SOLUTIONS OF THE EQUATIONS OF RADIATIVE TRANSFER

BY AN INVARIANT IMBEDDING APPROACH

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CHAPTER I

INTRODUCTION

The power in the invariant imbedding approach is that it replaces a system of linear integro-differential equations having two-point boundary conditions with a system of non-linear integro-differential equations possessing a set of initial conditions. It is the latter set of equations which are ideally suited for numerical calculations. With the conventional approach employing the X- and Y-functions, problems of unicity of the solutions are prevalent as was shown by Chandrasekhar (2) and Mullikin (4). Bellman et al. (1) successfully used the invariant imbedding technique for the solution of problems in radiative transfer in slabs of finite thickness with isotropic scattering. Kagiwada and Kalaba (3) have numerically solved the scalar equation of transfer for an atmosphere with a single scattering phase function of the form $p(\mu) = 1 + \mu$. This solution was accomplished by a Fourier expansion of the single scattering phase function in terms of associated Legendre polynomials. Similar expansions for the scattering and transmission functions yield Fourier coefficients with no explicit azimuth dependence. The resulting integro-differential equations have initial conditions which render them easily solvable by using highly accurate numerical integration schemes in double precision arithmetic. A similar technique will be used for scattering according to a Rayleigh phase function and Rayleigh phase matrix, resulting in an increase in the number of equations and their complexity. It should also be noted that when the solution corresponding to a certain optical depth is obtained, the solutions at all intermediate optical depths consistent with the step size used in the integration are also obtained.

The resulting calculations for both the scalar and matrix case illustrate the necessity for accounting for polarization effects in the theoretical calculations of resulting intensities.

CHAPTER BIBLIOGRAPHY

1. Bellman, R. E., R. E. Kalaba, and M. C. Prestrud, _Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness_, New York, American Elsevier Company, Inc., 1963.

2. Chandrasekhar, S., _Radiative Transfer_, New York, Dover Publications, 1960.

3. Kagiwada, H., and R. E. Kalaba, "Estimation of Local Anisotropic Scattering Properties using Measurements of Multiply Scattered Radiation," _The Journal of Quantitative Spectroscopy and Radiative Transfer_, VII (October, 1967), 295-303.

4. Mullikin, T. W., "A Nonlinear Integrodifferential Equation in Radiative Transfer," _The Journal of the Society for Industrial Mathematics_, XIII (June, 1965), 388-410.

CHAPTER II

REDUCTION OF THE MATRIX EQUATION OF TRANSFER

The equation of transfer for scattering according to a phase matrix can be written in the following form(1);

$$\mu \frac{d\underset{\sim}{I}(\tau,\mu,\phi)}{d\tau} = \underset{\sim}{I}(\tau,\mu,\phi) - \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} \underset{\sim}{P}(\mu,\phi;\mu',\phi')\underset{\sim}{I}(\tau,\mu',\phi')\,d\mu'\,d\phi' - \frac{1}{4\pi}\underset{\sim}{P}(\mu,\phi;-\mu_0,\phi_0)\underset{\sim}{F}e^{-\tau/\mu}$$

where $\pi\underset{\sim}{F}$ is the incident flux on the upper surface of the atmosphere in the direction $(-\mu_0, \phi_0)$ where $\theta_0 = \cos^{-1}\mu_0$ $(0 < \mu_0 \leq 1)$ and ϕ_0 defines the angle that the plane containing the z-axis and the incident beam makes with an arbitrary x-axis. The x-axis is taken to be in the direction of the incoming beam so that $\phi_0 = 0$. The matrix $\underset{\sim}{P}(\mu,\phi;\mu_0,\phi_0)$ is the single scattering phase matrix for the atmosphere. In the case of Rayleigh scattering, the single scattering matrix has the form

$$\underset{\sim}{P}(\mu,\phi;\mu_0,\phi_0) = \underset{\sim}{P}^{(0)}(\mu,\mu_0) + (1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\underset{\sim}{P}^{(1)}(\mu,\phi;\mu_0,\phi_0) + \underset{\sim}{P}^{(2)}(\mu,\phi;\mu_0,\phi_0)$$

where

$$\underset{\sim}{P}^{(0)}(\mu,\mu_0) = \frac{3}{4}\begin{pmatrix} 2(1-\mu^2)(1-\mu_0^2) + \mu^2\mu_0^2 & \mu^2 & 0 \\ \mu_0^2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\underset{\sim}{P}^{(1)}(\mu,\phi;\mu_0,\phi_0) = \frac{3}{4}\begin{pmatrix} 4\mu\mu_0\cos(\phi_0-\phi) & 0 & 2\mu\sin(\phi_0-\phi) \\ 0 & 0 & 0 \\ -4\mu_0\sin(\phi_0-\phi) & 0 & 2\cos(\phi_0-\phi) \end{pmatrix},$$

and

$$P^{(2)}(\mu,\phi;\mu_0,\phi_0) = \frac{3}{4}\begin{pmatrix} \mu^2\mu_0^2\cos 2(\phi_0-\phi) & -\mu^2\cos 2(\phi_0-\phi) & \mu^2\mu_0\sin 2(\phi_0-\phi) \\ -\mu_0^2\cos 2(\phi_0-\phi) & \cos 2(\phi_0-\phi) & -\mu_0\sin 2(\phi_0-\phi) \\ -\mu\mu_0^2\sin 2(\phi_0-\phi) & \mu\sin 2(\phi_0-\phi) & \mu\mu_0\cos 2(\phi_0-\phi) \end{pmatrix}.$$

Chandrasekhar (1) and Sekera (2) have introduced the properties of the Fourier components of the phase matrix which aid in the factorization of the matrix equations. For the case of the Rayleigh phase matrix a set of scalar equations was obtained. The resulting equations and their reduction may be shown as follows.

The diffusely reflected intensity $\underset{\sim}{I}(0,\mu,\phi)$, $0<\mu\leq 1$, $0\leq\phi\leq\pi$, is obtained from the scattering matrix by the following equation:

$$\underset{\sim}{I}(0,\mu,\phi) = \frac{1}{4\mu}\,\underset{\sim}{S}(\tau_1;\mu,\phi;\mu_0,\phi_0)\,\underset{\sim}{F}\,.$$

Similarly, the diffusely transmitted intensity $\underset{\sim}{I}(\tau_1,-\mu,\phi)$, $0<\mu\leq 1$, $0\leq\phi\leq\pi$, is obtained from the transmission matrix as follows:

$$\underset{\sim}{I}(\tau_1,\mu,\phi) = \frac{1}{4\mu}\,\underset{\sim}{T}(\tau_1;\mu,\phi;\mu_0,\phi_0)\,\underset{\sim}{F}$$

where τ_1 is the total optical thickness of the atmosphere.

Writing the scattering and transmission matrices in a form similar to that of the single scattering phase matrix results in the equations

$$\underset{\sim}{S}(\tau;\mu,\phi;\mu_0,\phi_0) = \underset{\sim}{S}^{(0)}(\tau;\mu,\mu_0) + (1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\,\underset{\sim}{S}^{(1)}(\tau;\mu,\phi;\mu_0,\phi_0) + \underset{\sim}{S}^{(2)}(\tau;\mu,\phi;\mu_0,\phi_0),$$

$$\underset{\sim}{T}(\tau;\mu,\phi;\mu_0,\phi_0) = \underset{\sim}{T}^{(0)}(\tau;\mu,\mu_0) + (1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\,\underset{\sim}{T}^{(1)}(\tau;\mu,\phi;\mu_0,\phi_0) + \underset{\sim}{T}^{(2)}(\tau;\mu,\phi;\mu_0,\phi_0).$$

where

$$\underset{\sim}{S}^{(k)}(\tau;\mu,\phi;\mu_0,\phi_0) = \underset{\sim}{P}^{(k)}(\mu,\phi;\mu_0,\phi_0)\,S_k(\tau;\mu,\mu_0) \quad k=1,2,$$

and

$$\underset{\sim}{T}^{(k)}(\tau;\mu,\phi;\mu_0,\phi_0) = \underset{\sim}{P}^{(k)}(\mu,\phi;\mu_0,\phi_0)\,T_k(\tau;\mu,\mu_0) \quad k=1,2.$$

The solution of the equation of transfer may readily be obtained by numericaly calculations of the resulting scalar equations for $S_k(\tau;\mu,\mu_0)$ and $T_k(\tau;\mu,\mu_0)$, k=1,2.

The procedure necessary for the explicit equations of the scalar functions $S_k(\tau;\mu,\mu_0)$ and $T_k(\tau;\mu,\mu_0)$ for k=1,2 and the matrix equations for the azimuthal independent matrices is shown in detail by Sekera (2). The resulting equations for the above functions are:

$$\left\{1/\mu + 1/\mu_0 + \partial/\partial\tau\right\} S_k(\tau;\mu,\mu_0) = \omega(\tau)\left\{1+\int_0^1 \psi^{(k)}(\mu')\,S(\tau;\mu,\mu')\,d\mu'/\mu'\right\}\times\left\{1+\int_0^1 \psi^{(k)}(\mu'')\,S(\tau;\mu_0,\mu'')\,d\mu''/\mu''\right\}$$

$$\left\{1/\mu + \partial/\partial\tau\right\} T_k(\tau;\mu,\mu_0) = \omega(\tau)\left\{e^{-\tau/\mu}+\int_0^1 \psi^{(k)}(\mu')\,T_k(\tau;\mu,\mu')\,d\mu'/\mu'\right\}\times\left\{1+\int_0^1 \psi^{(k)}(\mu'')\,S_k(\tau;\mu'',\mu_0)\,d\mu''/\mu''\right\}$$

for k=1,2,

$$\psi^{(1)}(\mu) = 3/8\,(1-\mu^2)(1+2\mu^2),$$

and

$$\psi^{(2)}(\mu) = 3/16\,(1+\mu^2)^2.$$

The resulting equations for the azimuthal independent components of $\underset{\sim}{S}(\tau;\mu,\phi;\mu_0,\phi_0)$ and $\underset{\sim}{T}(\tau;\mu,\phi;\mu_0,\phi_0)$ are

$$\left\{ \frac{1}{\mu_0} + \frac{1}{\mu} + \frac{\partial}{\partial\tau} \right\} \underset{\sim}{S}^{(0)}(\tau;\mu,\mu_0) = \frac{3}{4}\,\omega(\tau) \left\{ \underset{\sim}{M}(\mu) + \frac{1}{2}\int_0^1 \underset{\sim}{S}^{(0)}(\tau;\mu,\mu')\underset{\sim}{M}(\mu')\frac{d\mu'}{\mu'} \right\} \times$$
$$\times \left\{ \underset{\sim}{\tilde{M}}(\mu_0) + \frac{1}{2}\int_0^1 \underset{\sim}{\tilde{M}}(\mu'')\,\underset{\sim}{S}^{(0)}(\tau;\mu'',\mu_0)\frac{d\mu''}{\mu''} \right\}$$

and

$$\left\{ \frac{1}{\mu} + \frac{\partial}{\partial\tau} \right\} \underset{\sim}{T}^{(0)}(\tau;\mu,\mu_0) = \frac{3}{4}\,\omega(\tau) \left\{ \underset{\sim}{M}(\mu)e^{-\tau/\mu} + \frac{1}{2}\int_0^1 \underset{\sim}{T}^{(0)}(\tau;\mu,\mu')\underset{\sim}{M}(\mu')\frac{d\mu'}{\mu'} \right\} \times$$
$$\times \left\{ \underset{\sim}{\tilde{M}}(\mu_0)e^{-\tau/\mu} + \frac{1}{2}\int_0^1 \underset{\sim}{\tilde{M}}(\mu')\,\underset{\sim}{T}^{(0)}(\tau;\mu_0,\mu')\frac{d\mu'}{\mu'} \right\}$$

where

$$\underset{\sim}{M}(\mu) = \begin{pmatrix} \mu^2 & \sqrt{2}\,(1-\mu^2) \\ 1 & 0 \end{pmatrix} .$$

$\underset{\sim}{\tilde{M}}(\mu)$ is the transpose of the matrix $\underset{\sim}{M}(\mu)$.

