end at infinity. Further, all the singularities of these solutions are found to lie within the horizons. Geometries whose singularities are all isolated from infinity in this way are said to be black hole solutions. If there are singularities which are not contained within event horizons then the singularities are said to be naked.

In Sections II and III the analytic extension Majumdar-Papapetrou geometries is carried out. It is found that either the Majumdar-Papapetrou geometries have naked singularities or correspond to systems of charged black holes in equilibrium under their mutual gravitational and electrical forces.

Recently, Israel and Wilson [8] have generalized the techniques of Majumdar and Papapetrou to find a class of *stationary* solutions to the source free Einstein-Maxwell equations. In Section IV some typical solutions of this class are discussed. We find that if space-time is asymptotically Euclidean then all of the solutions analysed have naked singularities. We conjecture that the only stationary solutions of the source free Einstein-Maxwell equations with more than one black hole and without naked singularities are those static Majumdar-Papapetrou solutions which correspond to many black holes with charges and masses related by Eq. (1.1).

II. Majumdar-Papapetrou Geometries

The metric of the Majumdar-Papapetrou geometries has the form

$$ds^2 = -U^{-2}(\boldsymbol{x})\,dt^2 + U^2(\boldsymbol{x})\,d\boldsymbol{x}\cdot d\boldsymbol{x}\,. \tag{2.1}$$

Here, $\boldsymbol{x}$ denotes the position vector in a flat three dimensional space which we will call the background space. Thus for example, in Cartesian coordinates $\boldsymbol{x} = (x, y, z)$

$$d\boldsymbol{x}\cdot d\boldsymbol{x} = dx^2 + dy^2 + dz^2\,. \tag{2.2}$$

The only non-vanishing component of the vector potential $A_\mu(\boldsymbol{x})$ is the electrostatic potential $\Phi(\boldsymbol{x})$, which is related simply to the metric by[1]

$$\Phi(\boldsymbol{x}) = A_t(\boldsymbol{x}) = U^{-1}(\boldsymbol{x})\,. \tag{2.3}$$

In writing this relation we choose to normalize the electrostatic potential to unity at infinity rather than zero. The relation could have also been

[1] Here and in the following we use units in which $c = G = 1$. Charge and mass are both measured in units of length so that in the Newtonian limit the force laws of Newton and Coulomb have the form m^2/r^2 and e^2/r^2 respectively.

chosen with the opposite sign which would not affect the geometry but reverse the sign of all charges. Remarkably, the source-free Einstein-Maxwell equations then reduce to Laplace's equation for $U(\boldsymbol{x})$ on the background space.

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 . \tag{2.4}$$

In this coordinate system any solution may be extended until U has a singularity or until it vanishes. The singularities may be of several types corresponding to the source of U being point monopoles, point dipoles, line charges, etc. in the background space. To begin with let us consider the simplest examples when the sources are point monopoles,

$$\begin{aligned} U(\boldsymbol{x}) &= 1 + \sum_i \frac{m_i}{r_i} , \\ r_i &= [(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{\frac{1}{2}} . \end{aligned} \tag{2.5}$$

From the form of the metric in the weak field region when the sources are widely separated in the background space we may identify the m_i with the total mass enclosed by a large sphere surrounding each source. We, therefore, require

$$m_i > 0 . \tag{2.6}$$

Thus, in the coordinate patch obtain by letting $\boldsymbol{x}$ run over a complete background space, $U(\boldsymbol{x})$ is non-vanishing and the metric is regular except where $r_i = 0$. The charges inside a closed surface surrounding each source may be identified in a similar way from the electrostatic potential [Eq. (2.3)] or by computing the flux of electric field through such a surface. One finds

$$e_i = m_i . \tag{2.7}$$

In the case of a single point source one would expect to recover the equal charge and mass Reissner-Nordstrom solution. This is indeed the case as may be seen by expressing the metric in a more familiar form through the coordinate transformation $r \to r - m$. The analytic extension of this metric has been given by Carter [9].

