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SOLUTIONS TO ELECTROMAGNETIC INDUCTION PROBLEMS

(Thesis)

Caius Vernon Dodd

Submitted as a dissertation to the Graduate Council of the University of Tennessee in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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JUNE 1967

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ABSTRACT

This dissertation describes methods of solving electromagnetic induction problems by use of the vector potential. The differential equations for the vector potential are derived from Maxwell's equations, and cylindrical symmetry is assumed. This differential equation is then solved by two different basic techniques: a "closed form" solution and a relaxation solution.

For the "closed form" solution, sinusoidal driving currents and linear, isotropic, and homogeneous media are assumed. The differential equation is solved for two different conductor configurations; a rectangular cross-section coil above a plane with one conductor clad on another and a rectangular cross-section coil encircling a two conductor rod. The solutions for both configurations are given in terms of integrals of Bessel functions.

The relaxation method is used for three different cases, the first being that for an applied current with a sinusoidal time dependence in a linear, isotropic, and inhomogeneous medium. The second case is a time sequential relaxation for an applied current with any type of time dependence in a linear, isotropic, and inhomogeneous medium. The third case is the same as the second, except the medium may be nonlinear.

In the relaxation method finite difference approximations are made for all terms in the differential equation, and the vector potential at every point is expressed in terms of the vector potential at neighboring

points. The vector potential at every point may then be solved using a relaxation (iteration) technique for any particular configuration of coil and material.

Once the vector potential is determined, any physically observable electromagnetic phenomenon can be calculated from it. Equations are given for the eddy-current density, heating density, force density, induced voltages, and coil impedance for both perfect metals and metals with defects. Examples of the application of the relaxation technique to solve these problems and experimental verification of the answers are given. In most cases the agreement between calculated and measured values is good. These two techniques will allow the solution of a large number of electromagnetic induction problems.

CHAPTER I

INTRODUCTION

Electromagnetic problems are usually divided into three categories: low frequency, intermediate frequency, and high frequency. At low frequencies, static conditions are assumed; at high frequencies, wave equations are used. Both of these regions have been studied extensively. However, in the intermediate frequency range, where diffusion equations are used, very few problems have actually been solved. Induction problems fall into this intermediate frequency region. Electromagnetic induction has been used in industry for many years. As early as 1879, D. E. Hughes used an induction coil to sort metals. Since this time, induction coils have been used to test materials, to heat materials, and to form materials (Magnaforming). This thesis outlines a method of solving these induction problems.

There have been numerous articles on the testing of materials with eddy currents. Some of the first papers dealing with both the theory and the practical aspects of eddy-current testing are by Förster (1952), Förster and Stambke (1954), and Förster (1954). In this series of papers, analyses are made of a coil above a conducting surface, assuming the coil to be a magnetic dipole, and of an infinite coil encircling an infinite rod. Hochschild (1959) also gives an analysis of an infinite coil including some eddy-current distributions in the metal. Waidelich and Renken (1956) made an analysis of the coil impedance using an image approach. Their theoretical results agreed well with theory for relatively high frequencies. Libby (1959) presented a theory in which he assumed the coil was a transformer with a network tied to the secondary. This network representation gave good results when compared to experiment. The diffusion of eddy-current pulses [Atwood and Libby (1963)] can be represented in this manner. Russell, Schuster and Waidelich (1962) gave an analysis of a cup core coil where they assumed the flux was entirely coupled into the conductor. The semiempirical results agreed fairly well with the experimental measurements. Vein (1962), Cheng (1964), and Burrows (1964) gave treatments based on delta function coils, and Burrows continued with the development of an eddy-current flaw theory. Dodd and Deeds (1963) and Dodd (1965) gave a relaxation theory to calculate the vector potential of a coil with a finite cross section. Excerpts from the latter work appear here in parts of Chapter II, the first section of Chapter IV, and the first two figures in Chapter VI. The equation for the impedance of a defect is also given in an earlier work [Dodd and Deeds (1967)]. This dissertation extends the "closed form" solution to coils with finite cross sections and the relaxation calculation to include forces, nonsinusoidal currents, and nonlinear media.

The vector potential was used in this dissertation as opposed to the electric and magnetic fields. The differential equations for the vector potential will be derived from Maxwell's equations, with the assumption of cylindrical symmetry. This differential equation will then be solved by two different techniques to obtain a "closed form" solution and a relaxation solution.

For the "closed form" solution, sinusoidal driving currents and linear, isotropic, and homogeneous media will be assumed. Solutions will be obtained for two different conductor geometries: a rectangular cross-section coil above a plane with one conductor clad on another and a rectangular cross-section coil encircling a two conductor rod. The solutions for both geometries will be given in terms of integrals of Bessel functions.

The relaxation method will be used for three different cases, the first being that for an applied current with a sinusoidal time dependence in a linear, isotropic, inhomogeneous medium. The second case will involve a time sequential relaxation for an applied current with an arbitrary continuous time dependence in a linear, isotropic, inhomogeneous medium. The third case will be the same as the second, except that the medium will be assumed to be nonlinear.

In the relaxation method, finite difference approximations are made for all terms in the differential equation, and the vector potential at any point can be expressed in terms of the vector potential at neighboring points. The vector potential at every point can then be solved using a relaxation (iteration) technique for any particular configuration of coil and material.

Once the vector potential has been determined, either by a "closed form" solution or by a relaxation technique, any physically observable electromagnetic quantity can be calculated from it. Equations for the eddy-current density, heating density, force density, induced voltages, and coil impedance for both perfect metals and metals with defects will be derived. Examples of the application of the relaxation technique to solve these problems and experimental verification of the answers will be given. In most cases, the agreement between calculated and measured values is good.

