SOLUTIONS OF THE HELMHOLTZ EQUATION FOR A CLASS OF NON-SEPARABLE CYLINDRICAL AND ROTATIONAL COORDINATE SYSTEMS*

BY

VAUGHAN H. WESTON

University of Toronto

Abstract. In cylindrical and rotational coordinate systems, one of the variables can be separated out of the Helmholtz equation, leaving a second order partial differential equation in two variables. For a class of the coordinate systems, this equation is reducible to a recurrence set of ordinary differential equations in one variable, which are solvable by ordinary methods.

1. Introduction. The usual method of solving the Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0 \tag{1}$$

in three dimensions is the method of separation of variables, in which the equation is separated into three ordinary differential equations, each of which can be solved. However, it is shown by Eisenhart [1] that the method of separation of variables in euclidean space is applicable to only eleven coordinate systems, generated by confocal quadrics or their degenerate forms. Exact solutions of the Helmholtz equation have thus hitherto been limited to separable coordinate systems.

In this article a method is presented for solving (1) for a class of non-separable cylindrical and rotational coordinate systems.

2. Cylindrical and rotational coordinate systems. The definitions of cylindrical and rotational coordinate systems are those given by P. Moon and D. Spencer [2]. Let $z = F(w)$, where $z = x_1 + iy_1$, and F is any analytic function. Separating real and imaginary parts, one obtains

$$x_1 = \xi_1(u, v), \qquad y_1 = \xi_2(u, v). \tag{2}$$

The curves u = constant, and v = constant give rise to an orthogonal family of curves in the z-plane. For each function F, a cylindrical coordinate system and one or two rotational coordinate systems can be formed.

The cylindrical system (u_1, u_2, u_3) is given by the equations

$$\begin{aligned} x &= \xi_1(u_1, u_2), \\ y &= \xi_2(u_1, u_2), \\ z &= u_3, \end{aligned} \tag{3}$$

and the rotational coordinate systems $(u_1, u_2, u_3 = \phi)$ by

$$\begin{aligned} x &= \xi_1(u_1, u_2) \cos\phi, \\ y &= \xi_1(u_1, u_2) \sin\phi, \\ z &= \xi_2(u_1, u_2), \end{aligned} \tag{4}$$

*Received December 20, 1956; revised manuscript received April 8, 1957.

and

$$x = \xi_2(u_1, u_2) \cos \phi,$$
$$y = \xi_2(u_1, u_2) \sin \phi, \tag{5}$$
$$z = \xi_1(u_1, u_2).$$

System (4) is found by rotation about the y_1-axis and system (5) about the x_1-axis. The z-axis is then taken as the axis of rotation. The metric coefficients for the cylindrical coordinate systems have the property that

$$h_1^2 = h_2^2 = \left(\frac{\partial \xi_1}{\partial u_1}\right)^2 + \left(\frac{\partial \xi_1}{\partial u_2}\right)^2, \qquad h_3 = 1 \tag{6}$$

and for the rotational coordinate system (4)

$$h_1^2 = h_2^2 = \left(\frac{\partial \xi_1}{\partial u_1}\right)^2 + \left(\frac{\partial \xi_1}{\partial u_2}\right)^2$$
$$h_3 = \xi_1(u_1, u_2) \tag{7}$$

and rotational coordinate system (5)

$$h_1^2 = h_2^2 = \left(\frac{\partial \xi_2}{\partial u_1}\right)^2 + \left(\frac{\partial \xi_2}{\partial u_2}\right)^2$$
$$h_3 = \xi_2(u_1, u_2). \tag{8}$$

Also since F is an analytic function $\xi_i(u_1, u_2)$ has the property

$$\frac{\partial^2 \xi_i}{\partial u_1^2} + \frac{\partial^2 \xi_i}{\partial u_2^2} = 0,$$
$$\frac{\partial \xi_1}{\partial u_1} = \frac{\partial \xi_2}{\partial u_2}, \qquad \frac{\partial \xi_1}{\partial u_2} = -\frac{\partial \xi_2}{\partial u_1}. \tag{9}$$

Now rotational and cylindrical coordinates have the property that one of the variables (ϕ and u_3 respectively) can be separated out of (1) leaving a second order partial differential equation in the two variables u_1 and u_2. We shall show that there exists a class of coordinate systems for which this equation can be reduced to a recurrence set of ordinary differential equations in one variable.