For two sources $U(\boldsymbol{x})$ has the form

$$U(\boldsymbol{x}) = 1 + \frac{m_1}{r_1} + \frac{m_2}{r_2} . \tag{2.8}$$

In order to facilitate discussion of this metric we shall frequently refer to pictures of the background space such as Fig. 1. A region of space time with this metric and coordinates ranging over a complete flat background space will be called a Type I region.

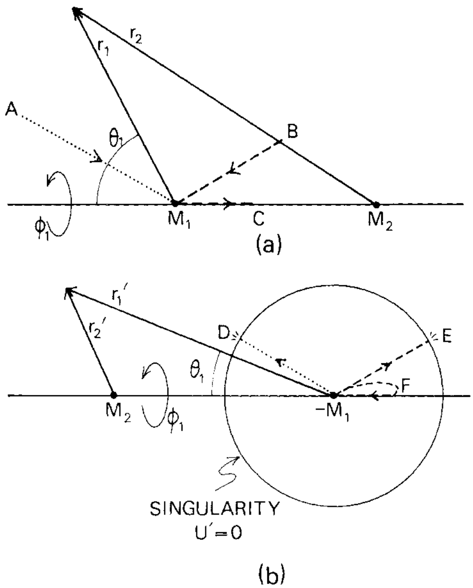

Fig. 1. The background space for two black holes with equal masses and charges is shown in a. The event horizons appear here as the points $r_1=0$ and $r_2=0$ although they have a finite surface area. If one passes through the event horizon associated with m_1, via a typical path shown by one of the dotted lines, then one emerges in a second background space shown in b. In this space one is prevented from reaching infinity by a real singularity at $U=0$. Once on the second sheets one can only either meet the singularities as in path AD and BE or re-emerge through the event horizon as in BFC

As in the Reissner-Nordstrom case the singularity at $U=0$ in the metric is only a singularity in the choice of coordinates. To see that the geometry is regular at $r_1=0$, for example, first transform the coordinates of the background space to spherical polar coordinates about $r_1=0$. It is then easily seen that $r_1=0$, $t=\text{const.}$ does not label a point but a surface of area $4\pi m_1^2$. The regular nature of the geometry at $r_1=0$ may be seen by making the coordinate transformation

$$\begin{aligned} t &= u + F(r_1)\,, \\ \frac{dF}{dr_1} &= \left(1+\frac{m_1}{r_1}+\frac{m_2}{a}\right)^2 \equiv V^2(r_1)\,, \end{aligned} \tag{2.9}$$

where a is the separation of the sources in the background coordinates. The metric then becomes

$$\begin{aligned} ds^2 = &-U^{-2}du^2 - 2du\,dr_1(V/U)^2 + [U^2 - V^2]\,dr_1^2 \\ &+ r_1^2U^2(d\theta^2+\sin^2\theta\,d\varphi^2)\,. \end{aligned} \tag{2.10}$$

Then, since near $r_1 = 0$, U has the form

$$U(r_1) = 1 + \frac{m_1}{r_1} + \frac{m_2}{a} + O(r_1), \tag{2.11}$$

it is not difficult to verify that the form of the metric given in (2.10) is nonsingular at $r_1 = 0$. Further, it follows from Eq. (2.10) that $r_1 = 0$ is a stationary null surface. If we write

$$r'_1 = -r_1, \quad r'_2 = [r_1'^2 + a^2 + 2ar'_1 \cos\theta]^{\frac{1}{2}}, \tag{2.12}$$

then when we continue r_1 through zero we reach a region which is described by a metric of the form (2.1) but with $U(\boldsymbol{x})$ replaced by

$$U'(\boldsymbol{x}') = 1 - \frac{m_1}{r'_1} + \frac{m_2}{r'_2}. \tag{2.13}$$

A diagram of the new background space is shown in Fig. 16. We call a region described by this metric Type II_1.