CHAPTER II

DERIVATION OF VECTOR POTENTIAL

The differential equations for the vector potential will be derived from Maxwell's equations which are [Dodd (1965)]:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad , \tag{2.1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad , \tag{2.2}$$

$$\nabla \cdot \vec{B} = 0 \quad , \tag{2.3}$$

$$\nabla \cdot \vec{D} = \rho \quad . \tag{2.4}$$

The medium is taken to be linear and isotropic, but not homogeneous. In a linear and isotropic medium, the following relations between $\vec{D}$ and $\vec{E}$ and $\vec{B}$ and $\vec{H}$ hold:

$$\vec{B} = \mu \vec{H} \quad , \tag{2.5}$$

$$\vec{D} = \epsilon \vec{E} \quad . \tag{2.6}$$

The current density $\vec{J}$ can be expressed in terms of Ohm's law:

$$\vec{J} = \sigma \vec{E} \quad . \tag{2.7}$$

Equations (2.6) and (2.7) may be substituted into equation (2.1) to obtain the curl of $\vec{H}$ in terms of $\vec{E}$:

$$\nabla \times \vec{H} = \sigma \vec{E} + \frac{\partial \epsilon \vec{E}}{\partial t} \quad . \tag{2.8}$$

The term $\sigma\vec{E}$ is much greater than $\frac{\partial\epsilon\vec{E}}{\partial t}$, so the latter may be neglected for frequencies below about ten megacycles per second.* The magnetic induction field $\vec{B}$ may be expressed as the curl of a vector potential $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A} \quad . \tag{2.9}$$

Substituting this into equation (2.2) gives

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\nabla \times \vec{A} = -\nabla \times \frac{\partial\vec{A}}{\partial t} \quad , \tag{2.10}$$

or

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla\psi \quad , \tag{2.11}$$

$$\sigma\vec{E} = -\sigma\frac{\partial\vec{A}}{\partial t} - \sigma\nabla\psi \quad . \tag{2.12}$$

The term ψ is a scalar potential. The coil may be driven by a voltage generator with an applied voltage ψ and an internal resistivity, $\frac{1}{\sigma}$. However, for the purpose of this problem the driving function is expressed as an alternating current density of constant amplitude, $\vec{J}_o$, rather than an applied potential, where

$$\underset{\substack{\sigma \to 0. \\ \nabla\psi \to \infty.}}{\text{Limit}} (-\sigma\nabla\psi) = \vec{J}_o \quad , \tag{2.13}$$

---

*For sinusoidal waves, $\frac{\partial\epsilon\vec{E}}{\partial t} = \frac{\epsilon\partial\vec{E}}{\partial t} = j\epsilon\omega\vec{E}$. The term σE is much greater than $\epsilon\omega E$ or $\sigma \gg \epsilon\omega$. $\sigma \approx 10^7$ mhos/meter for metals, $\epsilon \approx 10^{-11}$. For frequencies on the order of 10^7 cps, $\omega \approx 10^8$, $10^7 \gg 10^8 \times 10^{-11}$, or $\sigma \approx 10^{10}\,\epsilon\omega$.

This provides a current which is not affected by induced voltages or the presence of other coils. Making this substitution gives:

$$\sigma\vec{E} = -\sigma \frac{\partial \vec{A}}{\partial t} + \vec{J}_o \quad . \tag{2.14}$$

Substituting equations (2.5, page 5) and (2.9) into the left side of equation (2.8, page 5) and equation (2.14) into the right side gives:

$$\nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu} = \nabla \times [(1/\mu)\ \nabla \times \vec{A}] = -\sigma \frac{\partial \vec{A}}{\partial t} + \vec{J}_o \quad . \tag{2.15}$$

The vector identities (Morse and Feshbach, 1953)

$\nabla \times (\psi\vec{F}) = (\nabla\psi) \times \vec{F} + \psi\nabla \times \vec{F}$ and $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$, can be used to obtain the differential equation for $\vec{A}$:

$$\begin{aligned} \nabla \times (1/\mu)(\nabla \times \vec{A}) &= \nabla(1/\mu) \times (\nabla \times \vec{A}) + \frac{1}{\mu} \nabla \times (\nabla \times \vec{A}) \\ &= \nabla(1/\mu) \times (\nabla \times \vec{A}) + \frac{1}{\mu} \nabla\ (\nabla\cdot\vec{A}) - \frac{1}{\mu} \nabla^2\vec{A} \quad . \end{aligned} \tag{2.16}$$

In the definition of the vector potential the divergence of the vector potential was not defined, so it can be defined to be anything convenient. For induction problems $\nabla\cdot\vec{A}$ is set to zero. (This corresponds to the Coulomb gauge.) Equation (2.16) will then yield the following results when substituted into equation (2.15).

$$\nabla^2\vec{A} = -\mu\vec{J}_o + \mu\sigma \frac{\partial \vec{A}}{\partial t} + \mu\nabla\ (1/\mu) \times (\nabla \times \vec{A}) \quad . \tag{2.17}$$

This is the equation for the vector potential in an isotropic, linear, inhomogeneous medium. For most coil problems, it is possible to assume axial symmetry, as shown in Figure 1. The vector potential will be symmetric about the axis of the coil. Since this assumption is valid for most problems and the alternative to this assumption is a much