3. Solutions of the Helmholtz equation for rotational coordinate systems.

Theorem I. If for a rotational coordinate system there is a $u_i (i \neq 3)$ such that $h_3(u_1, u_2)$ has the property that

$$\frac{\partial h_3}{\partial u_i} = \sum_{s=N_1}^{M_1} f_s(u_i)[h_3(u_1, u_2)]^s, \tag{10}$$

where N_1, M_1 are integers, and $N_1 \leq M_1$ and if the metric coefficient h_1^2 has the form

$$h_1^2 = \sum_{s=N_2}^{M_2} c_s(u_i)[h_3(u_1, u_2)]^s, \tag{11}$$

where N_2, M_2 are integers and $N_2 \leq M_2$ then

$$\psi(u_1, u_2, \phi) = e^{i\mu\phi} \sum_r a_r(u_i)[h_3(u_1, u_2)]^r \tag{12}$$

is a solution of the equation $\nabla^2\psi + k^2\psi = 0$, where $a_r(u_i)$ satisfies the set of recurrence relations

$$\frac{d^2a_r}{du_i^2} + \sum_{q=r+1-N_1}^{r+1-M_1} \frac{da_q}{du_i}(2q+1)f_{r+1-q}(u_i) + \sum_{q=r+2-N_2}^{r+2-M_2}(q^2-\mu^2) \cdot a_q(u_i)c_{r+2-q}(u_i) + k^2\sum_{q=r-N_2}^{r-M_2} a_q(u_i)c_{r-q}(u_i) = 0. \tag{13}$$

Proof. The operator ∇^2 in rotational coordinate systems has the form

$$\frac{1}{h_1^2h_3}\left\{\frac{\partial}{\partial u_1}\left(h_3\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(h_3\frac{\partial}{\partial u_2}\right) + \frac{h_1^2}{h_3}\frac{\partial^2}{\partial\phi^2}\right\}.$$

Hence using the expression for ψ given by (12), Eq. (1) reduces to

$$e^{i\mu\phi}\left\{\frac{1}{h_1^2h_3}\left[\frac{\partial}{\partial u_1}\left(h_3\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(h_3\frac{\partial}{\partial u_2}\right)\right] - \frac{\mu^2}{h_3^2} + k^2\right\}\cdot\sum a_r(u_i)(h_3)^r = 0. \tag{14}$$

Multiplying (14) by $h_1^2\exp(-i\mu\phi)$ one obtains the following

$$\sum_r\left\{\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{1}{h_3}\frac{\partial h_3}{\partial u_1}\frac{\partial}{\partial u_1} + \frac{1}{h_3}\frac{\partial h_3}{\partial u_2}\frac{\partial}{\partial u_2} - \mu^2\frac{h_1^2}{h_3^2} + k^2h_1^2\right\}\cdot a_r(u_i)(h_3)^r = 0,$$

$$\sum_r\left\{(h_3)^r\frac{d^2a_r}{du_i^2} + 2\frac{da_r}{du_i}r(h_3)^{r-1}\frac{\partial h_3}{\partial u_i} + \left(k^2h_1^2 - \mu^2\frac{h_1^2}{h_3^2}\right)a_r(h_3)^r + r(h_3)^{r-2}a_r\left[(r-1)\left(\frac{\partial h_3}{\partial u_1}\right)^2 + (r-1)\left(\frac{\partial h_3}{\partial u_2}\right)^2 + h_3\frac{\partial^2h_3}{\partial u_1^2} + h_3\frac{\partial^2h_3}{\partial u_2^2}\right] + r(h_3)^{r-1}\frac{a_r}{h_3}\left[\frac{\partial h_3}{\partial u_1}\frac{\partial h_3}{\partial u_1} + \frac{\partial h_3}{\partial u_2}\frac{\partial h_3}{\partial u_2}\right] + \frac{(h_3)^r}{h_3}\frac{\partial h_3}{\partial u_i}\frac{da_r}{du_i}\right\} = 0.$$

Using relations (7), (8) and (9) one obtains

$$\sum_r\left\{(h_3)^r\frac{d^2a_r}{du_i^2} + (2r+1)\frac{da_r}{du_i}(h_3)^{r-1}\frac{\partial h_3}{\partial u_i} + r^2a_r(h_3)^{r-2}h_1^2 + \left(k^2h_1^2 - \mu^2\frac{h_1^2}{h_3^2}\right)a_r(h_3)^r\right\} = 0. \tag{15}$$