The function $U'(\boldsymbol{x}')$ has the same form as the electrostatic potential of a charge $-m_1$, at $r'_1 = 0$ and a charge $+m_2$ at $r'_2 = 0$. It is varying from large negative values near $r'_1 = 0$ to large positive values at $r'_2 = 0$ and must therefore vanish on some intermediate equipotential surface. On this surface the metric is singular. Since U' is normalized to 1 at large values of r'_1, r'_2, the surface $U'(\boldsymbol{x}') = 1$ divides those equipotential surfaces which completely enclose m_1 from those which completely enclose m_2. The singular surface $U'(\boldsymbol{x}') = 0$ therefore always completely encloses $r'_1 = 0$.

To see that the singularity of the metric at $U' = 0$ is a genuine singularity one has only to compute the field invariant

$$J = F_{\mu\nu} F^{\mu\nu} = \left(\frac{\nabla U'}{U'^2}\right)^2. \tag{2.14}$$

Where $U' = 0$, there J diverges.

While $U' = 0$ appears as a surface in the background coordinates, it actually is a point. To see this, compute the area of a surface just inside $U' = 0$ and let that surface tend to $U' = 0$. The surface area is

$$\int [U'(\boldsymbol{x}')]^2 \, da', \tag{2.15}$$

where da' is an element of surface area in the background space. On the surface $U' = 0$ this vanishes.

The metric given in Eq. (2.10) is extendable in other directions through $r_1 = 0$. Following Carter in the Reissner-Nordstrom case this may be accomplished by introducing in place of u a new variable w defined by

$$u + w = 2F(r_1). \tag{2.16}$$

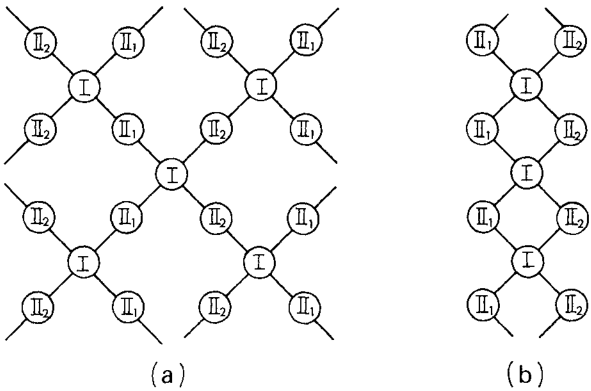

Fig. 2. a The most general possible extension of the two black hole Majumdar-Papapetrou metric. The three possible types of regions described in the text are I, II_1 and II_2. A complete solution may be built up by successively patching together these regions using the coordinate transformation in Eqs. (2.9) and (2.12). b A possible completion obtained by identifying some of the regions of a given type shown in a. In this identification two observers who fell into separate black holes could later meet again in a replica of their original region I.

The metric one obtains has formally the same form as Eq. (2.10) with u replaced by w. However, since the entire range of w can be obtained at fixed u by allowing r_1 to vary over either positive or negative values this is to be interpreted as two extensions of Eq. (2.10), one from the region $r_1 < 0$ and the other from $r_1 > 0$. In the case we start from $r_1 < 0$ new Type I region is reached by this extension. In the case we start from $r_1 > 0$ a new Type II_1 region is reached.

A Type II_1 region is bounded by the singularity and the null surface $r'_1 = 0$. It can only be extended by going through $r'_1 = 0$ in one of the two ways discussed above. A Type I region is bounded by $r_1 = 0$, $r_2 = 0$ and infinity. It can be extended either by going through the null surfaces $r_1 = 0$ or $r_2 = 0$. Extending through $r_2 = 0$ one reaches a region which we will call Type II_2 which clearly has the same general properties as Type II_1. In this way a complete structure of overlaping coordinate patches can be built up as in the Reissner-Nordstrom case. The general structure is quite complicated as is illustrated in Fig. 2a. Less general completions may be obtained by identifying sequences of regions of a given type. One of the more interesting of these has single sequence of regions of Type I as shown in Fig. 2b. By taking various two dimensional slices of this latter space we can represent it in terms of diagrams in which the null lines run at 45°. Such a diagram is illustrated in Fig. 3. We can also illustrate it by more schematic perspective diagrams such as Fig. 4.