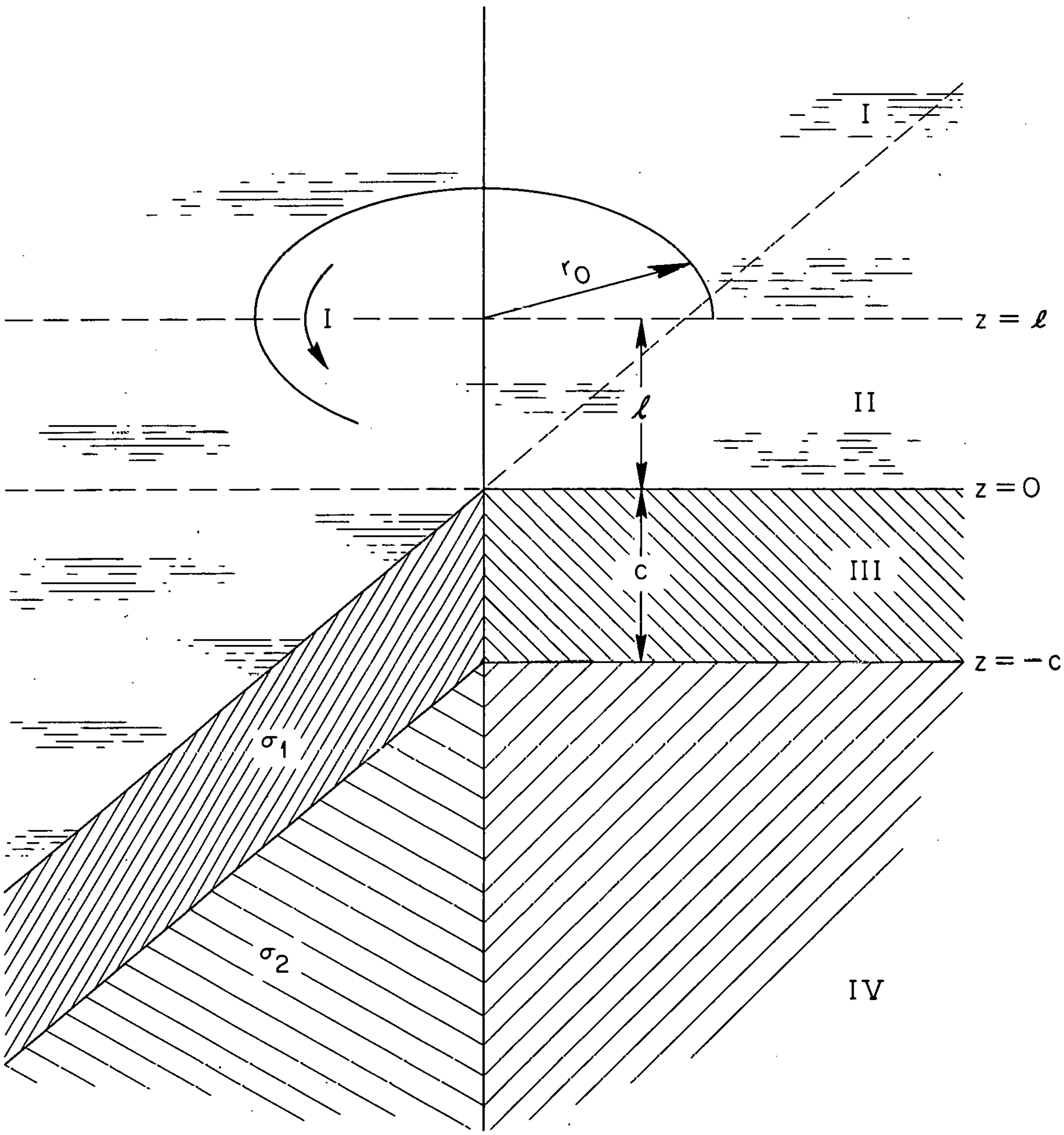

Figure 1. Delta function coil above a two conductor plane.

more complicated problem which is impractical at this time, axial symmetry is assumed. With axial symmetry, there is only a θ component of $\vec{I}$ and therefore $\vec{A}$. Expanding the θ component of equation (2.18) gives:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} = -\mu J_o + \mu\sigma\frac{\partial A}{\partial t} - \mu\left[\frac{\partial(1/\mu)}{\partial r}\left(\frac{1}{r}\frac{\partial rA}{\partial r}\right) + \left(\frac{\partial(1/\mu)}{\partial z}\right)\frac{\partial A}{\partial z}\right] \quad . \quad (2.18)$$

Assume that J_o is a sinusoidal function of time, $J_o = J'_o\ e^{j\omega t}$. Then the vector potential is likewise a sinusoidal function of time, $A = A'\ e^{j(\omega t + \phi)} = A''\ e^{j\omega t}$. Substituting into equation (2.18) gives:

$$\frac{\partial^2 A''}{\partial r^2} e^{j\omega t} + \frac{1}{r}\frac{\partial A''}{\partial r} e^{j\omega t} + \frac{\partial^2 A''}{\partial z^2}\ j\omega t - \frac{A''}{r^2} e^{j\omega t} = -\mu\ J'_o\ e^{j\omega t} + j\omega\mu\sigma A''\ e^{j\omega t} - \mu\left[\frac{\partial(1/\mu)}{\partial r}\left(\frac{1}{r}\frac{\partial rA''}{\partial r} e^{j\omega t}\right) + \left(\frac{\partial(1/\mu)}{\partial z}\right)\frac{\partial A''}{\partial z} e^{j\omega t}\right] .$$

Canceling out the term $e^{j\omega t}$ and dropping the prime gives:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} = -\mu J_o + j\omega\mu\sigma A - \mu\left[\frac{\partial(1/\mu)}{\partial r}\left(\frac{1}{r}\frac{\partial rA}{\partial r}\right) + \left(\frac{\partial(1/\mu)}{\partial z}\right)\frac{\partial A}{\partial z}\right] \quad . \quad (2.19)$$

This is the general differential equation for the vector potential in a linear, inhomogeneous medium with a sinusoidal driving current. We shall now solve the equation by two different methods, a "closed form" solution and a relaxation technique.

CHAPTER III

CLOSED FORM SOLUTION OF THE VECTOR POTENTIAL

For a closed form solution, we shall assume the medium to be linear, isotropic, and homogeneous. When I is the total driving current in a delta function coil at (r_o, z_o), the general equation (2.19) then becomes:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} - j\omega\mu\sigma A + \mu I\ \delta(r - r_o)\ \delta(z - z_o) = 0 \quad . \tag{3.1}$$

Once we have solved this linear differential equation for a particular conductor configuration, we can then superimpose any number of delta function coils to build up any desired shape of coil (provided that the current in each coil is known).

We shall solve the problem for two different conductor configurations: a coil above a two conductor plane and a coil encircling a two conductor rod. These two configurations apply to a large number of practical problems.

Coil above a Two Conductor Plane

The coil above a two conductor plane is shown in Figure 1, page 8. We have divided the problem into four regions. The differential equation in air (regions I and II) is:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} = 0 \quad . \tag{3.2}$$

The differential equation in a conductor (regions III and IV) is:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} - j\omega\mu\sigma_i A = 0 \quad . \tag{3.3}$$

Setting $A(r,z) = R(r)\ Z(z)$ and dividing by $R(r)\ Z(z)$ gives:

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{rR(r)}\frac{\partial R(r)}{\partial r} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} - \frac{1}{r^2} - j\omega\mu\sigma_i = 0 \quad . \tag{3.4}$$

We can write for the z dependence:

$$\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = \text{"constant"} = \alpha^2 + j\omega\mu\sigma_i \quad , \tag{3.5}$$

or

$$Z(z) = A\ e^{+\sqrt{\alpha^2+j\omega\mu\sigma_i}\ z} + B\ e^{-\sqrt{\alpha^2+j\omega\mu\sigma_i}\ z} \quad . \tag{3.6}$$

We shall define:

$$\alpha_i \equiv \sqrt{\alpha^2+j\omega\mu\sigma_i} \quad . \tag{3.7}$$

Equation (3.4) then becomes:

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{rR(r)}\frac{\partial R(r)}{\partial r} + \alpha^2 - \frac{1}{r^2} = 0 \quad . \tag{3.8}$$

This is a first-order Bessel equation and has the solutions:

$$R(r) = C\ J_1(\alpha r) + D\ Y_1\ (\alpha r) \quad . \tag{3.9}$$

Combining the solutions we have:

$$A(r,z) = [A\ e^{+\alpha_i z} + B\ e^{-\alpha_i z}][C\ J_1(\alpha r) + D\ Y_1(\alpha r)] \quad . \tag{3.10}$$