The addition of a Lambert surface at the lower surface of the atmosphere and the resulting change in reflected and transmitted intensities emerging from the atmosphere may be easily calculated from the previously computed functions (1).

CHAPTER BIBLIOGRAPHY

1. Chandrasekhar, S., *Radiative Transfer*, New York, Dover Publications, 1960.

2. Sekera, Z., "Reduction of the Equations of Radiative Transfer for a Plane-Parallel, Planetary Atmosphere: Parts I & II," The RAND Corporation (RM-4952-PR, RM-5056-PR), Santa Monica, California.

CHAPTER III

COMPUTATIONAL METHODS

The coupled nonlinear integro-differential equations obtained by reduction of the equation of transfer were solved using a combination of numerical means to obtain the greatest possible accuracy in calculating the resulting intensities. The final codes were written using double precision arithmetic on the International Business Machine 360/50 computer. The codes are capable of computing results for large optical depths in a relatively short time for any ground albedo less than one and greater than or equal to zero and any single scattering albedo which may vary with optical depth in the atmosphere. Mullikin (6) has shown that singular solutions exist near the desired solutions and that all numerical integration must be done with high precision to avoid them.

The integrals with respect to μ are approximated by Gauss quadrature normalized to the interval from zero to one. A change in the order from seven to nine in the Gaussian integration results in a change of at most one part in 300,000 in the total intensity for a μ and μ_0 value of 0.500 for all optical depth considered. Final values obtained for large optical depths up to a depth of five were done with a Gauss quadrature of order nine so that cosine values near those computed by Coulson *et al.* (5) could be obtained for comparison. The integration with respect to the optical depth is performed by using Runge-Kutta integration as a starting procedure and then switching to the fifth-order Adams-Bashforth predictor coupled with the fifth-order Adams-Moulton corrector. As a test of the numerical procedure, the integro-differential equations of Chandrasekhar's X- and Y-functions for an isotropic phase function were solved. These results are shown in Table I with the numerical results of Carlstedt and Mullikin (2) and Bellman *et al.* (1). The results of Bellman *et al.* were calculated by a similar technique using invariant imbedding. As can be seen from this table, the agreement is excellent.

TABLE I

COMPARISON OF COMPUTATIONAL RESULTS FOR X- AND Y-FUNCTIONS FOR CONSERVATIVE ISOTROPIC SCATTERING WITH $\mu_0 = 0.50$

τ		Present Scheme	Carlstedt-Mullikin	Bellman
0.2	X	1.24475	1.24480	1.24456
	Y	8.98554 -1	8.98582 -1	8.98214 -1
0.6	X	1.45999	1.46000	1.46003
	Y	6.57025 -1	6.57032 -1	6.57095 -1
1.0	X	1.57403	1.57404	1.57403
	Y	5.00054 -1	5.00045 -1	5.00032 -1
1.4	X	1.64578	1.64578	1.64578
	Y	4.00627 -1	4.00621 -1	4.00626 -1
1.6	X	1.67272	1.67272	1.67272
	Y	3.64711 -1	3.64707 -1	3.64715 -1
2.0	X	1.69561	1.69561	1.69561
	Y	3.35177 -1	3.35174 -1	3.35181 -1
2.4	X	1.74783	1.74783	1.74783
	Y	2.71921 -1	2.71919 -1	2.71921 -1
2.8	X	1.77358	1.77358	1.77358
	Y	2.42915 -1	2.42913 -1	2.42914 -1
3.0	X	1.78459	1.78459	1.78459
	Y	2.30908 -1	2.30907 -1	2.30908 -1
3.5	X	1.80803	1.80803	1.80803
	Y	2.06009 -1	2.06008 -1	2.06009 -1

A check of results from the calculations of the Stoke's vector in the matrix case was also accomplished. Chandrasekhar (1) has shown that $\underset{\sim}{S}^{(0)}(\tau;\mu,\mu_0)$ and $\underset{\sim}{T}^{(0)}(\tau;\mu,\mu_0)$ must be expressible in terms of functions of μ and μ_0. These functions are $\psi(\mu)$, $\phi(\mu)$, $\chi(\mu)$, $\zeta(\mu)$, $\xi(\mu)$, $\eta(\mu)$, $\sigma(\mu)$, $\theta(\mu)$, $\gamma_l^{(1)}(\mu)$, and $\gamma_r^{(1)}(\mu)$, where

$$\psi(\mu) = \mu^2 + \tfrac{1}{2}\int_0^1 [\mu'^2 S_{ll}^{(0)}(\mu,\mu') + S_{lr}^{(0)}(\mu,\mu')]\, d\mu'/\mu' ,$$

$$\phi(\mu) = 1-\mu^2 + \tfrac{1}{2}\int_0^1 (1-\mu'^2)\, S_{ll}^{(0)}(\mu,\mu')\, d\mu'/\mu' ,$$

$$\chi(\mu) = 1 + \tfrac{1}{2}\int_0^1 [\mu'^2 S_{rl}^{(0)}(\mu,\mu') + S_{rr}^{(0)}(\mu,\mu')]\, d\mu'/\mu' ,$$

$$\zeta(\mu) = \tfrac{1}{2}\int_0^1 (1-\mu'^2)\, S^{(0)}(\mu,\mu')\, d\mu'/\mu' ,$$

$$\xi(\mu) = \mu^2 e^{-\tau/\mu} + \tfrac{1}{2}\int_0^1 [\mu'^2 T_{ll}^{(0)}(\mu,\mu') + T_{lr}^{(0)}(\mu,\mu')]\, d\mu'/\mu' ,$$

$$\eta(\mu) = (1-\mu^2) e^{-\tau/\mu} + \tfrac{1}{2}\int_0^1 (1-\mu'^2)\, T^{(0)}(\mu,\mu')\, d\mu'/\mu' ,$$

$$\sigma(\mu) = e^{-\tau/\mu} + \tfrac{1}{2}\int [\mu'^2 T^{(0)}(\mu,\mu') + T^{(0)}(\mu,\mu')]\, d\mu'/\mu' ,$$

$$\theta(\mu) = \tfrac{1}{2}\int_0^1 (1-\mu'^2)\, T_{rl}^{(0)}(\mu,\mu') ,$$

$$t_l(\mu) = \tfrac{1}{2}\int_0^1 [T_{ll}^{(0)}(\mu',\mu) + T_{rl}^{(0)}(\mu',\mu)]\, d\mu' ,$$

$$t_r(\mu) = \tfrac{1}{2}\int_0^1 [T_{lr}^{(0)}(\mu',\mu) + T_{rr}^{(0)}(\mu',\mu)]\, d\mu' ,$$

$$\gamma_l^{(1)}(\mu) = t_l(\mu)/\mu + e^{-\tau/\mu} ,$$

$$\gamma_r^{(1)}(\mu) = t_r(\mu)/\mu + e^{-\tau/\mu} ,$$

$$\bar{S} = \int_0^1\int_0^1 S^{(0)}(\mu',\mu'')\, d\mu'\, d\mu'' .$$

TABLE II

THE INTEGRAL FUNCTIONS OF CHANDRASEKHAR
FOR μ=0.500 AND ω_0=1.00

	τ = 0.02		τ = 0.50		τ = 1.00	
	ADAMS	ELBERT	ADAMS	ELBERT	ADAMS	ELBERT
$\bar{\zeta}$	0.01909	0.0194	0.29603	0.2931	0.44690	0.4401
ψ	0.26184	0.26176	0.41038	0.41132	0.49804	0.49868
ϕ	0.79356	0.79302	1.08305	1.08010	1.16621	1.15646
χ	1.03555	1.03517	1.3227	1.32297	1.43395	1.43443
ζ	0.00196	0.00197	0.05029	0.05030	0.08557	0.08512
ξ	0.25200	0.25192	0.23117	0.22980	0.21481	0.20563
η	0.76365	0.76340	0.52978	0.53607	0.32380	0.35580
σ	0.99601	0.99582	0.63121	0.63397	0.41140	0.42160
ϵ	0.00196	0.00197	0.04618	0.04499	0.06798	0.06282
$\gamma_l^{(1)}$	0.98036	0.98039	0.66378	0.66825	0.49695	0.50734
$\gamma_r^{(1)}$	0.98037	0.98039	0.66693	0.66803	0.50468	0.50708

The functions of μ are of a form slightly different than those of Chandrasekhar (1) due to the difference of form for the reflected and transmitted intensities with no azimuthal dependence. Shown also in Table II are the resulting calculations obtained by Chandrasekhar and Elbert (4). The agreement between corresponding functions is very good for an optical depth of 0.02, while noticeable differences occur for optical depths of 0.5 and 1.0. In the discussion of their results, Chandrasekhar and Elbert note that for increasing optical depths their values lacked stability and were subject to a greater error.

A final check of the numerical results was obtained for the diffusely reflected and transmitted flux normal to the upper and lower boundaries in the scalar and matrix cases. The sum of the upward flux F_u, the downward flux F_d, and the direct solar flux $F_s = \mu_0 \pi e^{-\tau/\mu_0} F$ is equal to $\mu_0 \pi F$ in the absence of absorption in the atmosphere. With this notation $(F_u + F_d + F_s)/\mu_0 \pi F = 1.0$. The resulting calculations are shown in Table III and agreement is quite good. The close agreement of the upward and downward fluxes for both the matrix and scalar approach should be noted. In both cases for the optical depths shown, the relative stability of the two cases for increasing optical depth may be noted. The largest error occurs for small values of μ, which is to be expected; roundoff error does not appear to be significant in either case.

TABLE III

COMPARISON OF FLUXES FOR CONSERVATIVE RAYLEIGH SCATTERING

τ	μ_0	CASE	F_u	F_d	$(F_u + F_d + F_s)/\mu_0 \pi F$
1.00	0.01592	SCALAR	0.03762	0.01241	1.00058
		MATRIX	.03769	.01234	1.00041
	0.50000	SCALAR	.78331	.57447	1.00004
		MATRIX	.78416	.57410	1.00004
	0.98408	SCALAR	1.06376	.90882	1.00003
		MATRIX	1.06286	.90968	1.00001
5.00	0.01592	SCALAR	0.04543	.00459	1.00021
		MATRIX	.04548	.00454	1.00011
	0.50000	SCALAR	1.28741	.28342	1.00006
		MATRIX	1.28827	.28251	1.00004
	0.98408	SCALAR	2.28850	.78412	1.00003
		MATRIX	2.28588	0.78660	1.00004

CHAPTER BIBLIOGRAPHY

1. Bellman, R. E., R. E. Kalaba, and M. C. Prestrud. Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness, New York, American Elsevier Company, Inc., 1963.

2. Carlstedt, J. L., and T. W. Mullikin, "The X- and Y-functions of Chandrasekhar," Astrophysical Journal Supplement, XII (June,1966), 449-501.

3. Chandrasekhar, S., Radiative Transfer, New York, Dover Publications, 1960.

4. Chandrasekhar, S., and D. D. Elbert, "The Illumination and Polarization of the Sunlit Sky on Rayleigh Scattering," The Transactions of the American Philosophical Society, XLIV (December, 1954), 643-729.

5. Coulson, K. L., J. V. Dave, and Z. Sekera. Tables Relating to Radiation Emerging from a Planetary Atmosphere with Rayleigh Scattering, California, University of California Press, 1960.