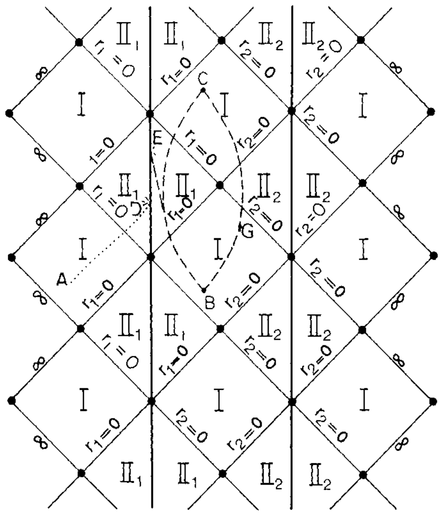

Fig. 3. This diagram shows the geometry of the axis of the two black hole Majumdar-Papapetrou solutions with the coordinate patches identified as in Fig. 2b. The upward direction is timelike and the horizontal direction spacelike. Light rays move on 45° lines. The heavy vertical lines represent the point singularities (excepting the dotted points which are at infinity). The projections of the typical paths shown in Fig. 1 are shown here by corresponding dotted lines. Note that with this identification a second observer could pass through the event horizon $r_2 = 0$ via path BGC and meet an observer who followed BFG through $r_1 = 0$

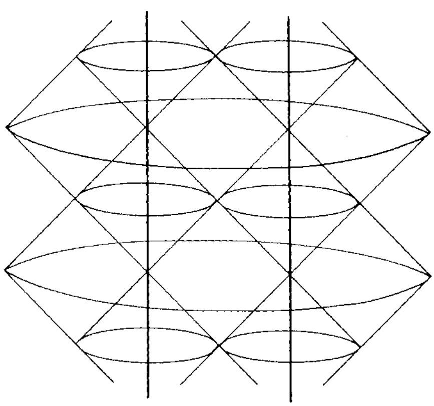

Fig. 4. In this diagram is a schematic portrayal of the two black hole Majumdar-Papapetrou solution. The singularities are once again represented as heavy vertical lines. The event horizon are cones and infinity is the outer surface

The Majumdar-Papapetrou geometry illustrated in Figs. 2–4 is clearly very special. Not only do its singularities have a special relation between their masses and charges but its parts have been arranged in such a way that by passing through $r_1 = 0$ and back again in a timelike direction one can reach the same points as are accessible in a similar way through $r_2 = 0$.

Nevertheless, it illustrates a geometry in which the two null surfaces $r_1 = 0$ and $r_2 = 0$ are the boundary of the set of events which can be reached by any observer who starts from infinity and returns there. The surfaces $r_1 = 0$ and $r_2 = 0$ are thus two separate components of the event horizon. As all the singularities are contained within these surfaces this Majumdar-Papapetrou geometry has two distinct black holes. The way to generalize the solution through more complicated diagrams of the type of Fig. 2 or by adding more black holes as in Eq. (2.5) should now be clear.

III. Majumdar-Papapetrou Geometries with Naked Singularities

There are other possible geometries which can be generated by the Majumdar-Papapetrou prescription besides those where U has only discrete point sources. In this section all of these alternatives will be shown to have naked singularities.

Suppose outside of a large sphere in the background space one is given a solution U of Laplace's equation which approaches unity. This function U generates an asymptotically flat Majumdar-Papapetrou metric which can be analytically extended inward in the coordinates of Eq. (2.1) until either U vanishes or becomes infinite. In the case U vanishes $1/U$ becomes infinite. Then, along at least some curve which approaches the point of vanishing U, $J = [\nabla(1/U)]^2$ will diverge giving a naked singularity.