We now need to determine the constants A, B, C, and D. They are functions of the separation "constant" α and are usually different for each value of α. Our complete solution would be a sum of all the individual solutions if α were a discrete variable, but, since α is a continuous variable, the complete solution is an integral over the entire range of α. Thus, the general solution is:

$$A(r,z) = \int_0^\infty [A(\alpha)\ e^{\alpha_i z} + B(\alpha)\ e^{-\alpha_i z}][C(\alpha)\ J_i(\alpha r) + D(\alpha)\ Y_1(\alpha r)]\ d\alpha\ . \quad (3.11)$$

We must take $A(\alpha) = 0$ in region I, where z goes to plus infinity. Due to the divergence of Y_1 at the origin, $D(\alpha) = 0$ in all regions. In region IV, where z goes to minus infinity, $B(\alpha)$ must vanish. The solutions in each region then become:

$$A^{(1)}(r,z) = \int_0^\infty B_1(\alpha) e^{-\alpha z}\ J_1(\alpha r)\ d\alpha \quad , \qquad (3.12)$$

$$A^{(2)}(r,z) = \int_0^\infty [C_2(\alpha)\ e^{+\alpha z} + B_2(\alpha)\ e^{-\alpha z}]\ J_1(\alpha r)\ d\alpha \quad , \qquad (3.13)$$

$$A^{(3)}(r,z) = \int_0^\infty [C_3(\alpha)\ e^{\alpha_1 z} + B_3(\alpha)\ e^{-\alpha_1 z}]\ J_1(\alpha r)\ d\alpha \quad , \qquad (3.14)$$

$$A^{(4)}(r,z) = \int_0^\infty [C_4(\alpha)\ e^{\alpha_2 z}\ J_1(\alpha r)\ d\alpha \quad . \qquad (3.15)$$

The boundary conditions between the different regions are:

$$A^{(1)}(r,\ell) = A^{(2)}(r,\ell) \quad , \qquad (3.16)$$

$$\frac{\partial A^{(1)}}{\partial z}(r,z)\Big]_{z=\ell} = \frac{\partial A^{(2)}}{\partial z}(r,z)\Big]_{z=\ell} - \mu I\ \delta(r - r_o) \quad , \qquad (3.17)$$

$$A^{(2)}(r,o) = A^{(3)}(r,o) \quad , \qquad (3.18)$$

$$\frac{\partial A^{(2)}}{\partial z}(r,z)\Big]_{z=o} = \frac{\partial A^{(3)}}{\partial z}(r,z)\Big]_{z=o} \quad , \qquad (3.19)$$

$$A^{(3)}(r,-c) = A^{(4)}(r,-c) \quad , \tag{3.20}$$

$$\frac{\partial A^{(3)}}{\partial z}(r,z)\Big]_{z=-c} = \frac{\partial A^{(4)}}{\partial z}(r,z)\Big]_{z=-c} \quad . \tag{3.21}$$

Equation (3.16) gives:

$$\int_0^\infty B_1(\alpha)\ e^{-\alpha\ell}\ J_1(\alpha r)\ d\alpha = \int_0^\infty [C_2(\alpha)\ e^{\alpha\ell} + B_2(\alpha)\ e^{-\alpha\ell}]\ J_1(\alpha r)\ d\alpha \quad . \tag{3.22}$$

If we multiply both sides of equation (3.22) by $\int_0^\infty J_1(\alpha' r)r\ dr$ and then reverse the order of integration, we obtain:

$$\int_0^\infty \frac{B_1(\alpha)\ e^{-\alpha\ell}}{\alpha}\left[\int_0^\infty J_1(\alpha r)\ J_1(\alpha' r)\ \alpha r\ dr\right] d\alpha$$

$$= \int_0^\infty \frac{1}{\alpha}\ [C_2(\alpha)\ e^{\alpha\ell} + B_2(\alpha)\ e^{-\alpha\ell}]\left[\int_0^\infty J_1(\alpha r)\ J_1(\alpha' r)\ \alpha r\ dr\right] d\alpha \quad . \tag{3.23}$$

We can simplify equation (3.23) by use of the Fourier-Bessel equation, which is:

$$F(\alpha') = \int_0^\infty F(\alpha) \int_0^\infty J_1\ (\alpha r)\ J_1(\alpha' r)\ \alpha r\ dr\ d\alpha \quad . \tag{3.24}$$

Equation (3.23) then becomes:

$$\frac{B_1}{\alpha'}\ e^{-\alpha'\ell} = \frac{C_2}{\alpha'}\ e^{\alpha'\ell} + \frac{B_2}{\alpha'}\ e^{-\alpha'\ell} \quad . \tag{3.25}$$

We can evaluate the other integral equations in a similar manner. We get (after dropping the primes on the α):

$$-B_1\ e^{-\alpha\ell} = C_2\ e^{\alpha\ell} - B_2\ e^{-\alpha\ell} - \mu I r_o\ J_1(\alpha r_o) \quad , \tag{3.26}$$

$$\frac{C_2}{\alpha} + \frac{B_2}{\alpha} = \frac{C_3}{\alpha} + \frac{B_3}{\alpha} \quad , \tag{3.27}$$

$$C_2 - B_2 = \frac{\alpha_1}{\alpha} C_3 - \frac{\alpha_1}{\alpha} B_3 \quad , \tag{3.28}$$

$$\frac{C_3}{\alpha} e^{-\alpha_1 c} + \frac{B_3}{\alpha} e^{+\alpha_1 c} = \frac{C_4}{\alpha} e^{-\alpha_2 c} \quad , \tag{3.29}$$

$$\frac{\alpha_1}{\alpha} C_3 e^{-\alpha_1 c} - \frac{\alpha_1}{\alpha} B_3 e^{\alpha_1 c} = \frac{\alpha_2}{\alpha} C_4 e^{-\alpha_2 c} \quad . \tag{3.30}$$