Use relations (10) and (11), and arrange (15) in power series in $h_3(u_1, u_2)$ to obtain

$$\sum_r(h_3)^r\left\{\frac{d^2a_r}{du_i^2} + \sum_{q=r+1-N_1}^{r+1-M_1}\frac{da_q}{du_i}(2q+1)f_{r+1-q}(u_i) + \sum_{q=r+2-N2}^{r+2-M_2}(q^2-\mu^2)a_q(u_i)c_{r+2-q}(u_i) + \sum_{q=r-N_2}^{r-M_2}k^2a_q(u_i)c_{r-q}(u_i)\right\} = 0.$$

Hence equate coefficients of $(h_3)^r$ to zero. A recurrence set of ordinary differential equations is then obtained relating the functions $a_r(u_i)$

$$\frac{d^2a_r}{du_i^2} + \sum_{q=r+1-N_1}^{r+1-M_1}\frac{da_q}{du_i}(2q+1)f_{r+1-q}(u_i) + k^2\sum_{q=r-N_2}^{r-M_2}a_q(u_i)c_{r-q}(u_i) + \sum_{q=r-2-N_2}^{r+2-M_2}(q^2-\mu^2)a_q(u_i)c_{r+2-q}(u_i) = 0. \tag{16}$$

Thus provided that the ordinary differential equations given by (16) can be solved for the $a_r(u_i)$, the Helmholtz equation has the solution

$$\psi(u_1, u_2, \phi) = e^{i\mu\phi} \sum_r a_r(u_i)[h_3(u_1, u_2)]^r. \tag{17}$$

Here the summation over r represents a power series in the function (h_3).

There are two cases of practical importance, (i) lower termination of series (17) when $M_2 \geq N_2 \geq 2$ and $M_1 \geq N_1 \geq 1$ and (ii) upper termination when $N_2 \leq M_2 \leq 0$ and $N_1 \leq M_1 \leq 1$.

For the case of lower termination the differential equation (16) is an inhomogeneous equation, the homogeneous part involving a_r and the inhomogeneous involving $a_{r-1}, a_{r-2}, \cdots$ i.e. terms a_n such that $n < r$. Thus, if $a_{r-1}, a_{r-2}, \cdots$ are known then a_r can be found. Now there is some number p such that $a_{p-1} \equiv a_{p-2} \equiv a_{p-3} \equiv 0$. Hence the inhomogeneous portion of (16) for $r = p$, is zero. Thus a_p is a solution of the homogeneous equation and can be found. Since Eq. (16) for $r = p + 1$ involves only a_p and a_{p+1} and a_p is now known, then a_{p+1} can be found. Hence if $a_p, a_{p+1}, \cdots, a_{r-1}$ are known, then a_r can be found.

Hence one has a solution of the form

$$\left.\begin{aligned}\psi_p^\mu(u_1, u_2, \phi) &= e^{i\mu\phi} \sum_{r=p,p+1}^{\infty} a_r(u_i)(h_3)^r \\ &= e^{i\mu\phi}(h_3)^p \sum_{N=0}^{\infty} a_{p+N}(h_3)^N.\end{aligned}\right\} \tag{18}$$

Similarly it can be shown for the case of upper termination that if q is such that $a_{q+1} \equiv a_{q+2} \equiv a_{q+3} \equiv 0$, the expression (17) becomes

$$\left.\begin{aligned}\psi_q^\mu(u_1, u_2, \phi) &= e^{i\mu\phi} \sum_{-\infty}^{r=q,q-1} a_r(u_i)(h_3)^r \\ &= e^{i\mu\phi}(h_3)^q \sum_{N=0}^{\infty} a_{-N+q}(u_i)(h_3)^{-N}.\end{aligned}\right\} \tag{19}$$

The numbers p and q are determined from boundary conditions. In addition to the solutions of the form (17) there may exist other solutions described in the following theorem.