The cases where U is infinite divide naturally into those where it is infinite at a point or a line in the background space. If the infinities of U are not to be naked singularities then there must be no equipotential surface of U which contains a point where U approaches infinity. If this were the case approaching the point of infinity along this surface U would be constant while $(\nabla U)^2$ would diverge. This would imply an infinite value of the electromagnetic invariant J and a naked singularity. To avoid a naked singularity U must therefore become infinite from all directions at once if it becomes infinite at all. Put in another way, $|U|$ must be bounded below at a singularity. In the case of point infinities a theorem from potential theory [10] shows that U must then be of the form

$$U(x) = w(x) + c/r \tag{3.1}$$

where c is a constant, r the distance from the singularity and w is regular there. This, however, is the case already discussed in Section II.

In the case of line singularities, if it is assumed that the potential close to the line approaches that of some straight line singularity, then it is not difficult to generalize the potential theory result mentioned above to show that if no equipotentials intersect the singularity, then U must diverge no faster than

$$U(x) \approx f(z) \log(\varrho), \quad \varrho \to 0, \tag{3.2}$$

close to the line. Here, ϱ and z are cylindrical coordinates with axis tangent to the line singularity at the point in question. The behavior or Eq. (3.2), however, itself gives rise to a divergent J.

The only Majumdar-Papapetrou geometries which represent black holes are then those for which U has only point monopoles as sources. A consequence of this is that all these black hole solutions have event horizons with spherical topology as is required by a theorem of Hawking [11].

IV. The Israel-Wilson Metrics

Recently, Israel and Wilson have generalized the methods of Majumdar and Papapetrou and found a class of *stationary* solutions to the source free Einstein-Maxwell equations. Their solutions have the metric

$$ds^2 = -|U|^{-2}(dt + \boldsymbol{\omega} \cdot d\boldsymbol{x})^2 + |U|^2 d\boldsymbol{x} \cdot d\boldsymbol{x} \tag{4.1}$$

where $U(\boldsymbol{x})$ is any *complex* solution to Laplace's equation in the background space

$$\nabla^2 U = 0. \tag{4.2}$$

The vector $\boldsymbol{\omega}$ is found by solving the equation

$$\nabla \times \boldsymbol{\omega} = i[U \nabla \bar{U} - \bar{U} \nabla U] \equiv \boldsymbol{\tau} \tag{4.3}$$

the electrostatic potential Φ and a magnetic scalar potential χ, defined by the relations

$$F_{ti} = \partial_i \Phi \tag{4.4}$$

$$F^{ij} = |U|^2 \varepsilon^{ijk} \partial_k \chi, \tag{4.5}$$

are related to the function U by

$$\Phi + i\chi = 1/U. \tag{4.6}$$

For convenience U and Φ will be always taken to be normalized to 1 at infinity in the background space and χ to be normalized to zero.

We do not propose here to examine for black holes all of the solutions which can be generated by the Israel-Wilson method but only some of the more interesting examples. In particular, only those solutions will be considered for which the sources of U in the background space are discrete points,

$$U = 1 + \sum_i \frac{m_i}{r_i}$$
$$r_i = [(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^{\frac{1}{2}} . \tag{4.7}$$

The numbers m_i and $\boldsymbol{x}_i = (x_i, y_i, z_i)$ may be complex.

Israel and Wilson have examined the cases where the m_i are real and the $\boldsymbol{x}_i$ complex. For the case of a single point source of U they find the solution to be the charged Kerr metric [12] with equal charge and mass. This solution has a naked singularity and no horizons. The singularity occurs on the ring where $r = 0$ and can be seen from the divergence of the electromagnetic invariants there. It is not difficult to see that this is also the case if m is simultaneously complex. Superpositions of discrete sources of U with the $\boldsymbol{x}_i$ complex also generally[2] result in naked singularities where U is infinite. This is because the infinities arise because of the vanishing of the r_i. Sufficiently near such points the geometry will behave like that of a single source for U which is singular there. In a search for black hole solutions of the type of Eq. (4.7) we may therefore restrict ourselves to those cases where the $\boldsymbol{x}_i$ are real and the m_i are complex

$$m_i = M_i + iN_i . \tag{4.8}$$

The simplest example of this type is when U has a single discrete source,

$$U = 1 + \frac{M + iN}{r} . \tag{4.9}$$

Using polar coordinates for the background space, the solution of Eq. (4.3) for ω can be taken as

$$\boldsymbol{\omega} = \boldsymbol{e}_{(\varphi)} \left(\frac{2N}{r} \right) \left(\frac{\cos\theta - 1}{\sin\theta} \right) . \tag{4.10}$$