We now have six equations with six unknowns. Their solution is:

$$B_1 = \frac{1}{2} \mu I r_o J_1(\alpha r_o) \left\{ e^{\alpha \ell} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}} \right] e^{-\alpha \ell} \right\} \quad , \tag{3.31}$$

$$C_2 = \frac{1}{2} \mu I r_o J_1(\alpha r_o) e^{-\alpha \ell} \quad , \tag{3.32}$$

$$B_2 = \frac{1}{2} \mu I r_o J_1(\alpha r_o) \left\{ \frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}} \right\} e^{-\alpha \ell} \quad , \tag{3.33}$$

$$C_3 = \mu I r_o J_1(\alpha r_o) \left\{ \frac{\alpha(\alpha_2+\alpha_1) e^{-\alpha \ell + 2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}} \right\} \quad , \tag{3.34}$$

$$B_3 = \mu I r_o J_1(\alpha r_o) \left\{ \frac{\alpha(\alpha_1-\alpha_2) e^{-\alpha \ell}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}} \right\} \quad , \tag{3.35}$$

$$C_4 = \mu I r_o J_1(\alpha r_o) \left\{ \frac{2\alpha_1 \alpha \, e^{(\alpha_2+\alpha_1)c} e^{-\alpha \ell}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}} \right\} \quad . \tag{3.36}$$

We can now write the expressions for the vector potential in each region:

$$A^{(1)}(r,z) = \frac{\mu I r_o}{2} \int_0^\infty J_1(\alpha r_o) J_1(\alpha r) e^{-\alpha \ell - \alpha z}$$

$$\times \left\{ e^{2\alpha \ell} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1) e^{2\alpha_1 c}} \right] \right\} d\alpha \quad , \tag{3.37}$$

$$A^{(2)}(r,z) = \frac{\mu I r_o}{2} \int_0^\infty J_1(\alpha r_o) J_1(\alpha r)\ e^{-\alpha \ell}$$

$$\times \left\{ e^{\alpha z} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}} \right] e^{-\alpha z} \right\} d\alpha, \quad (3.38)$$

$$A^{(3)}(r,z) = \mu I r_o \int_0^\infty J_1(\alpha r_o) J_1(\alpha r)\ e^{-\alpha \ell} \alpha$$

$$\times \left\{ \frac{(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}\ e^{\alpha_1 z} + (\alpha_1-\alpha_2)\ e^{-\alpha_1 z}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}} \right\} d\alpha \quad , \quad (3.39)$$

$$A^{(4)}(r,z) = \mu I r_o \int_0^\infty J_1(\alpha r_o) J_1(\alpha r)\ e^{-\alpha \ell} \alpha$$

$$\times \left\{ \frac{2\alpha_1\ e^{(\alpha_2+\alpha_1)c}\ e^{+\alpha_2 z}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}} \right\} d\alpha \ . \quad (3.40)$$

These give the vector potential produced by a single delta function coil. We can approximate any coil, as shown in Figure 2, by a number of delta function coils. In general, we shall have:

$$A(r,z)(\text{total}) = \sum_{i=1}^{n} A_i(r,z) = \sum_{i=1}^{n} A(r,z,\ \ell_i,\ r_i) \quad . \quad (3.41)$$

We shall now make the further restricting assumptions that the phase and amplitude of the current in each loop are identical and the coil has a rectangular cross section, as shown in Figure 2. We can now approximate equation (3.41) by:

$$A(r,z)(\text{total}) \approx \int_{r_1}^{r_2}\int_{\ell_1}^{\ell_2} A(r,z,\ r_o,\ \ell)\ dr_o\ d\ell \quad , \quad (3.42)$$

where $A(r,z,\ \ell,\ r_o)$ is the vector potential produced by a current density $I(\ell,\ r_o)$.

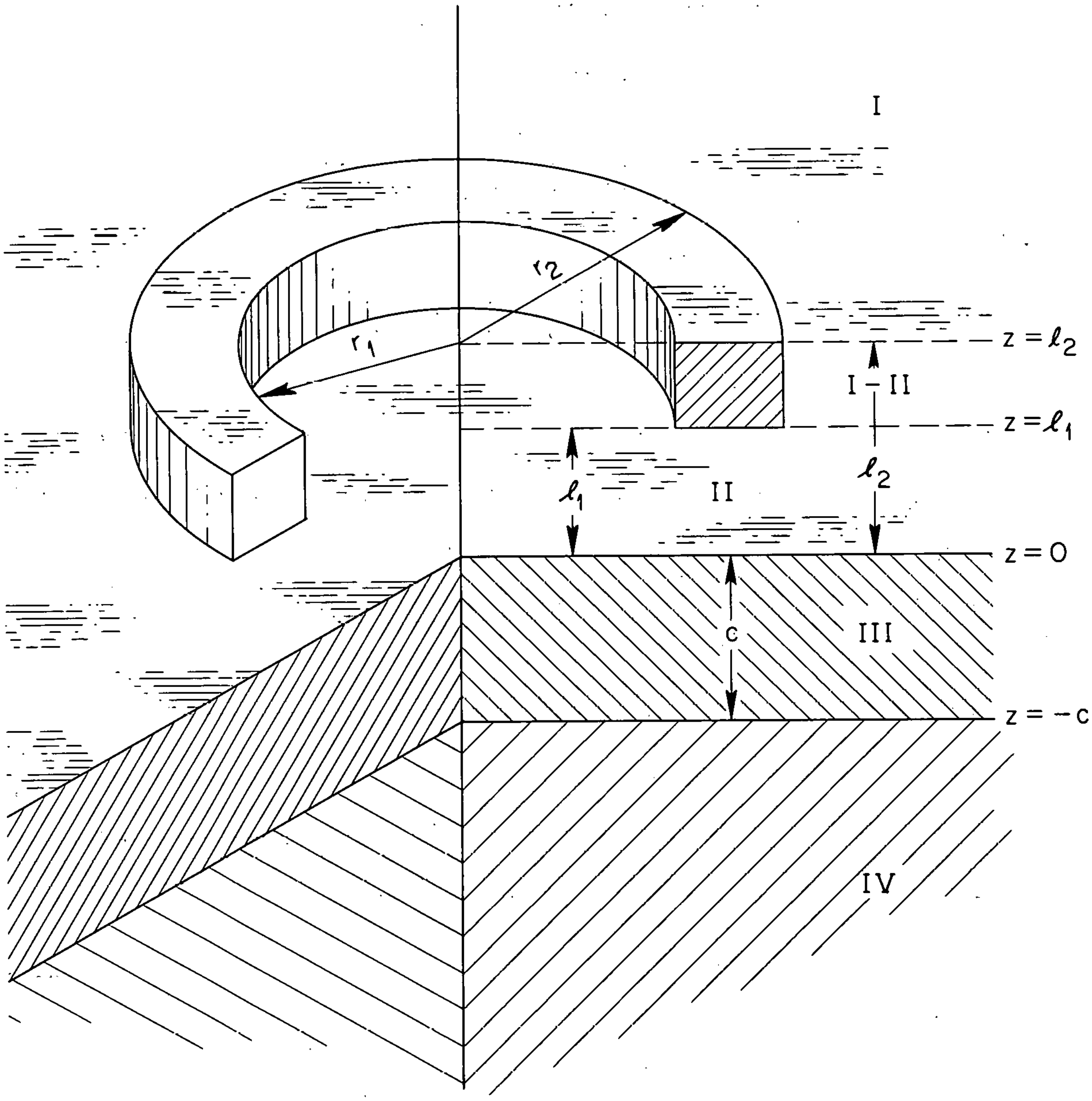

Figure 2. Rectangular cross-section coil above a two conductor plane.