6. Mullikin, T. W., "A Nonlinear Integro-Differential Equation in Radiative Transfer," The Journal of the Society for Industrial and Applied Mathematics, XIII (June,1965), 388-409.

CHAPTER IV

DISCUSSION

In Figures 1-8, the reflected intensities for both the scalar and vector solutions are plotted as a function of μ for $\phi=0$ and $\phi=1.57$ with a ground albedo of zero for some selected values of μ_0 and tau. Since it is difficult to read accurate values from graphs, Table IV was developed to show the percentage error between the two calculations. The percentage error is defined as $(I_v - I_s) \times 100/I_v$ where I_v is the intensity computed from the correct vector field approach and I_s is the intensity computed from the scalar theory. As can be seen from this table the percentage error can be quite significant, reaching a maximum value of approximately twelve per cent in absolute value for the largest μ and μ_0 with $\phi=0$ and tau of one. For fixed μ and μ_0 the error as a function of tau tends to reach a maximum and decrease monotonically to some asymptotic value for $\tau \to \infty$. This behavior is entirely expected since as the atmosphere becomes very optically thick the very-high-order collisions give smaller contributions to the total. For corresponding values of μ, μ_0, and τ the percentage error is almost always smaller in absolute value for $\phi=1.57$ than for $\phi=0$.

TABLE IV

PERCENT RELATIVE ERRORS BETWEEN THE VECTOR AND SCALAR REFLECTED INTENSITIES

μ	$\mu_0 = 0.1933$, $\phi=0$, A=0			$\mu_0 = 0.1933$, $\phi=\pi/2$, A=0		
	$\tau=0.50$	$\tau=1.00$	$\tau=5.00$	$\tau=0.50$	$\tau=1.00$	$\tau=5.00$
0.0159	4.32	4.66	4.63	0.07	0.11	- 0.01
0.5000	2.77	3.80	3.67	- 1.81	- 1.68	- 1.52
0.9841	-11.30	-11.70	- 8.11	- 9.77	- 9.30	- 7.18

μ	$\mu_0 = 0.5000$, $\phi=0$, A=0			$\mu_0 = 0.5000$, $\phi=\pi/2$, A=0		
	$\tau=0.50$	$\tau=1.00$	$\tau=5.00$	$\tau=0.50$	$\tau=1.00$	$\tau=5.00$
0.0159	3.87	4.62	4.51	- 1.79	- 1.90	- 1.87
0.5000	- 2.63	- 2.11	- 1.13	- 2.37	- 2.41	- 1.85
0.9841	- 7.02	- 8.63	- 5.91	- 4.06	- 4.90	- 3.40

μ	$\mu_0 = 0.8067$, $\phi=0$, A=0			$\mu_0 = 0.8067$, $\phi=\pi/2$, A=0		
	$\tau=0.50$	$\tau=1.00$	$\tau=5.00$	$\tau=0.50$	$\tau=1.00$	$\tau=5.00$
0.0159	- 1.20	- 0.86	- 0.50	- 5.74	- 5.74	- 4.76
0.5000	-10.10	-10.50	- 6.71	- 3.33	- 3.81	- 2.72
0.9841	2.23	0.65	- 0.54	3.82	3.40	1.67

μ	$\mu_0 = 0.9841$, $\phi=0$, A=0			$\mu_0 = 0.9841$, $\phi=\pi/2$, A=0		
	$\tau=0.50$	$\tau=1.00$	$\tau=5.00$	$\tau=0.50$	$\tau=1.00$	$\tau=5.00$
0.0159	- 8.27	- 8.52	- 6.47	- 8.75	- 9.04	- 6.87
0.5000	- 7.02	- 8.60	- 5.91	- 4.06	- 4.90	- 3.40
0.9841	7.50	7.44	4.03	8.16	8.16	4.74

In Figures 9-16 the corresponding transmitted intensities are plotted. Table V gives the corresponding percentage errors in the transmitted intensity. It should be noted that errors in excess of seventeen per cent can be encountered by using the incorrect scalar theory. As the optical thickness of the atmosphere increases, the percentage error will decrease since the photons reaching the bottom will have undergone thousands of collisions and polarization effects are virtually destroyed.

TABLE V

PERCENT ERRORS BETWEEN THE VECTOR AND SCALAR TRANSMITTED INTENSITIES

μ	μ_0 = 0.1933, ϕ=0, A=0			μ_0 = 0.1933, ϕ=π/2, A=0		
	τ=0.50	τ=1.00	τ=5.00	τ=0.50	τ=1.00	τ=5.00
0.0159	13.30	17.30	- 0.12	- 1.87	- 4.61	- 3.74
0.5000	7.86	11.20	- 0.94	- 2.18	- 2.63	- 2.24
0.9841	- 7.73	- 8.17	1.01	-10.40	-11.10	- 1.45
μ	μ_0 = 0.5000, ϕ=0, A=0			μ_0 = 0.5000, ϕ=π/2, A=0		
	τ=0.50	τ=1.00	τ=5.00	τ=0.50	τ=1.00	τ=5.00
0.0159	6.60	9.60	0.42	- 3.00	- 4.10	- 2.79
0.5000	8.45	11.30	1.85	- 2.55	- 2.85	- 1.42
0.9841	- 0.58	- 0.96	0.11	- 4.21	- 5.23	- 0.68
μ	μ_0 = 0.8067, ϕ=0, A=0			μ_0 = 0.8067, ϕ=π/2, A=0		
	τ=0.50	τ=1.00	τ=5.00	τ=0.50	τ=1.00	τ=5.00
0.0159	- 0.72	0.29	- 0.15	- 6.94	- 7.43	- 2.00
0.5000	6.26	7.90	1.89	- 3.47	- 4.14	- 0.85
0.9841	6.21	6.45	1.46	3.87	3.40	0.56
μ	μ_0 = 0.9841, ϕ=0, A=0			μ_0 = 0.9841, ϕ=π/2, A=0		
	τ=0.50	τ=1.00	τ=5.00	τ=0.50	τ=1.00	τ=5.00
0.0159	- 9.55	- 9.65	- 1.50	-10.50	-10.70	- 1.71
0.5000	- 0.58	- 0.96	0.11	- 4.21	- 5.23	- 0.68
0.9841	8.90	9.33	2.05	8.26	8.45	1.71

APPENDIX

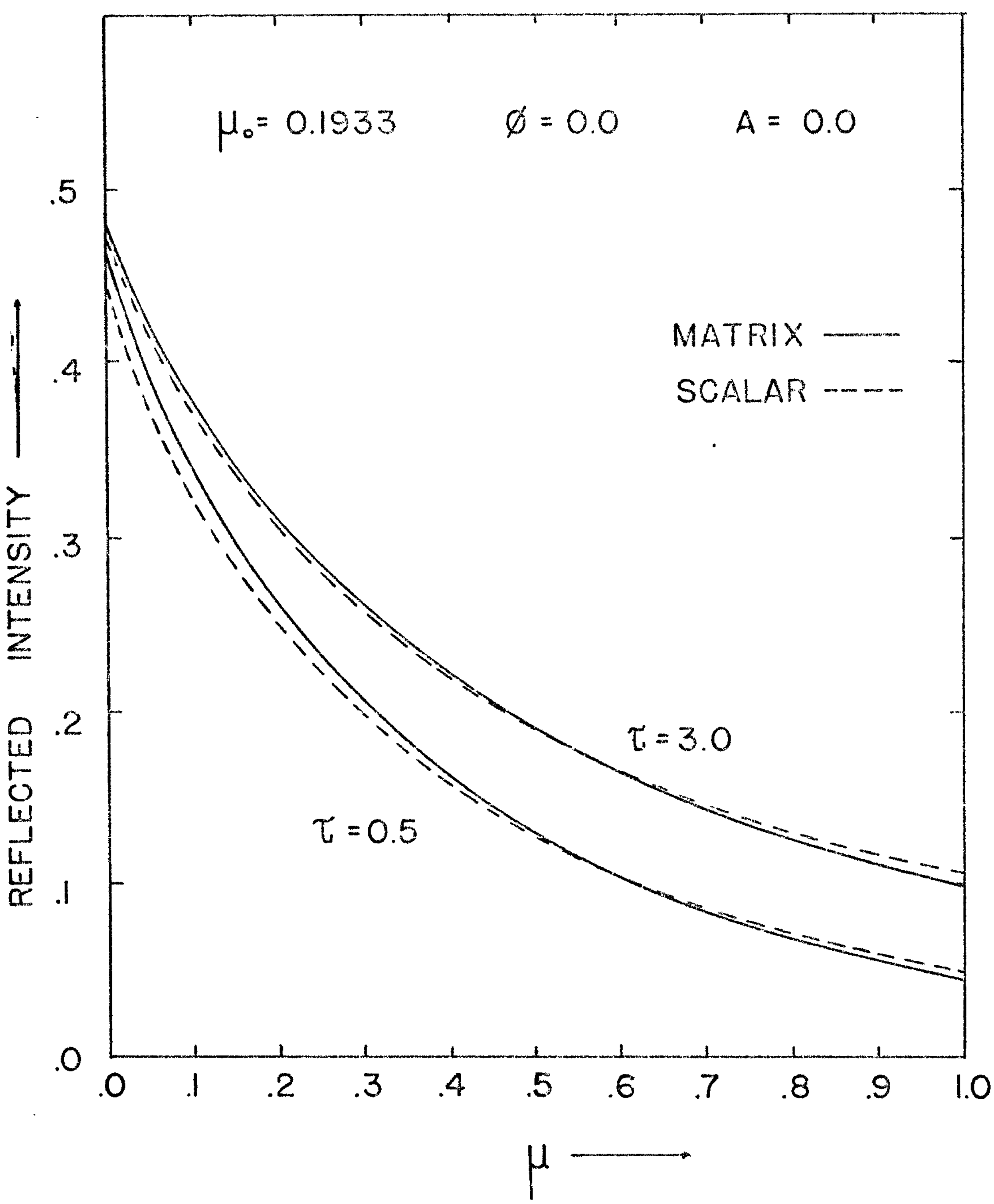

Fig.1--Reflected intensity for $\mu_\circ$=0.1933, Ø=0.0, and A=0.