If we introduce a new radial coordinate R defined by

$$R = r + M , \tag{4.11}$$

[2] The special cases where a sum of the form of Eq. (4.7) can be arranged to have no infinites are static and have already been considered.

then the metric takes the form

$$\begin{aligned} ds^2 = & -V^2\left(dt + 4N\sin^2\frac{\theta}{2}\right)^2 + V^2 dR^2 \\ & + (R^2+N^2)(d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \tag{4.12}$$

$$V^2 = \left(1+\frac{M}{R}\right)^2 + \frac{N^2}{R^2}. \tag{4.13}$$

This is just the electromagnetic generalization of NUT space [13] given by Brill [14] for the special case that his constant Φ has the value $8(N^2+M^2)$. The properties of this solution have been extensively discussed [13–16]. The most striking fact which emerges from these discussions is that in the $r>0$ region the geometry is regular if one assumes that the topology of the $r=$ const. hypersurfaces is S^3 while it is singular if one assumes the usual topology $R\times S^2$. The singularity in this latter case lies along the axis $\theta=\pi$. The presence of the singularity is signaled by an infinity in g^{tt} along this axis. The singularity on this axis may be removed by following Misner [15] and making the coordinate transformation

$$t = t' + 4N\varphi. \tag{4.14}$$

If φ is to have the interpretation of an angular coordinate, points which differ in coordinate φ by 2π must be identified. Points which differ in values of t or t' by $8\pi N$ must then also necessarily be identified and this leads directly to the topology S^3 for the $r=$const hypersurfaces. If one insists on preserving the topology $R\times S^2$ then the space will be singular on the axis $\theta=\pi$.

This type of singularity appears generally in the Israel-Wilson solutions we are discussing. To see this imagine introducing a polar coordinate system with one of the singularities of U as its center and consider the behavior of the projection of $\boldsymbol{\omega}$ along a unit vector $\boldsymbol{e}_{(\varphi)}$ of the background geometry in the φ-direction

$$\omega_{(\varphi)} = \boldsymbol{\omega}\cdot\boldsymbol{e}_{(\varphi)}. \tag{4.15}$$

The value of $\omega_{(\varphi)}$ at one point (z_1) on the polar axis may be related to the value of $\omega_{(\varphi)}$ at another point (z_2) on the axis through Stokes' theorem. Consider a circular loop of radius ϱ about the axis at point z. If

$$I(\varrho, z) = \oint \boldsymbol{\omega}\cdot d\boldsymbol{x} \tag{4.16}$$

is the line integral about this loop, then

$$I(\varrho, z_1) - I(\varrho, z_2) = \int_S \boldsymbol{\tau}\cdot d\boldsymbol{a}, \tag{4.17}$$

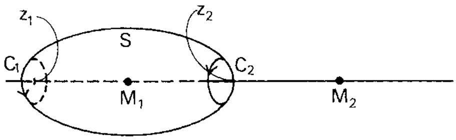

Fig. 5. A picture of the background space for the Israel-Wilson space with two point sources for U showing how a loop C_1 around the axis at one point can be distorted to a loop at another point C_2 sweeping out a two surface S. When the loops are shrunk to zero S will completely enclose the singularity of U at $r_1 = 0$

where S is a two surface in the background space bounded by the loop at z_1, and z_2 (see Fig. 5). One therefore has

$$\lim_{\varrho \to 0} 2\pi\varrho[\omega_{(\varphi)}(\varrho, z_1) - \omega_{(\varphi)}(\varrho, z_2)] = \int_V d^3x(\nabla \cdot \tau) \tag{4.18}$$