After reversing the order of integration, we can write:

$$A^{(1)}(r,z) = \int_0^\infty \int_{r_1}^{r_2} \int_{\ell_1}^{\ell_2} \frac{\mu I r_o}{2} J_1(\alpha r_o) J_1(\alpha r)\, e^{-\alpha\ell}\, e^{-\alpha z}$$

$$\left\{ e^{2\alpha\ell} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}\right]\right\} d\alpha\, dr_o\, d\ell \;, \quad (3.43)$$

Integration with respect to the ℓ variable gives:

$$A^{(1)}(r,z) = \int_0^\infty \int_{r_1}^{r_2} \frac{\mu I r_o}{2\alpha} J_1(\alpha r_o) J_1(\alpha r)\, e^{-\alpha z}$$

$$\times \left\{ e^{\alpha\ell_2} - e^{\alpha\ell_1} - \left(e^{-\alpha\ell_2} - e^{-\alpha\ell_1}\right)\right.$$

$$\left.\times \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}\right]\right\} d\alpha\, dr \;. \quad (3.44)$$

We can now integrate over the r_o variable [with the help of equation (11.1.1) in the National Bureau of Standards Handbook of Mathematical Functions No. 55] and get:

$$A^{(1)}(r,z) = \mu I \sum_{k=o}^{\infty} \frac{2(k+1)}{(2k+1)(2k+3)} \int_0^\infty J_{2k+2}(\alpha r_2) J_1(\alpha r) \frac{r_2\, e^{-\alpha z}}{\alpha^2} \times \left\{ e^{\alpha\ell_2} - e^{\alpha\ell_1}\right.$$

$$\left. - \left(e^{-\alpha\ell_2} - e^{-\alpha\ell_1}\right)\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}\right]\right\}$$

$$- \mu I \sum_{k=o}^{\infty} \frac{2(k+1)}{(2k+1)(2k+3)} \int_0^\infty J_{2k+2}(\alpha r_1) J_1(\alpha r) \frac{r_1\, e^{-\alpha z}}{\alpha^2} \left\{ e^{\alpha\ell_2} - e^{\alpha\ell_1} - \left(e^{-\alpha\ell_2} - e^{-\alpha\ell_1}\right)\right.$$

$$\left.\times \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}\right]\right\} d\alpha \quad . \qquad (3.45)$$

While equation (3.45) is quite complicated, we can make some definitions which will help simplify it. Also, in actual computations it will be more useful to work with dimensionless quantities. We therefore define:

$$\bar{r} \equiv \frac{r_1+r_2}{2} \quad , \tag{3.46}$$

and

$$\bar{r}^2\, \omega\mu\sigma_i \equiv M_i \quad . \tag{3.47}$$

If we divide all dimensions by $\bar{r}$ and multiply α, α_1 and α_2 by $\bar{r}$, we then have equation (3.45) in terms of dimensionless quantities. We can also break equation (3.45) into the sum of four integrals. They may be written:

$$F_1(r, r_m, z, \ell_n, M_1, M_2) \equiv \mu I\bar{r}^2 \sum_{k=0}^{\infty} \frac{2(k+1)}{(2k+1)(2k+3)} \int_0^\infty J_{2k+2}\left\{\bar{r}\alpha\left(\frac{r_m}{\bar{r}}\right)\right\}$$
$$\times J_1\left\{\alpha\bar{r}\left(\frac{r}{\bar{r}}\right)\right\}\left(\frac{r_m}{\bar{r}}\right) \frac{e^{-(\bar{r}\alpha)\left(\frac{z}{\bar{r}}\right)}}{\bar{r}^2\alpha^2} \left\{e^{\bar{r}\alpha\left(\frac{\ell_n}{\bar{r}}\right)} - e^{-\alpha\bar{r}\left(\frac{\ell_n}{\bar{r}}\right)}\right.$$
$$\left.\times\left[\frac{(\bar{r}\alpha+\bar{r}\alpha_1)(\bar{r}\alpha_1-\bar{r}\alpha_2)+(\bar{r}\alpha-\bar{r}\alpha_1)(\bar{r}\alpha_2+\bar{r}\alpha_1)\ e^{2\bar{r}\alpha_1\left(\frac{c}{\bar{r}}\right)}}{(\bar{r}\alpha-\bar{r}\alpha_1)(\bar{r}\alpha_1-\bar{r}\alpha_2)+(\bar{r}\alpha+\bar{r}\alpha_1)(\bar{r}\alpha_2+\bar{r}\alpha_1)e^{2\bar{r}\alpha_1\left(\frac{c}{\bar{r}}\right)}}\right]\right\} d(\bar{r}\alpha) \quad . \tag{3.48}$$

We can now redefine the expressions in equation (3.48): $\bar{r}\alpha \equiv \alpha$, $\frac{z}{\bar{r}} \equiv z$, etc. We shall now consider all dimensions normalized in terms of $\bar{r}$ (except $\bar{r}$). Equation (3.48) then becomes:

$$F_1(r,\ r_m,\ z,\ \ell_n,\ M_1,\ M_2) = \mu I\bar{r}^2 \sum_{k=0}^{\infty} \frac{2(k+1)}{(2k+1)(2k+3)} \int_0^\infty J_{2k+2}(\alpha r_m) J_1(\alpha r) \frac{r_m}{\alpha^2}\, e^{-\alpha z}$$
$$\times\left\{e^{\alpha\ell_n} - e^{-\alpha\ell_n}\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2)+(\alpha-\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}\right]\right\} d\alpha \quad . \tag{3.49}$$