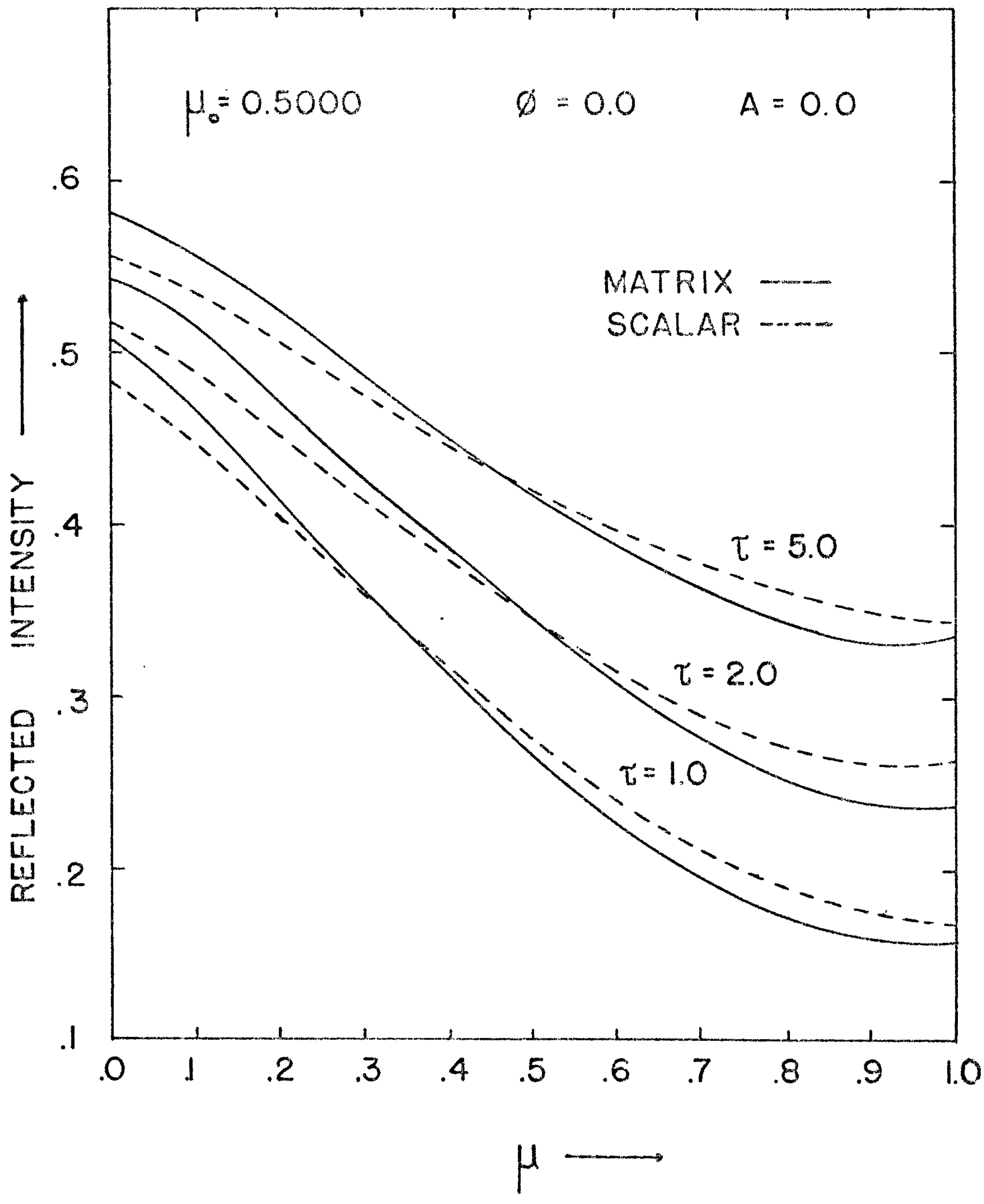

Fig.2--Reflected intensity for μ_0=0.5000, Ø=0.0, and A=0.

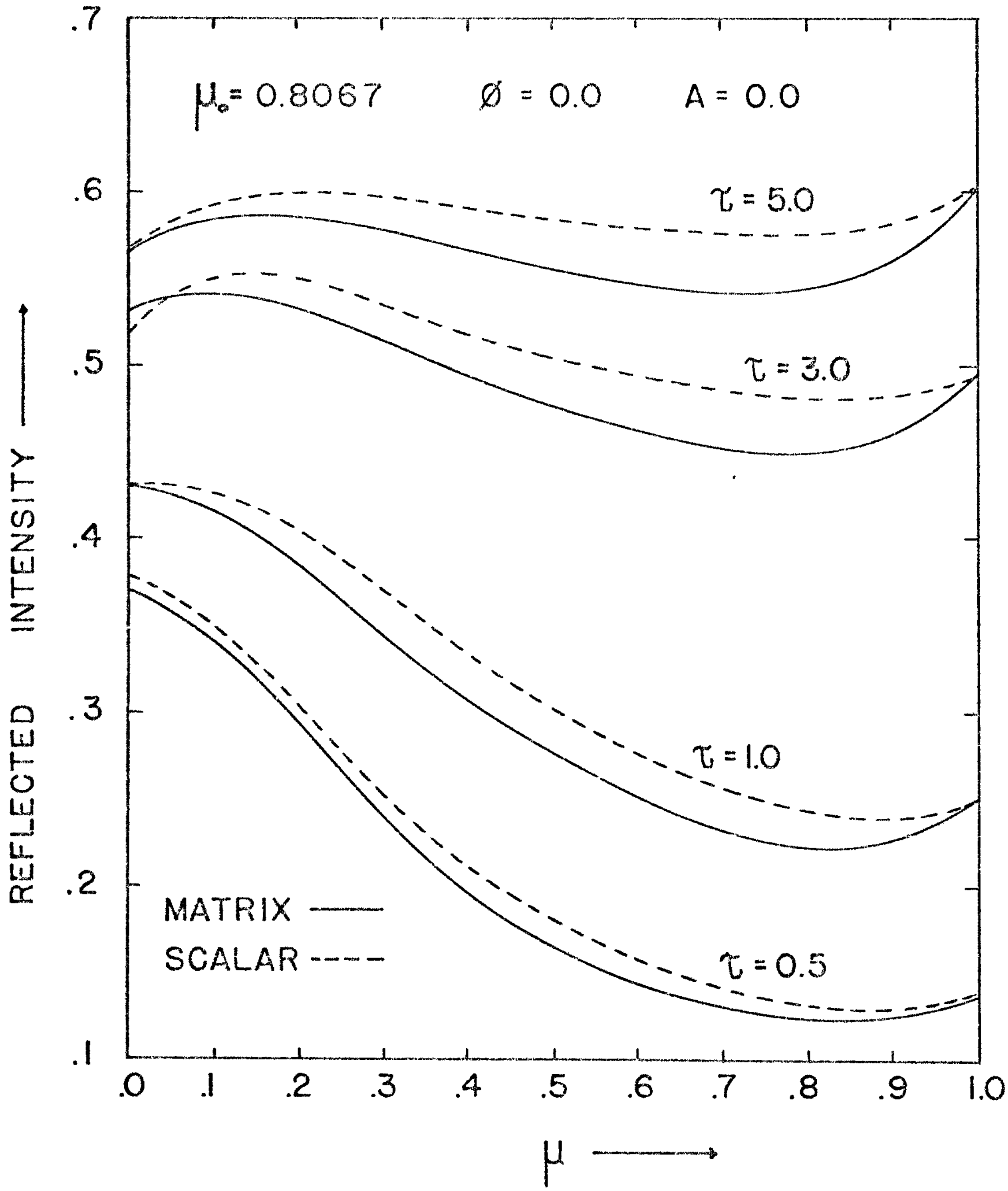

Fig.3--Reflected intensity for μ_0=0.8067, ϕ=0.0, and A=0.

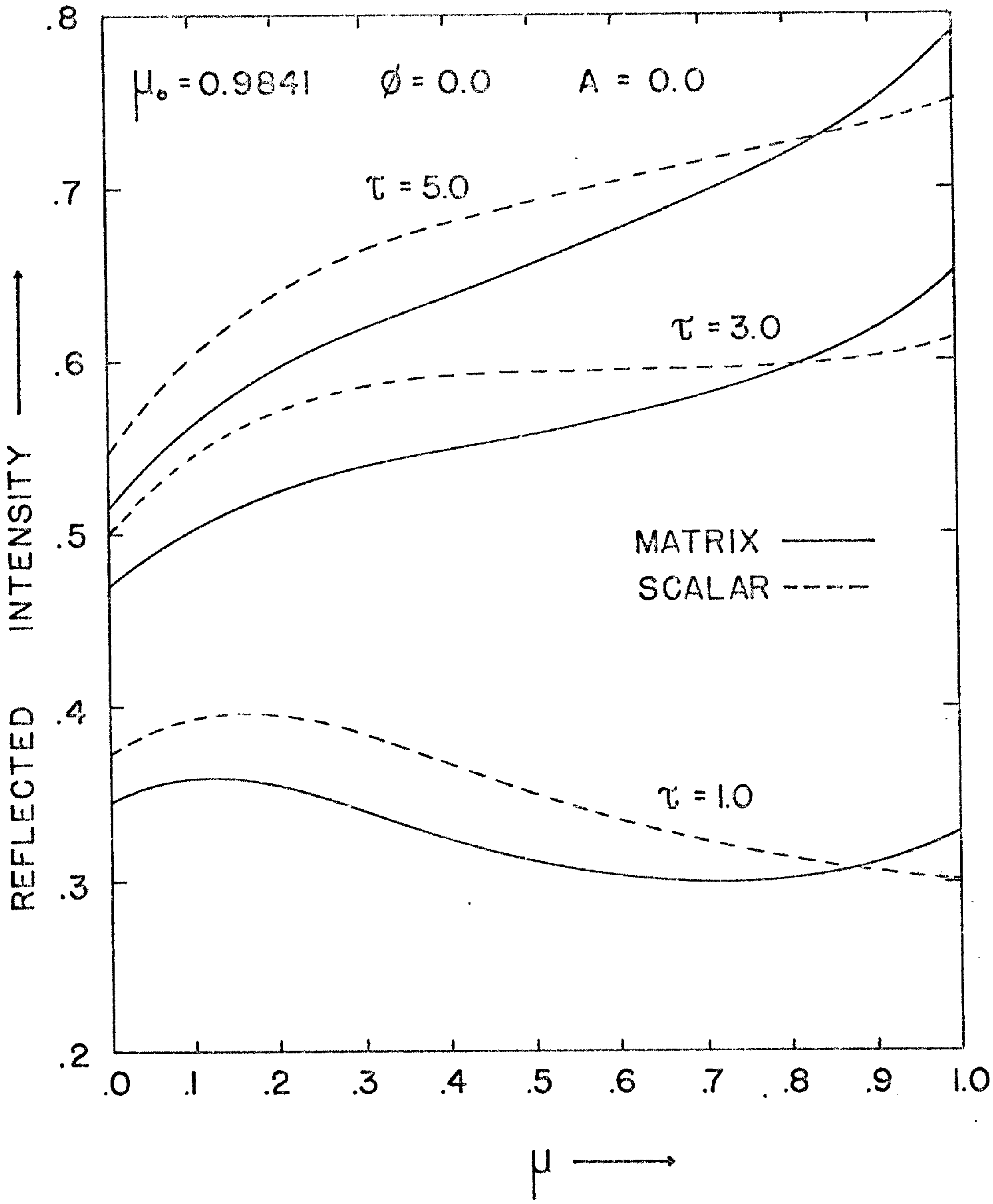

Fig.4--Reflected intensity for μ_0=0.9841, ϕ=0.0, and A=0.