where V is the volume inside any closed surface containing the axis and intersecting it a z_1 and z_2. Since $\nabla \cdot \tau = 0$ except at the singularities of U it is not difficult to evaluate this integral to find

$$\lim_{\varrho \to 0} 2\pi\varrho[\omega_{(\varphi)}(\varrho, z_1) - \omega_{(\varphi)}(\varrho, z_2)] = 8\pi \sum_{i \in V} \left(N_i + \sum_j \frac{N_i M_j - N_j M_i}{|\boldsymbol{x}_i - \boldsymbol{x}_j|}\right). \tag{4.19}$$

The function $\omega_{(\varphi)}$ will thus in general be singular along some part of the axis. This will be reflected in the metric by a singularity in

$$g^{tt} = |U|^2 - |U|^{-2}\,\omega^2 . \tag{4.20}$$

As in the charged NUT case this singularity will be a real singularity of the geometry if the space has an asymptotically Euclidean topology.

The generic situation is illustrated by the Israel-Wilson space in which U has two point sources with imaginary strength.

$$U = 1 + \frac{m_1}{r_1} + \frac{m_2}{r_2} . \tag{4.21}$$

At large distances in the background space this solution will behave like the charged-NUT space discussed above, unless $N_1 = -N_2 \equiv N$. Only with this condition can the space be asymptotically flat and non-singular. To see this we examine the large r behavior of ω in a polar coordinate system whose axis is the line connecting the charges in the background space. Eq. (4.3) determines ω up to the gradient of a scalar. This reflects the fact that the gradient of a scalar $\lambda(x)$ can always be added to ω by the coordinate transformation $t \to t + \lambda(x)$. If the time coordinate is chosen properly ω can then be arranged to have only a φ-component

$$\omega = \boldsymbol{e}_{(\varphi)}\omega_{(\varphi)} , \tag{4.22}$$

with the asymptotic behavior

$$\omega_{(\varphi)} = \frac{2Na\sin\theta}{r^2} + O\left(\frac{1}{r^3}\right). \tag{4.23}$$

In these coordinates the space is clearly asymptotically flat with a total angular momentum Na (see Ref. [17], § 103). Having chosen the large r behavior of Eq. (4.16) for the function $\omega_{(\varphi)}$ we therefore find from Eq. (4.19) that $\omega_{(\varphi)}(0, z) = 0$ on the parts of the axis which stretch to infinity. Between $r_1 = 0$ and $r_2 = 0$, however, we have

$$\lim_{\varrho \to 0} [\varrho\,\omega_{(\varphi)}(\varrho, z)] = 4N\left(1 + \frac{M_1 + M_2}{a}\right) \equiv P, \tag{4.24}$$

and $\omega_{(\varphi)}$ diverges on the axis between the sources.

As in the charged NUT case this singularity can be removed by the transformation

$$t = t' + P\varphi. \tag{4.25}$$

If, however, we are to have a consistent interpretation of our coordinates at infinity φ must be an angular coordinate. Points which differ in values of t and t' by $2\pi P$ must now be identified and the surfaces of constant r will no longer have the topology $R \times S^2$.

It is straightforward to generalize this argument to the case of more than two sources for U and show that if some of the N_i are non-zero then the space must either be singular along some curve or asymptotically non-Euclidean in the sense that the topology of the surfaces of constant large r will no longer be $R \times S^2$.

We are concerned here with examining the Israel-Wilson metrics for possible black hole solutions. For this we shall require first that space have a well defined infinity. By this we will mean that the space is asymptotically flat with the usual topology of $R \times S^2$ for the hypersurfaces of constant large r in the background space. Second, we shall require that there are no singularities which are not hidden from infinity by event horizons. Under these conditions the only Israel-Wilson solutions with discrete sources for U which represent black hole solutions are those with all the $N_i = 0$. These, however, are just the static Majumdar-Papapetrou solutions discussed in Section II.