Now we have for equation (3.45, page 17):

$$A^{(1)}(r,z) = F_1(r,\ r_2,\ z,\ \ell_2,\ M_1,\ M_2) - F_1(r,\ r_2,\ z,\ \ell_1,\ M_1,\ M_2)$$
$$- F_1(r,\ r_1,\ z,\ \ell_2,\ M_1,\ M_2) + F_1(r,\ r_1,\ z,\ \ell_1,\ M_1,\ M_2) \quad . \qquad (3.50)$$

In general we have:

$$A^{(i)}(r,z) = F_i(r,\ r_2,\ z,\ \ell_2,\ M_1,\ M_2) - F_i(r,\ r_2,\ z,\ \ell_1,\ M_1,\ M_2)$$
$$- F_i(r,\ r_1,\ z,\ \ell_2,\ M_1,\ M_2) + F_i(r,\ r_1,\ z,\ \ell_1,\ M_1,\ M_2) \quad .$$

In region II, i=2, and $F_2(r,\ r_m,\ z,\ \ell_n,\ M_1,\ M_2)$ is given by:

$$F_2(r,\ r_m,\ z,\ \ell_n,\ M_1,\ M_2) = -\mu I\bar{r}^2 \sum_{k=0}^{\infty} \frac{2(k+1)}{(2k+1)(2k+3)} \int_0^{\infty} J_{2k+2}\,(\alpha r_m) J_1(\alpha r)$$

$$\times\ e^{-\alpha\ell_n} \frac{r_m}{\alpha^2} \left\{ e^{\alpha z} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}} \right] e^{-\alpha z} \right\} d\alpha \ . \qquad (3.51)$$

In region III, i=3, and $F_3(r,\ r_m,\ z,\ \ell_n,\ M_1,\ M_2)$ is given by:

$$F_3(r,\ r_m,\ z,\ \ell_n,\ M_1,\ M_2) = -\mu I\bar{r}^2 \sum_{k=0}^{\infty} \frac{4(k+1)}{(2k+1)(2k+3)} \int_0^{\infty} J_{2k+2}\,(\alpha r_m) J_1(\alpha r)$$

$$\times\ e^{-\alpha\ell_n} \frac{r_m}{\alpha} \left\{ \frac{(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}\ e^{\alpha_1 z} + (\alpha_1-\alpha_2)\ e^{-\alpha_1 z}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}} \right\} d\alpha \quad . \qquad (3.52)$$

In region IV, i=4, and $A^{(4)}(r,z)$ is given in terms of the functions:

$$F_4(r,\ r_m,\ z,\ \ell_n,\ M_1,\ M_2) = -\mu I\bar{r}^2 \sum_{k=0}^{\infty} \frac{4(k+1)}{(2k+1)(2k+3)} \int_c^{\infty} J_{2k+2}(\alpha r_m)$$

$$\times\ J_1(\alpha r)\ e^{-\alpha\ell_n} \frac{r_m}{\alpha} \left\{ \frac{2\alpha_1\ e^{(\alpha_2+\alpha_1)c}\ e^{\alpha_2 z}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}} \right\} d\alpha \quad . \qquad (3.53)$$

In region I-II, between the top and bottom of the coil, we have to break the problem into two parts. We shall use equation (3.50) to determine the vector potential due to the part of the coil below the point of interest and equation (3.51) to determine the vector potential due to the part of the coil above the point. The total vector potential is the sum of the two:

$$\begin{aligned}A^{(12)}(r,z) = {} & F_1(r,\ r_2,\ z,\ \ell_2{=}z,\ M_1,\ M_2) - F_1(r,\ r_1,\ z,\ \ell_2{=}z,\ M_1,\ M_2) \\ & - F_1(r,\ r_2,\ z,\ \ell_1,\ M_1,\ M_2) + F_1(r,\ r_1,\ z,\ \ell_1,\ M_1,\ M_2) \\ & + F_2(r,\ r_2,\ z,\ \ell_2,\ M_1,\ M_2) - F_2(r,\ r_1,\ z,\ \ell_2,\ M_1,\ M_2) \\ & - F_2(r,\ r_2,\ z,\ \ell_2{=}z,\ M_1,\ M_2) + F_2(r,\ r_1,\ z,\ \ell_1{=}z,\ M_1,\ M_2) \quad . \quad (3.54)\end{aligned}$$

Thus, in principle, we have determined the vector potential at any point in space for a coil above any two plane conductors. However, there still remains the task of evaluating the integral equations (3.49, page 18), (3.51), (3.52), and (3.53). In order to generate a table with enough values of the F functions to be really useful in practice, we would have to evaluate the integral equation about 10^{10} times. This would require a computer program and is left for a later date.

These equations will reduce somewhat for more simple geometrical configurations. For instance, if we let the conductivities in the two metals be the same, we get the case of a coil above a single conductor:

$$F_1(r,\ r_m,\ z,\ \ell_n,\ M_2) = \mu I \bar{r}^2 \sum_{k=0}^{\infty} \frac{2(k+1)}{(2k+1)(2k+3)}$$

$$\times \int_0^\infty J_{2k+2}(\alpha r_m) J_1(\alpha r)\, \frac{r_m}{\alpha^2}\, e^{-\alpha z} \left\{ e^{\alpha \ell_n} - e^{-\alpha \ell_n} \frac{\alpha-\alpha_2}{\alpha+\alpha_2} \right\} d\alpha \ . \qquad (3.55)$$

$$F_2(r,\ r_m,\ z,\ \ell_n,\ M_2) = -\,\mu I\bar{r}^2 \sum_{k=0}^{\infty} \frac{2(k+1)}{(2k+1)(2k+3)}$$

$$\times \int_0^{\infty} J_{2k+2}(\alpha r_m) J_1(\alpha r)\ e^{-\alpha \ell_n}\ \frac{r_m}{\alpha^2} \left\{ e^{\alpha z} + e^{-\alpha z}\, \frac{\alpha-\alpha_2}{\alpha+\alpha_2} \right\} d\alpha\ . \qquad (3.56)$$

$$F_3(r,\ r_m,\ z,\ \ell_n,\ M_2) = F_4(r,\ r_m,\ z,\ \ell_n,\ M_2) = -\,\mu I\bar{r}^2 \sum_{k=0}^{\infty} \frac{4(k+1)}{(2k+1)(2k+3)}$$

$$\times \int_0^{\infty} J_{2k+2}(\alpha r_m) J_1(\alpha r)\ \frac{r_m\ e^{-\alpha \ell_n}\ e^{\alpha_2 z}}{\alpha\ (\alpha+\alpha_2)}\ d\alpha\ . \qquad (3.57)$$

If, instead, we let c, the thickness of the metal in region III, go to zero rather than let $\alpha_1 = \alpha_2$, we obtain exactly the same equations for F_1, F_2, and F_4. The value for F_3 is different, but it is for a region that no longer exists. We also get similar solutions if we let c approach infinity. The only difference in this case is that the conductivity is in terms of σ_1 instead of σ_2, and the vector potential in region IV vanishes.