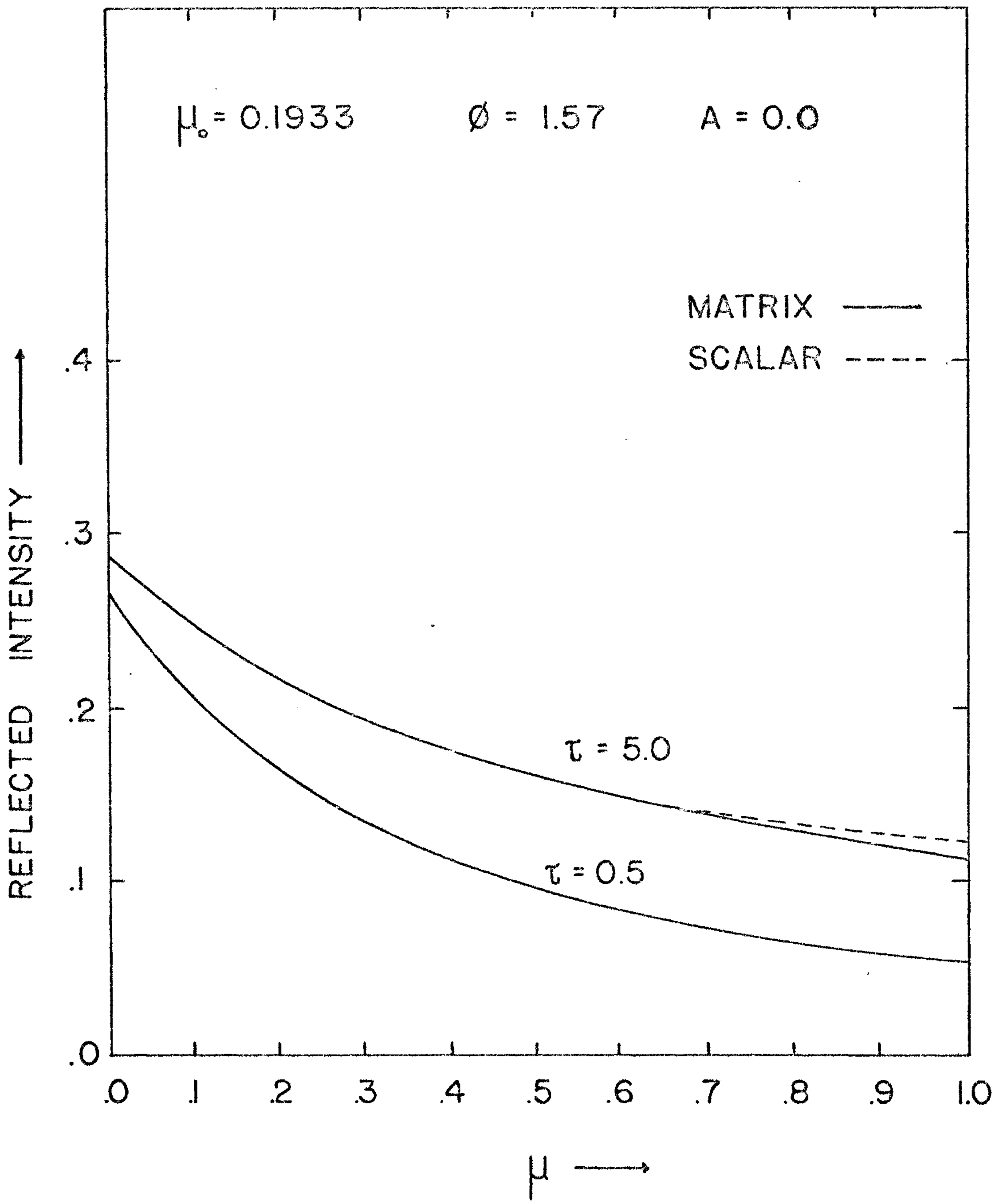

Fig. 5--Reflected intensity for μ_o=0.1933, ϕ=1.57, and A=0.

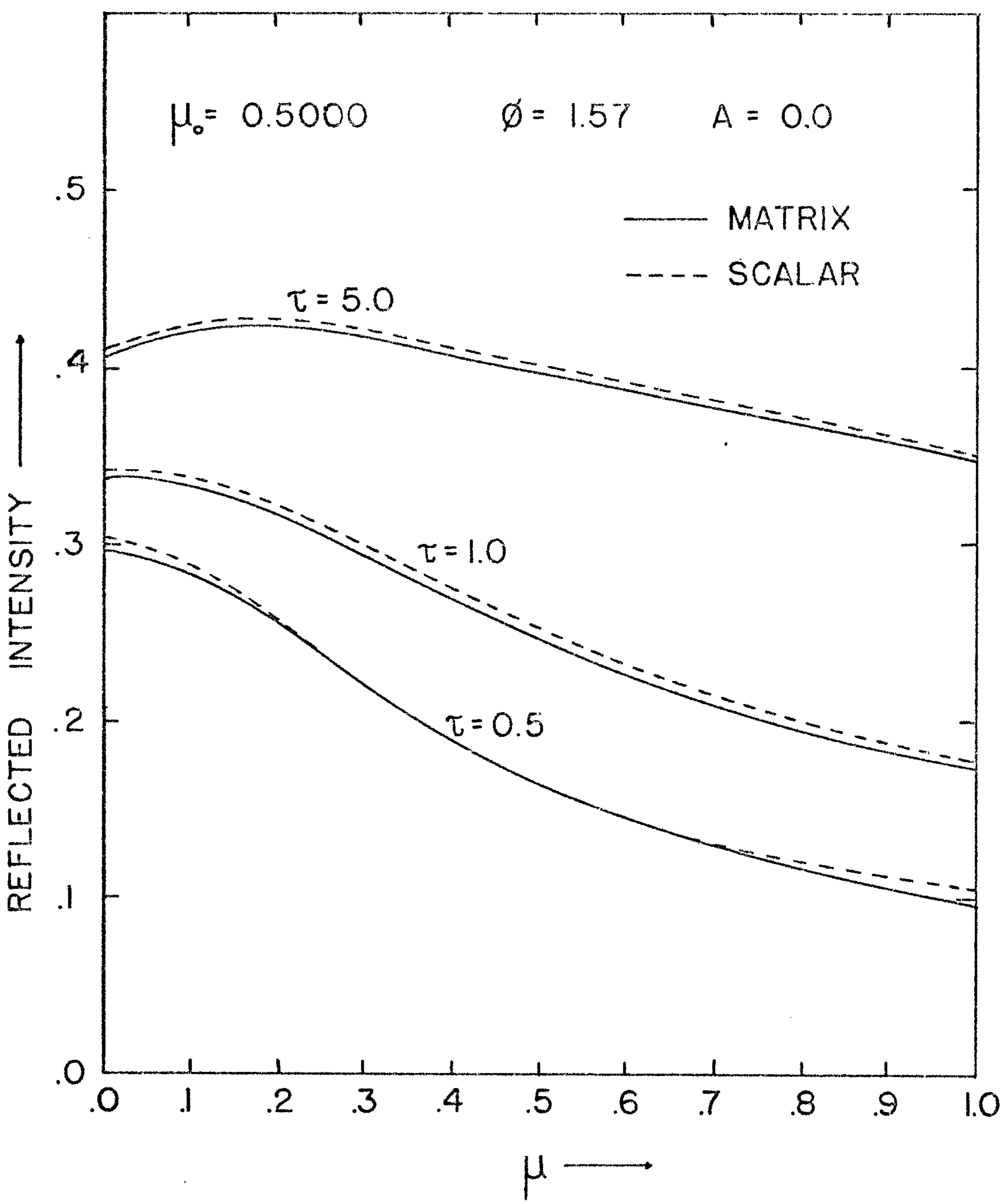

Fig.6--Reflected intensity for μ_0=0.500, Ø=1.57, and A=0.

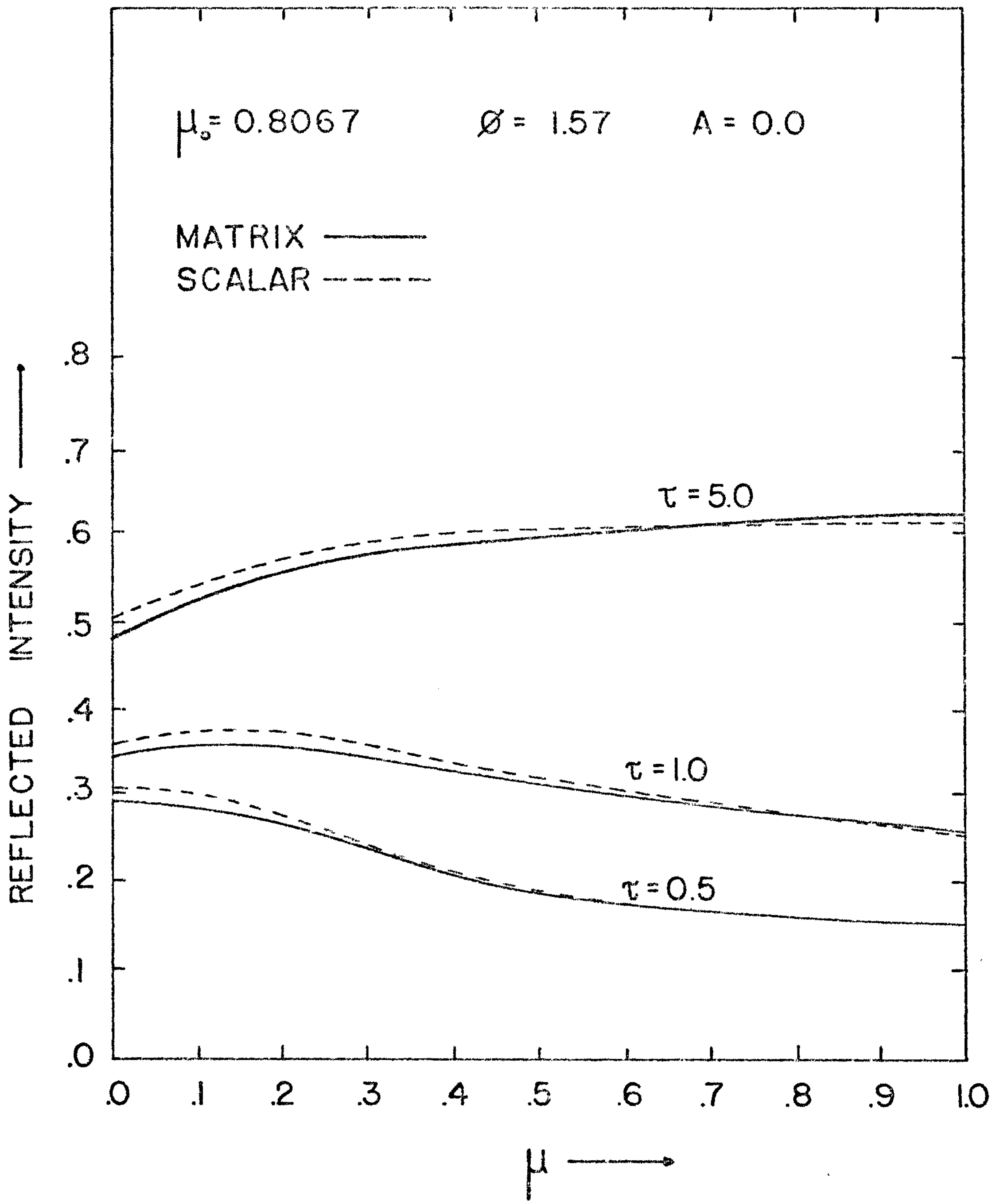

Fig.7--Reflected intensity for μ_0=0.8067, Ø=1.57, and A=0.