Theorem II. If for a rotational coordinate system there is a $u_i (i \neq 3)$ such that $h_3(u_1, u_2)$ and h_1^2 have the properties given by (10) and (11) respectively, and if there is a function $B(u_j)$ where $i \neq j \neq 3$; such that

$$\frac{dB}{du_j}\frac{\partial h_3}{\partial u_j} = B(u_j) \sum_{s=N_3}^{M_3} d_s(u_i)(h_3)^s, \tag{20}$$

where N_3 and M_3 are integers and $N_3 \leq M_3$ and

$$\frac{d^2B}{du_j^2} = B(u_j) \sum_{s=N_4}^{M_4} e_s(u_i)(h_3)^s \tag{21}$$

where N_4 and M_4 are integers and $N_4 \leq M_4$ then

$$\psi(u_1, u_2, \phi) = e^{i\mu\phi}B(u_j) \sum_r b_r(u_i)[h_3(u_1, u_2)]^r \tag{22}$$

is a solution of the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$ where $b_r(u_i)$ must satisfy the set of recurrence relations

$$\frac{d^2b_r}{du_i^2} + \sum_{q=r+1-N_1}^{r+1-M_1} \frac{db_q}{du_i}(2q+1)f_{r+1-q} + \sum_{q=r+2-N_2}^{r+2-M_2} (q^2-\mu^2)b_qc_{r+2-q}$$
$$+ k^2 \sum_{q=r-N_2}^{r-M_2} b_qc_{r-q} + \sum_{r-N_4}^{r-M_4} b_qe_{r-q} + \sum_{q=r-M_3+1}^{r-M_3+1} b_q(2q+1)\,d_{r+1-q} = 0.$$

The proof is similar to that of Theorem 1. As before, there are two cases of practical importance for (22); upper termination and lower termination of the series.

When $M_2 \geq N_2 \geq 2$, $M_1 \geq N_1 \geq 1$, $M_4 \geq N_4 \geq 0$ and $N_3 \geq N_3 \geq 1$ there are solutions of the Helmholtz equation of the form

$$\psi^{\mu}_{p'}(u_1, u_2, \phi) = e^{i\mu\phi}B(u_j) \sum_{r=p'}^{\infty} b_r(u_i)[h_3(u_1, u_2)]^r \tag{23}$$

and when $N_2 \leq M_2 \leq 0$, $N_1 \leq M_1 \leq 1$, $N_4 \leq M_4 \leq 0$ and $N_3 \leq M_3 \leq 1$ there are solutions of the form

$$\psi^{\mu}_{q'}(u_1, u_2, \phi) = e^{i\mu\phi}B(u_j) \sum_{-\infty}^{r=q'} b_r(u_i)[h_3(u_1, u_2)]^r. \tag{24}$$

The second type of solutions given by (23) and (24) are essential if h_3 is an even function of the variable u_j. Then the first type of solution given by (17) are even functions of u_j, and if odd solutions in the variable u_j are required, then this $B(u_j)$ is chosen such that $B(u_j)$ is an odd function of u_j.

4. Solutions of the Helmholtz equation for cylindrical coordinate systems. In a similar manner as for the rotational case, solutions of the Helmholtz equation can be obtained and are stated in the following theorem (where to simplify presentation the two cases corresponding to Theorem I and Theorem II are combined):

Theorem III. If for a cylindrical coordinate system there is a $u_i (i \neq 3)$ and $\xi_k(u_1, u_2)$ $(k \neq 3)$ such that

$$\frac{\partial \xi_k}{\partial u_i} = \sum_{s=N_1}^{M_1} b_s(u_i)[\xi_k(u_1, u_2)]^s, \tag{25}$$

where N_1 and M_1 are integers and $N_1 \leq M_1$ and if the metric coefficient h_1 has the property that

$$h_1^2 = \sum_{s=N_2}^{M_2} c_s(u_i)[\xi_k(u_1, u_2)]^s, \tag{26}$$

where N_2 and M_2 are integers and $N_2 \leq M_2$ and there exists a function $B(u_j)$ where $i \neq j \neq 3$ such that

$$\frac{d^2B(u_j)}{du_j^2} = B(u_j) \sum_{s=N_4}^{M_4} e_s(u_i)[\xi_k(u_1, u_2)]^s \tag{27}$$

and

$$\frac{dB}{du_j}\frac{\partial \xi_k}{\partial u_j} = B(u_j) \sum_{s=N_3}^{M_3} d_s(u_i)[\xi_k(u_1, u_2)]^s, \tag{28}$$

where N_3, M_3, N_4 and M_4 are integers and $N_3 \leq M_3$, $N_4 \leq M_4$. Then

$$\psi(u_1, u_2, u_3) = e^{i\mu u_3}B(u_j) \sum_r a_r(u_i)[\xi_k(u_1, u_2)]^r \tag{29}$$

is a solution of the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$ and $a_r(u_i)$ satisfies the set of recurrence relations