While there are no black hole Israel-Wilson metrics it is still of interest to investigate then analytic extension through $r_i = 0$. Wo confine ourselves here to the case when U has two point sources of equal and opposite imaginary strength. The extension through $r_1 = 0$ can be accomplished by techniques similar to those used for the Majumdar-

Papapetrou geometries. The relevant coordinate transformation is

$$t = u + F(r_1), \tag{4.26}$$

$$\frac{dF}{dr_1} = \left|1 + \frac{m_1}{r_1} + \frac{m_2}{a}\right|^2. \tag{4.27}$$

Since $\boldsymbol{\omega} \cdot d\boldsymbol{x}$ is always finite it is easily checked that the metric in these coordinates is finite as r_1 passes to negative values. A similar transformation will also remove the singularity at $r_1 = 0$ starting from the t' coordinate patch.

If we write $r'_1 = -r_1, r'_2 = [r'^2_1 + a^2 + 2ar'_1 \cos\theta_1]^{\frac{1}{2}}$ the metric for $r_1 < 0$ can be put in a form similar to Eq. (4.1) except that now

$$U = 1 - \frac{M_1}{r'_1} + \frac{M_2}{r'_2} - iN\left(\frac{1}{r'_1} + \frac{1}{r'_2}\right). \tag{4.28}$$

By continuing from the original region through $r_2 = 0$ one would reach a region where U has the same form except M_1 and M_2 would be interchanged. Finally, by continuing the metric given in Eq. (4.29) through $r'_2 = 0$ and defining $r''_2 = -r'_2$

$$r''_1 = [r'^2_2 + a^2 - 2ar'_2 \cos\theta_2]^{\frac{1}{2}} \tag{4.29}$$

one finds a region where

$$U = 1 - \frac{M_1}{r''_1} - \frac{M_2}{r''_2} + iN\left(\frac{1}{r''_2} - \frac{1}{r''_1}\right). \tag{4.30}$$

The complete space will be built up of these three types of patches.

If all the NUT type singularities are to be removed by making the time periodic then one must have simultaneous periodicities with periods

$$\begin{gathered} 8\pi N, \quad 8\pi N\left(1 \pm \frac{M_S}{a}\right), \quad 8\pi N\left(1 \pm \frac{M_D}{a}\right) \\ M_S = M_1 + M_2, \quad M_D = M_1 - M_2, \quad M_1 \geqq M_2. \end{gathered} \tag{4.31}$$

For these to be commensurate one must have

$$\begin{aligned} M_S &= \left(1 - \frac{1}{n}\right)a, \qquad n = 2, 3, \ldots, \\ M_D &= \left(1 - \frac{k}{n}\right)a, \qquad k = 1, 2, \ldots, n. \end{aligned} \tag{4.32}$$

With these restrictions the only remaining possibilities for singularities are the rings

$$r''_1 = r''_2 = M_S$$

where U vanishes in a region of the type (4.30). The conditions (4.32) show that there will always be such a ring consistent with $r''_1 + r''_2 \geqq a$. A straightforward calculation of the electromagnetic field invariants reveals that these points are indeed singular.

V. Conclusions

An extensive class of stationary solutions of the matter-free Einstein-Maxwell equations has been examined for black hole solutions. Of the enormous variety of possible metrics which can be generated by the Majumdar-Papapetrou-Israel-Wilson techniques we were able to find only one class in which the singularities were isolated from a reasonable infinity by event horizons. These were the static Majumdar-Papapetrou solutions corresponding to many black holes each with a spherical topology, each with $e=m$, remaining in equilibrium by the consequent balance of their electrostatic repulsion and gravitational attraction. In view of ones experience with the Reissner-Nordstrom and Newman-Kerr solutions [18, 19, 11] it seems reasonable conjecture that these together with the Majumdar-Papapetrou solutions discussed above are the only stationary asymptotically Euclidean solutions of the source free Einstein-Maxwell equations with no naked singularities.

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