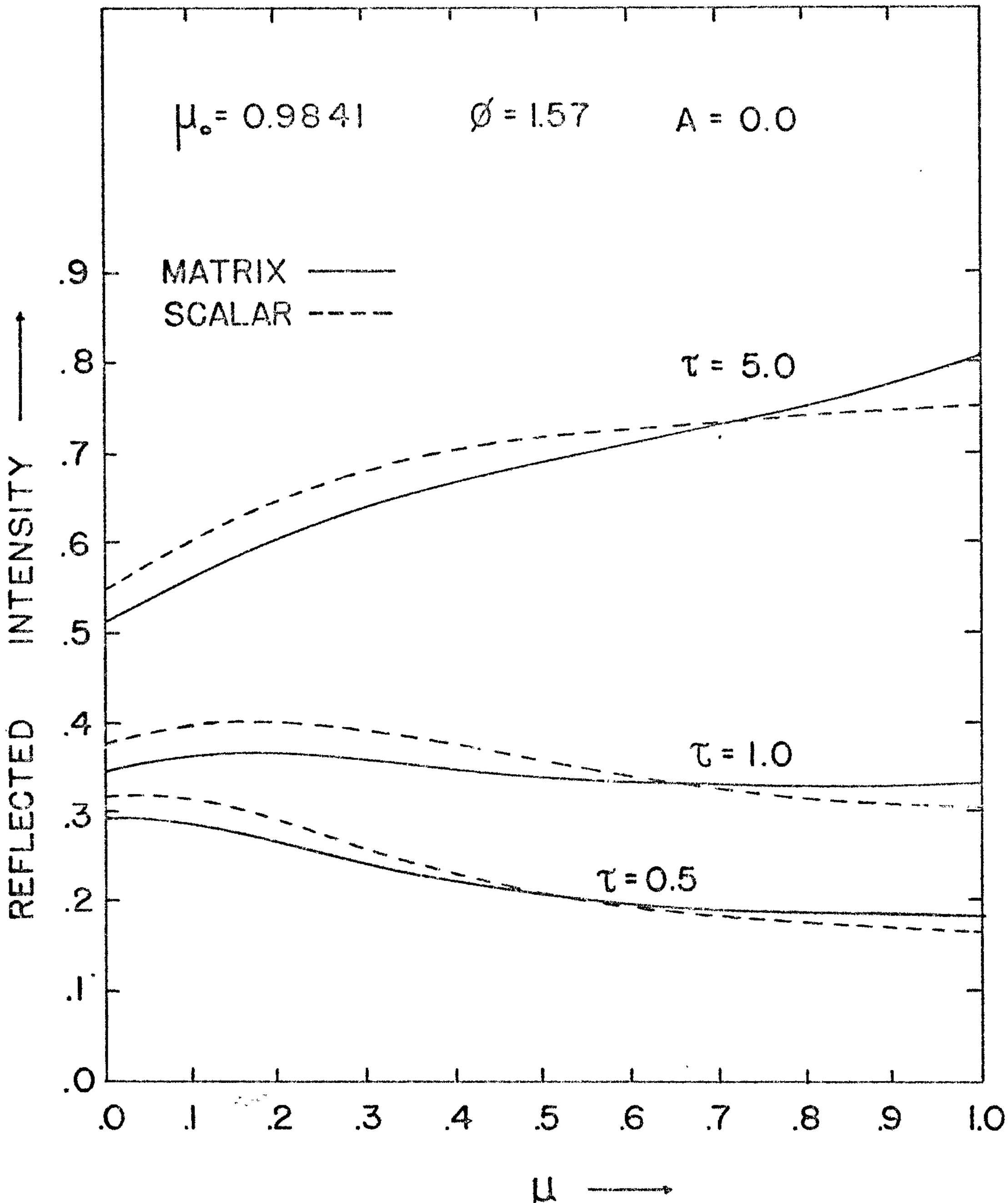

Fig.8--Reflected intensity for μ_0=0.9841, ϕ=1.57, and A=0.

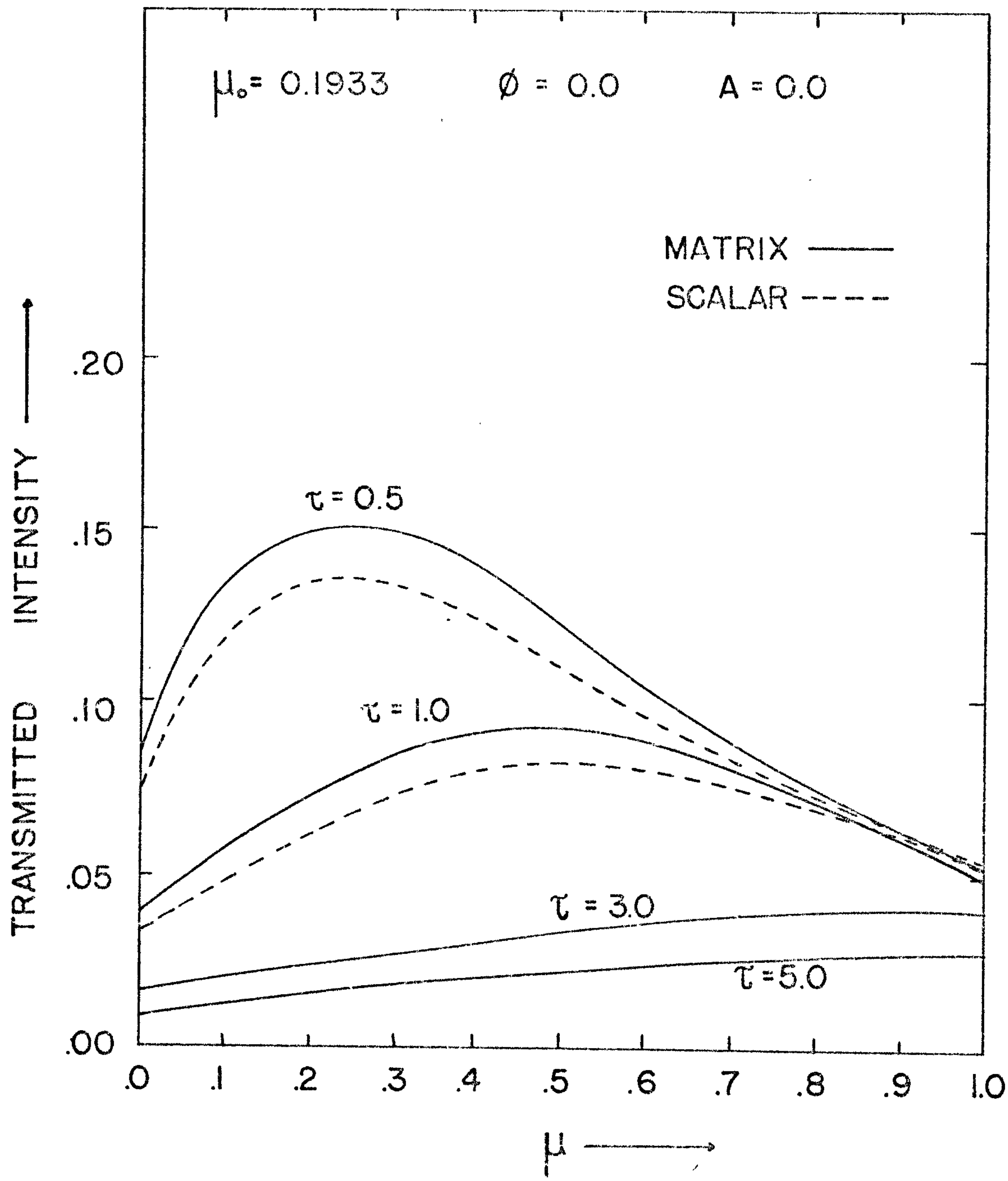

Fig.9--Transmitted intensity for μ_0=0.1933, ϕ=1.57, and A=0.

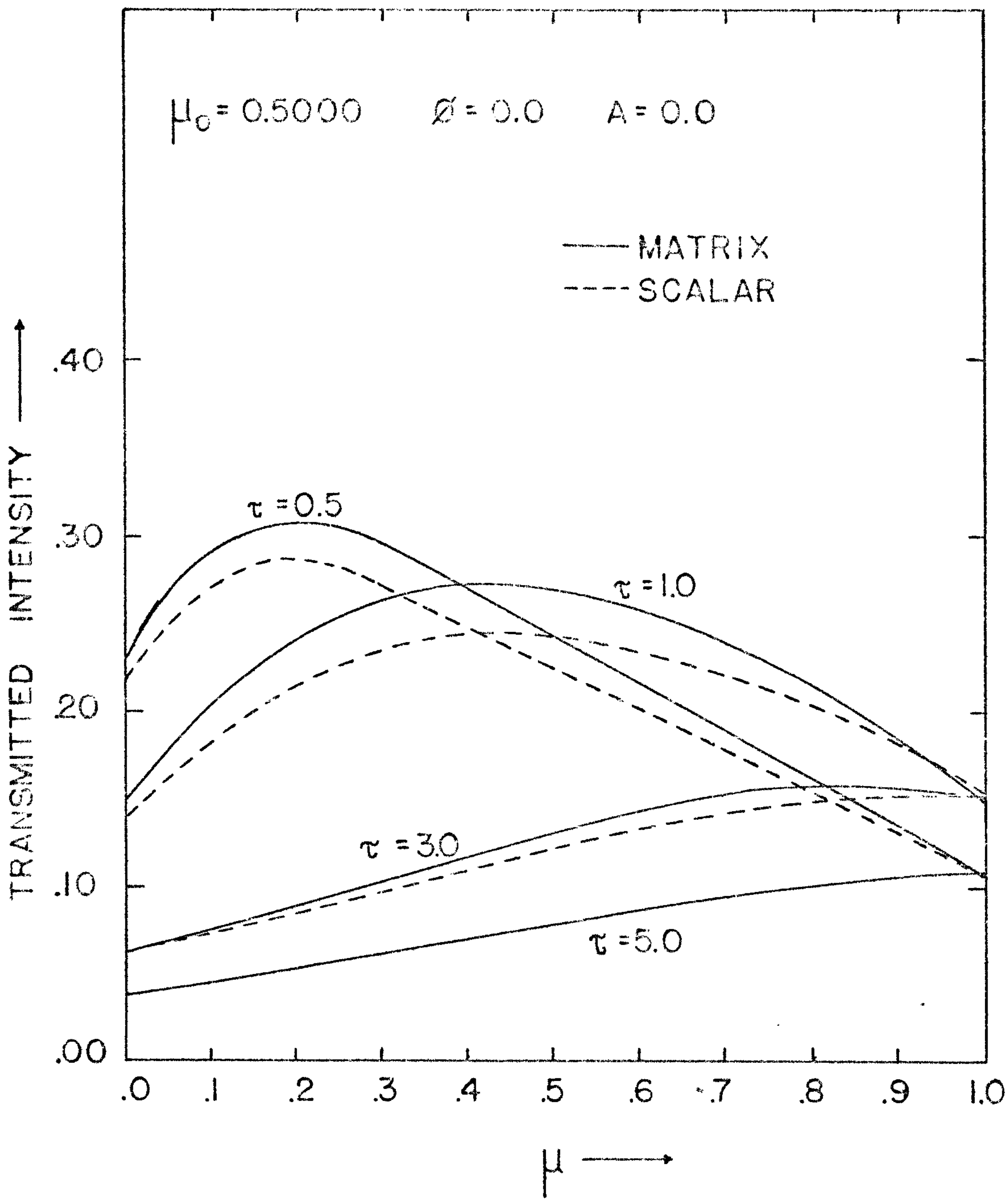

Fig.10--Transmitted intensity for μ_0=0.5000, ϕ=0.0, and A=0.