$$\begin{aligned} \frac{d^2a_r}{du_i^2} + \sum_{q=r+1-N_1}^{r+1-M_1} \frac{da_q}{du_i}(2q)b_{r+1-q}(u_i) + \sum_{q=r+2-N_2}^{r+2-M_2} q(q-1)a_q(u_i)c_{r+2-q}(u_i) \\ + (k^2 - \mu^2) \sum_{q=r-N_2}^{r-M_2} a_q(u_i)c_{r-q}(u_i) + \sum_{q=r-N_4}^{r-M_4} a_q(u_i)c_{r-q}(u_i) \\ + \sum_{q=r-1-N_3}^{r-1-M_3} 2qa_q(u_i)\, d_{r-q+1} = 0. \end{aligned} \tag{30}$$

A special case of the expression given by (29) corresponding to (17) for rotational coordinates, is given when $B(u_j) = 1$ and $d_s(u_i) = 0$ for $s = N_3, N_3 + 1, \cdots, M_3$ and $e_s(u_i) = 0$ for $s = N_4, N_4 + 1, \cdots, M_4$. There are two cases of practical importance:

(i) Lower termination, $M_2 \geq N_2 \geq 2$, $M_1 \geq N_1 \geq 1$, $M_4 \geq N_4 \geq 0$, $M_3 \geq N_3 \geq 1$

(ii) Upper termination, $N_2 \leq M_2 \leq 0$, $N_1 \leq M_1 \leq 1$, $N_4 \leq M_4 \leq 0$, $N_3 \leq M_3 \leq 1$.

For lower termination the expression given by (29) reduces to

$$\psi_p^\mu(u_1, u_2, u_3) = e^{i\mu u_3}B(u_j) \sum_{r=p}^{\infty} a_r(u_i)[\xi_k(u_1, u_2)]^r \tag{31}$$

and for upper termination

$$\psi_q^\mu(u_1, u_2, u_3) = e^{i\mu u_3}B(u_j) \sum_{-\infty}^{r=q} a_r(u_i)[\xi_k(u_1, u_2)]^r. \tag{32}$$

5. Particular coordinate systems. The question of practicability of the above method of solving Eq. (1) for the particular non-separable coordinate systems under consideration is answered by the application of the method to the toroidal coordinate system. The author has obtained a complete set of solutions of Eq. (1) in toroidal coordinates, which satisfy the radiation condition and possess a ring singularity [3]. Besides toroidal coordinates, Eq. (1) can be solved for other well-known coordinate systems, as is shown in Table 1. The solutions given there, are independent of prescribed boundary conditions. The expressions given are in terms of power series of $f(u_i)\ h_3(u_1, u_2)$ and $g(u_i)\ \xi_k(u_1, u_2)$ instead of $h_3(u_1, u_2)$ and $\xi_k(u_1, u_2)$ according to whether the system is rotational or cylindrical. An appropriate choice of $f(u_i)$ or $g(u_i)$ simplifies the differential equations (13) and (30), transforming the homogeneous part into a recognizable form.

Acknowledgement. The author wishes to thank the National Research Council of Canada for financial assistance through scholarships.

References

1. L. Eisenhart, *Separable systems of Stäckel*, Ann. Math. **35**, 284 (1934)
2. P. Moon and D. E. Spencer, *Separability in a class of coordinate systems*, Franklin Inst. J. **254**, 227 (1952)
3. V. Weston, *Solutions of the toroidal wave equation and their applications*, Ph.D. thesis, Univ. of Toronto. 1956