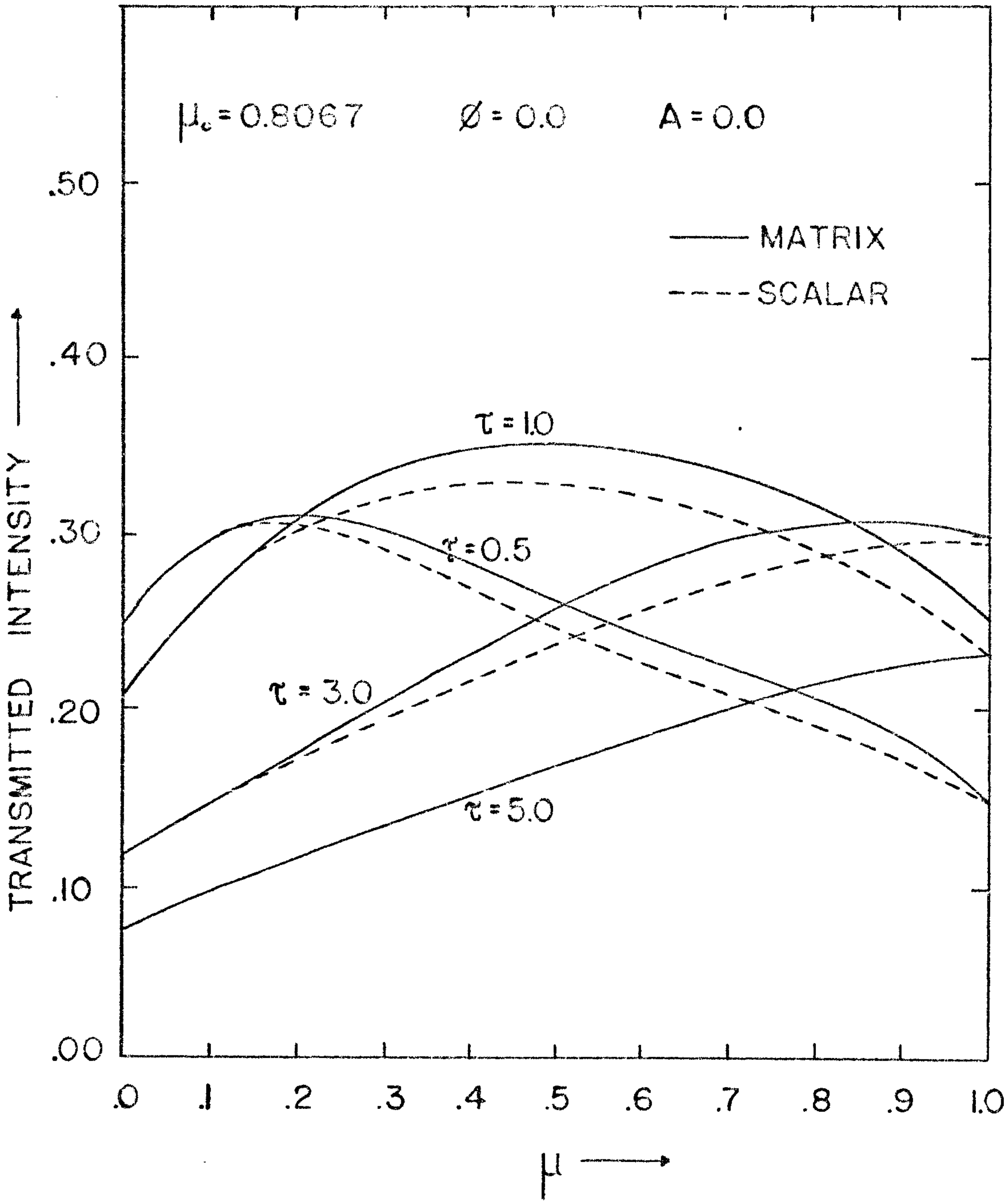

Fig.11--Transmitted intensity for μ_0=0.8067, Ø=0.0, and A=0.

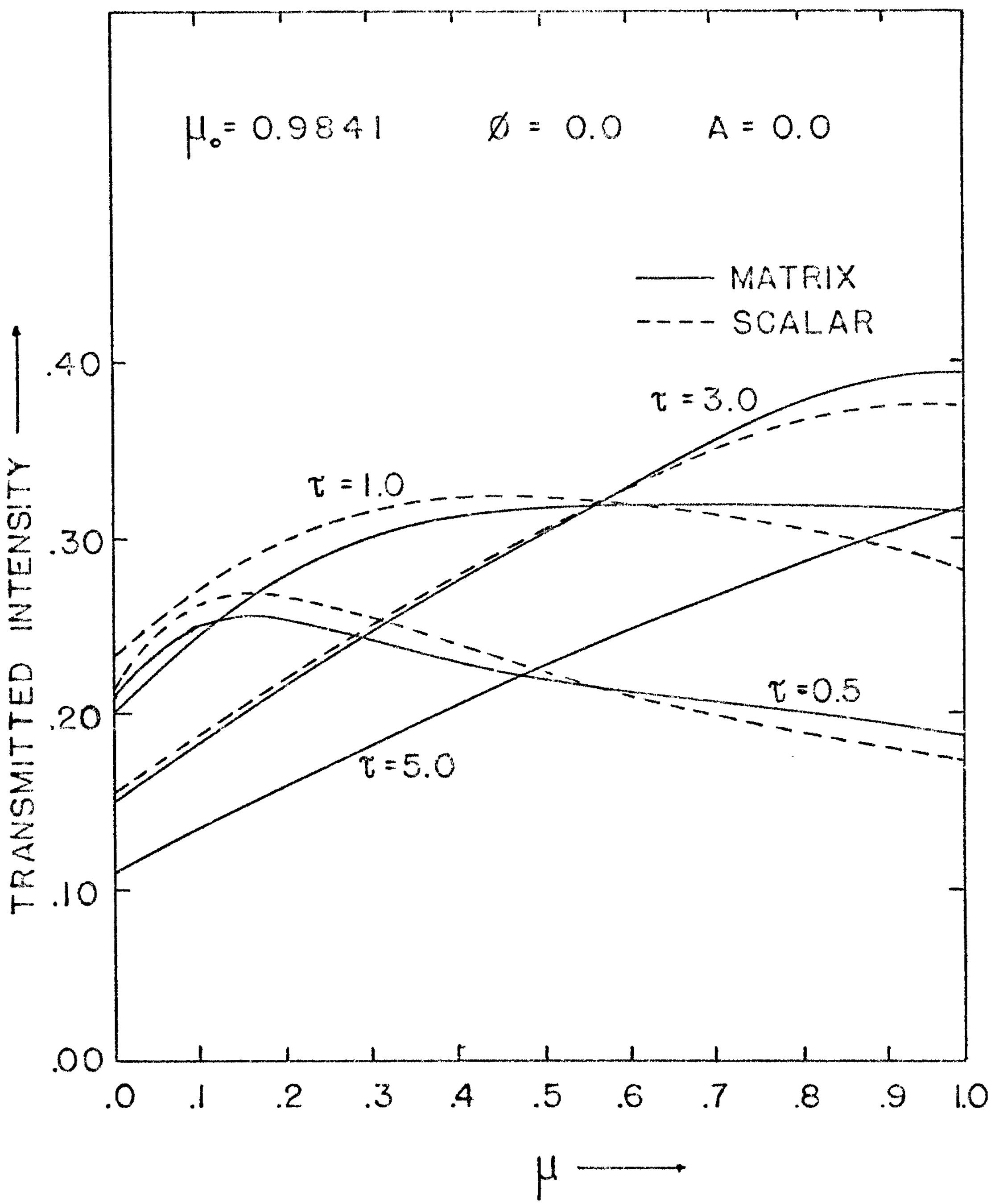

Fig.12==Transmitted intensity for μ_o=0.9841, O=0.0, and A=0.

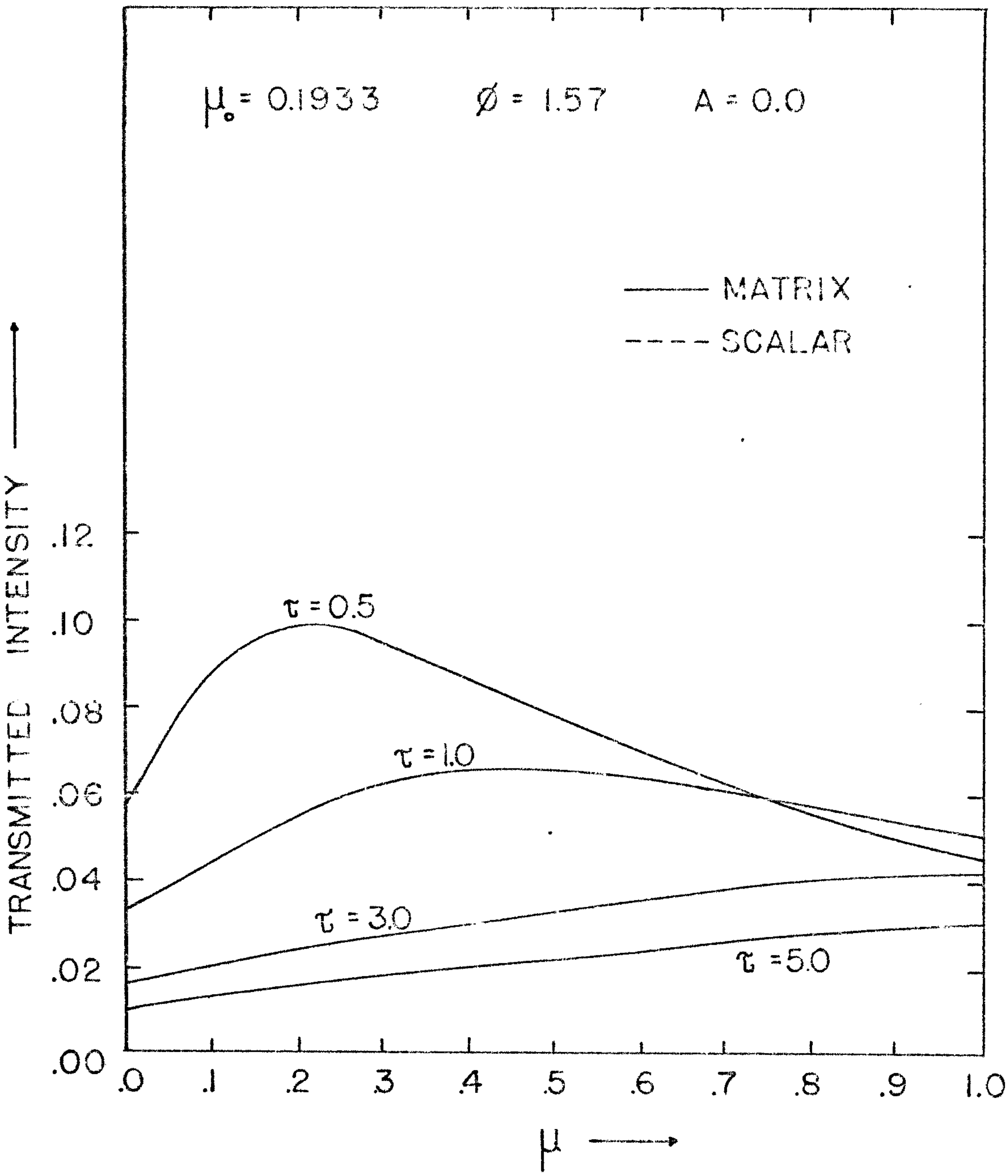

Fig.13--Transmitted intensity for μ_0=0.1933, Ø=1.57, and A=0.

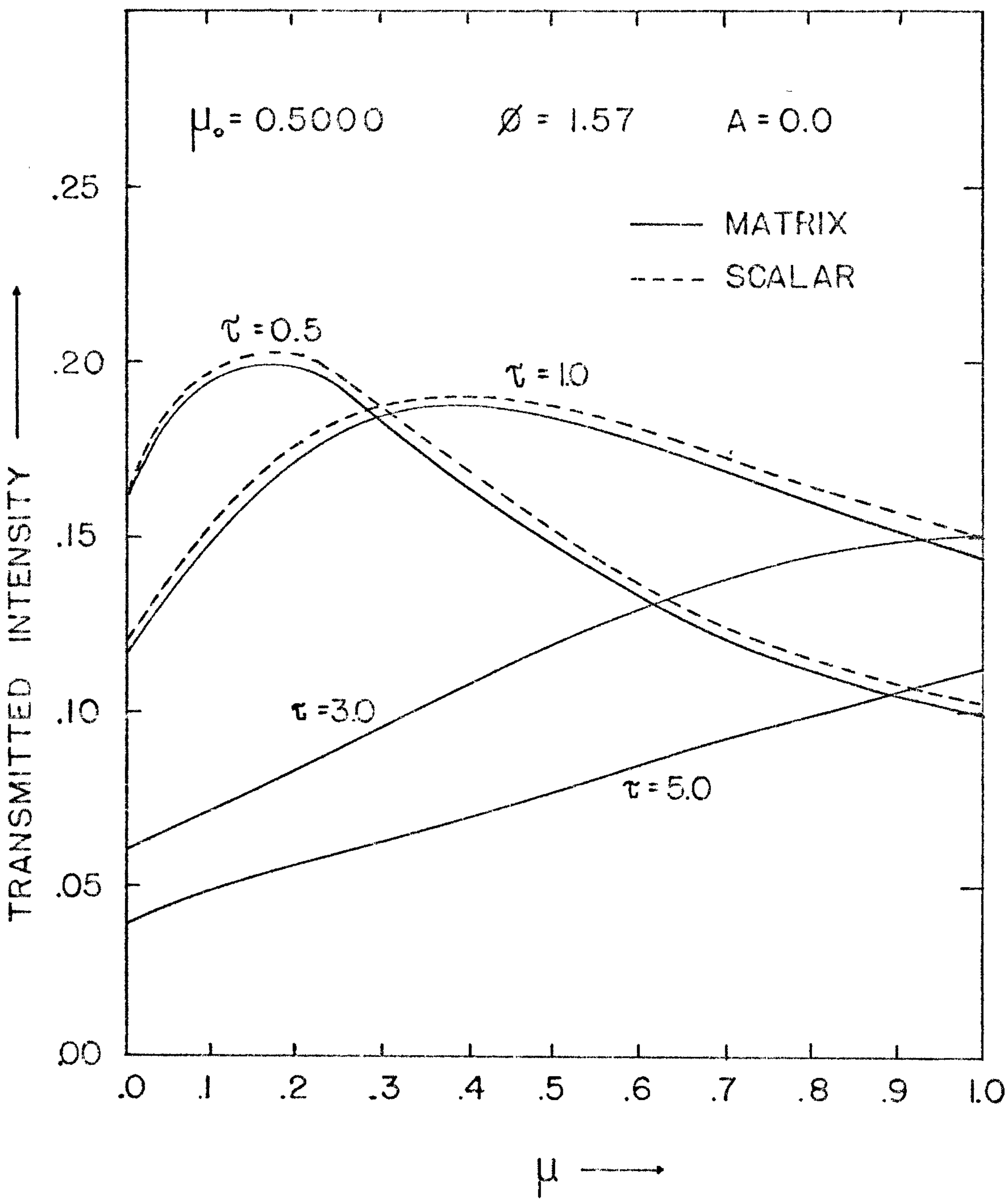

Fig.14--Transmitted intensity for μ_o=0.5000, Ø=1.57, and A=0.

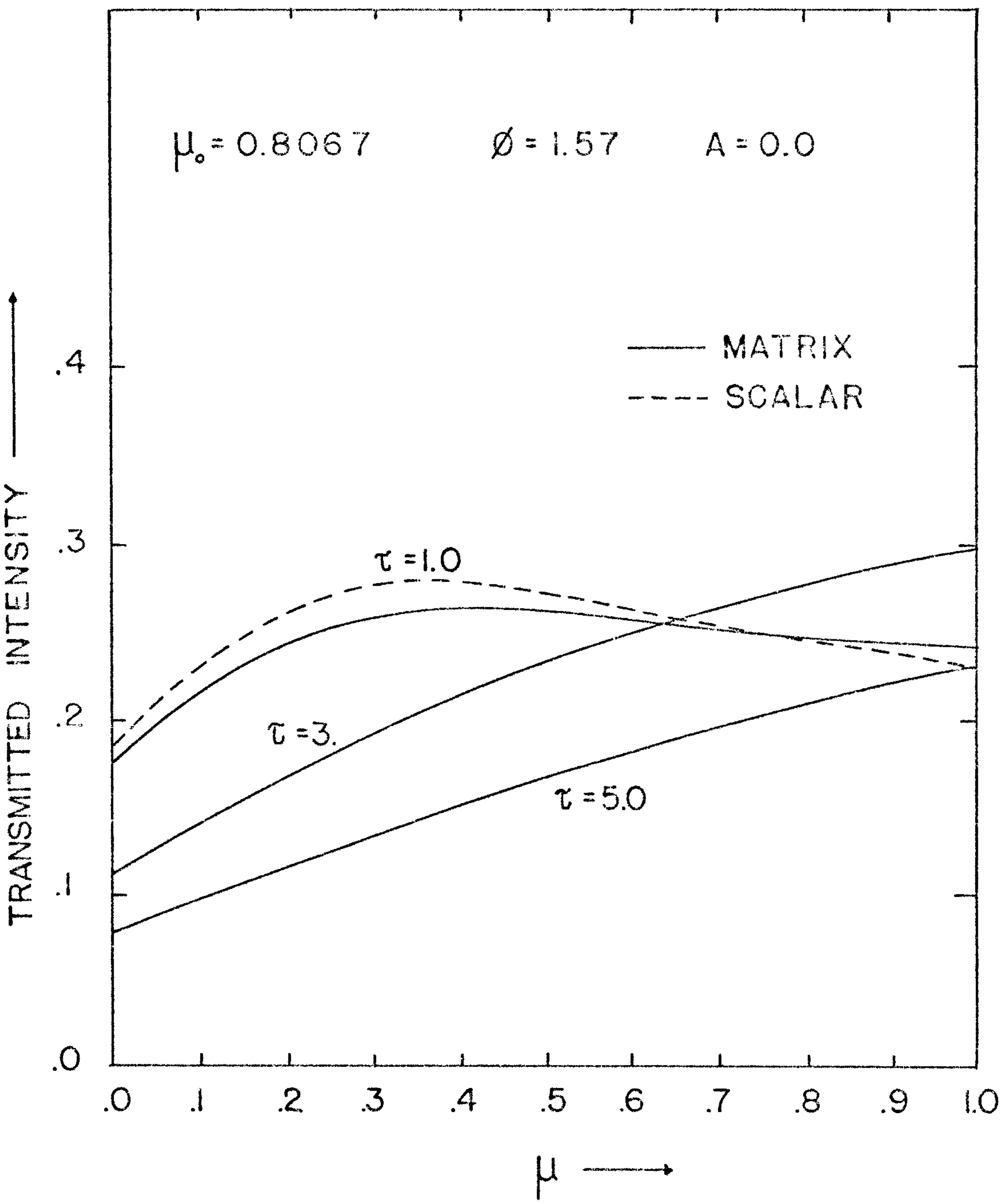

Fig.15--Transmitted intensity for μ_0=0.8067, ϕ=1.57, and A=0.

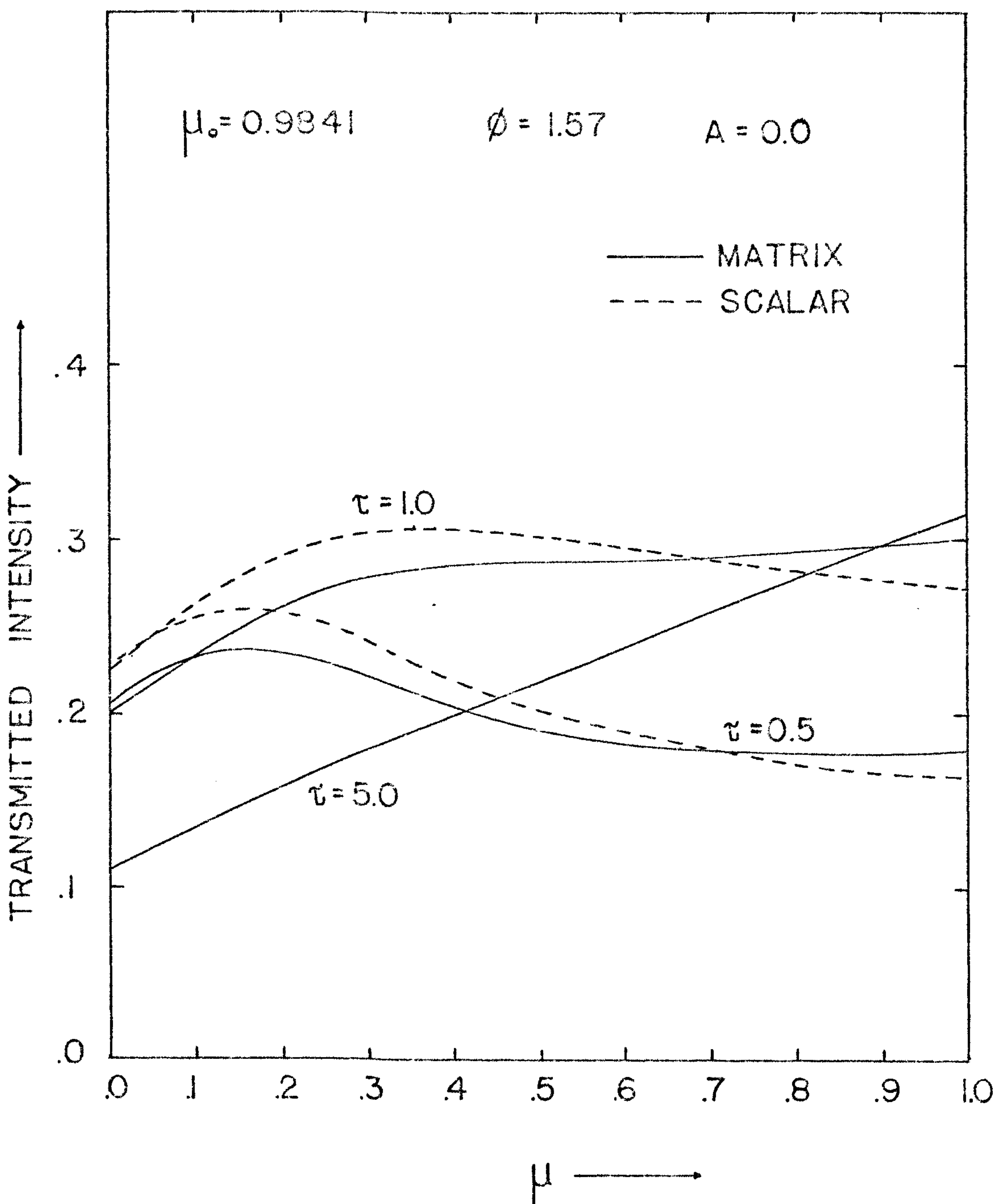

Fig.16--Transmitted intensity for μ_0=0.9841, ϕ=1.57, and A=0.

BIBLIOGRAPHY

Books

Bellman, R. E., R. E. Kalaba, and M. C. Prestrud. Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness, New York, American Elsevier Company, Inc., 1963.

Chandrasekhar, S., Radiative Transfer, New York, Dover Publications, 1960.

Coulson, K. L., J. V. Dave, and Z. Sekera. Tables Relating to Radiation Emerging from a Planetary Atmosphere with Rayleigh Scattering, California, University of California Press, 1960.

Articles

Carlstedt, J. L., and T. W. Mullikin, "The X- and Y-Functions of Chandrasekhar," Astrophysical Journal Supplement, XII (June,1966), 449-501.

Chandrasekhar, S., and D. D. Elbert, "The Illumination and Polarization of the Sunlit Sky on Rayleigh Scattering," The Transactions of the American Philosophical Society, XLIV (December,1954), 643-729.

Kagiwada, H., and R. E. Kalaba, "Estimation of Local Anisotropic Scattering Properties using Measurements of Multiply Scattered Radiation," The Journal of Quantitative Spectroscopy and Radiative Transfer, VII (October,1967), 295-303.

Mullikin, T. W., "A Nonlinear Integrodifferential Equation in Radiative Transfer," The Journal of the Society for Industrial and Applied Mathematics, XIII (June,1965), 388-410.

Sekera, Z., "Reduction of the Equations of Radiative Transfer for a Plane-Parallel, Planetary Atmosphere: Parts I & II," The RAND Corporation (RM-4952-PR,RM-5056-PR), Santa Monica, California.