TABLE 1

Coordinate system	*Series solution of* $\nabla^2\psi + k^2\psi = 0$	*Set of recurrence equations relating the coefficients*	*Solutions of the corresponding homogeneous equation*
Bipolar (ξ, η, z) $x = \dfrac{d \sinh \xi}{\cosh \xi - \cos \eta}$ $y = \dfrac{d \sin \eta}{\cosh \xi - \cos \eta}$ $z = z$	$\psi_e(s, \eta, z) = e^{i\mu z} \sum_{r=\tau}^{\infty} A_r(s) (s - \cos \eta)^{-r}$ where $s = \cosh \xi$	$(s^2 - 1) \dfrac{d^2 A_r}{ds^2} + s \dfrac{dA_r}{ds} - r^2 A_r + (k^2 - \mu^2) d^2 A_{r-2} - 2(r-1)\left[(s^2 - 1)\dfrac{dA_{r-1}}{ds} - (r-1)sA_{r-1}\right] = 0$	$(s \mp [s^2 - 1]^{\frac{1}{2}})^{r}$ $(s \mp [s^2 - 1]^{\frac{1}{2}})^{-r}$
	$\psi_0(s, \eta, z) = e^{i\mu z} \sin \eta \sum_{r=\tau'}^{\infty} B_r(s) (s - \cos \eta)^{-r}$	$(s^2 - 1) \dfrac{d^2 B_r}{ds^2} + s \dfrac{dB_r}{ds} - (r-1)^2 B_r + d^2(k^2 - \mu^2) B_{r-2} - 2(r-1)\left[(s^2 - 1)\dfrac{dB_{r-1}}{ds} - (r-2)sB_{r-1}\right] = 0$	$(s \mp [s^2 - 1]^{\frac{1}{2}})^{r-1}$ $(s \mp [s^2 - 1]^{\frac{1}{2}})^{-r+1}$
Bispherical (ξ, η, ϕ) $x = \dfrac{d \sin \eta \cos \phi}{\cosh \xi - \cos \eta}$	$\psi_c(s, \eta, \phi) = c^{i\mu\phi} \sum_{r=\tau}^{\infty} A_r(x) (s - \cos \eta)^{-r}$ where $x = \cos \eta$	$(1 - x^2) \dfrac{d^2 A_r}{dx^2} - 2x \dfrac{dA_r}{dx} + A_r\left[\dfrac{-\mu^2}{1 - x^2} + r(r+1)\right] + d^2 k^2 A_{r-2} + (2r-1)\left[(1 - x^2) \cdot \dfrac{dA_{r-1}}{dx} + (r-1)xA_{r-1}\right] = 0$	$P_r^{\mu}(x)$ $Q_r^{\mu}(x)$

$y = \dfrac{d \sin \eta \sin \phi}{\cosh \xi - \cos \eta}$ $z = \dfrac{d \sinh \xi}{\cosh \xi - \cos \eta}$	$\psi_0(s, \eta, \phi) = e^{i\mu\phi} \sinh \xi \sum_{r=\tau'}^{\infty} B_r(x)\,(s - \cos \eta)^{-r}$	$(1 - x^2) \dfrac{d^2B_r}{dx^2} - 2x \dfrac{dB_r}{dx} + B_r\left[\dfrac{-\mu^2}{1 - x^2} + r(r - 1)\right] + d^2k^2B_{r-2} + (2r - 1)\left[(1 - x^2) \cdot \dfrac{dB_{r-1}}{dx} + (r - 2)xB_{r-1}\right] = 0$	$P^{\mu}_{r-1}(x)$ $Q^{\mu}_{r-1}(x)$
Toroidal (ξ, η, ϕ) $x = \dfrac{d \sinh \xi \cos \phi}{\cosh \xi - \cos \eta}$ $y = \dfrac{d \sinh \xi \sin \phi}{\cosh \xi - \cos \eta}$ $z = \dfrac{d \sin \eta}{\cosh \xi - \cos \eta}$	$\psi_e(s, \eta, \phi) = e^{i\mu\phi} \sum_{r=\tau}^{\infty} A_r(s)\,(s - \cos \eta)^{-r}$	$(s^2 - 1) \dfrac{d^2A_r}{ds^2} + 2s \dfrac{dA_r}{ds} - A_r\left[\dfrac{+\mu^2}{(s^2 - 1)} + r(r + 1)\right] + d^2k^2A_{r-2} - (2r - 1)\left[(s^2 - 1) \cdot \dfrac{dA_{r-1}}{ds} - s(r - 1)A_{r-1}\right] = 0$	$P^{\mu}_{r}(s)$ $Q^{\mu}_{r}(s)$
	$\psi_0(s, \eta, \phi) = e^{i\mu\phi} \sin \eta \sum_{r=\tau'}^{\infty} B_r(s)\,(s - \cos \eta)^{-r}$	$(s^2 - 1) \dfrac{d^2B_r}{ds^2} + 2s \dfrac{dB_r}{ds} - B_r\left[\dfrac{+\mu^2}{s^2 - 1} + r(r - 1)\right] + d^2k^2B_{r-2} - (2r - 1)\left[(s^2 - 1) \cdot \dfrac{dB_{r-1}}{ds} - s(r - 2)B_{r-1}\right] = 0$	$P^{\mu}_{r-1}(s)$ $Q^{\mu}_{r-1}(s)$