We shall now consider the special case where the second conductor becomes an insulator, that is, $\sigma_2 = 0$ and $\alpha_2 = \alpha$. This gives the case of a finite cross-section coil above a plane sheet of finite thickness:

$$F_1(r,\ r_m,\ z,\ \ell_n,\ M_1) = +\,\mu I\bar{r}^2 \sum_{k=0}^{\infty} \frac{2(k+1)}{(2k+1)(2k+3)} \int_0^{\infty} J_{2k+2}(\alpha r_m) J_1(\alpha r)\ \frac{r_m}{\alpha^2}\ e^{-\alpha z}$$

$$\times \left\{ e^{\alpha \ell_n} - e^{-\alpha \ell_n} \left[\frac{-\,(\alpha+\alpha_1)(\alpha-\alpha_1) + (\alpha+\alpha_1)(\alpha-\alpha_1)\ e^{2\alpha_1 c}}{-\,(\alpha-\alpha_1)^2 + (\alpha+\alpha_1)^2\ e^{2\alpha_1 c}} \right] \right\} d\alpha\ . \qquad (3.58)$$

$$F_2(r, r_m, z, \ell_n, M_1) = -\mu I\overline{r}^2 \sum_{k=o}^{\infty} \frac{2(k+1)}{(2k+1)(2k+3)} \int_o^{\infty} J_{2k+2}(\alpha r_m) J_1(\alpha r) \frac{r_m}{\alpha^2} e^{-\alpha \ell_n}$$
$$\times \left\{ e^{\alpha z} + e^{-\alpha z} \left[\frac{-(\alpha+\alpha_1)(\alpha-\alpha_1) + (\alpha+\alpha_1)(\alpha-\alpha_1)\ e^{2\alpha_1 c}}{-(\alpha-\alpha_1)^2 + (\alpha+\alpha_1)^2\ e^{2\alpha_1 c}} \right] \right\} d\alpha \ , \tag{3.59}$$

$$F_3(r, r_m, z, \ell_n, M_1) = -\mu I\overline{r}^2 \sum_{k=o}^{\infty} \frac{4(k+1)}{(2k+1)(2k+3)} \int_o^{\infty} J_{2k+2}(\alpha r_m) J_1(\alpha r) \frac{r_m}{\alpha} e^{-\alpha \ell_n}$$
$$\times \left[\frac{(\alpha+\alpha_1)\ e^{2\alpha_1 c}\ e^{\alpha_1 z} + (\alpha_1-\alpha)\ e^{-\alpha_1 z}}{-(\alpha-\alpha_1)^2 + (\alpha+\alpha_1)^2\ e^{2\alpha_1 c}} \right] d\alpha \ , \tag{3.60}$$

$$F_4(r, r_m, z, \ell_n, M_1) = -\mu I\overline{r}^2 \sum_{k=o}^{\infty} \frac{4(k+1)}{(2k+1)(2k+3)} \int_o^{\infty} J_{2k+2}(\alpha r_m) J_1(\alpha r) \frac{r_m}{\alpha} e^{-\alpha \ell_n}$$
$$\times \left[\frac{2\alpha_1\ e^{(\alpha+\alpha_1)c}\ e^{\alpha z}}{-(\alpha-\alpha_1)^2 + (\alpha+\alpha_1)^2\ e^{2\alpha_1 c}} \right] d\alpha \ . \tag{3.61}$$

Considerable simplification results if we specialize to the case of a single delta function coil in a nonconducting medium. We can let $\sigma_1 = \sigma_2 = o$ and obtain the following reduction for equation (3.37, page 14):

$$A^{(1)}(r,z) = \frac{\mu I r_o}{2} \int_o^{\infty} J_1(\alpha r_o) J_1(\alpha r)\ e^{-\alpha(z-\ell)}\ d\alpha \ . \tag{3.62}$$

If we take ℓ=o, we get the same equation for the vector potential as given in equation (8-54) of Panofsky and Phillips (1956). We also get the same equation if we shift the origin to the coil and move the metal away to infinity.

If we let the conductivity in equation (3.37) approach infinity, we obtain:

$$A^{(1)}(r,z) = \frac{\mu I r_o}{2} \int_o^{\infty} J_1(\alpha r_o) J_1(\alpha r)\ e^{-\alpha z} \left(e^{\alpha \ell} - e^{-\alpha \ell} \right) d\alpha \ . \tag{3.63}$$

This is the equation for a coil at $z=+\ell$ and an image at $z=-\ell$, 180 degrees out of phase. Equation (3.38, page 15) likewise reduces to a coil and its image when the conductivity approaches infinity:

$$A^{(2)}(r,z) = \frac{\mu I r_0}{2} \int_0^\infty J_1(\alpha r_0) J_1(\alpha r) \left[e^{-\alpha(\ell-z)} - e^{-\alpha(\ell+z)} \right] d\alpha \quad . \tag{3.64}$$

The vector potential given by equation (3.64) goes to zero at the face of the metal, as it should. The vector potential inside the metal, as given by equations (3.39) and (3.40, page 15) also goes to zero, as it should, when the conductivity is infinite.

Coil Encircling a Two Conductor Rod

We shall assume a delta function coil encircling an infinitely long, two conductor rod, as shown in Figure 3.

The general differential equation is the same as equation (3.4, page 11) in the case of a coil above a conducting plane.

$$\frac{1}{R(r)} \frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{rR(r)} \frac{\partial R(r)}{\partial r} + \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} - \frac{1}{r^2} - j\omega\mu\sigma = 0 \quad . \tag{3.65}$$

Now, however, we shall assume the separation constant to be negative:

$$\frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} = \text{constant} = -\alpha^2 \quad . \tag{3.66}$$

Then

$$Z(z) = F \sin\alpha(z-z_0) + G \cos\alpha(z-z_0) \quad , \tag{3.67}$$

and equation (3.65) becomes:

$$r^2 \frac{\partial^2 R(r)}{\partial r^2} + \frac{r\partial R(r)}{\partial r} - \left[(\alpha^2 + j\omega\mu\sigma) r^2 + 1 \right] R(r) = 0 \quad . \tag{3.68}$$

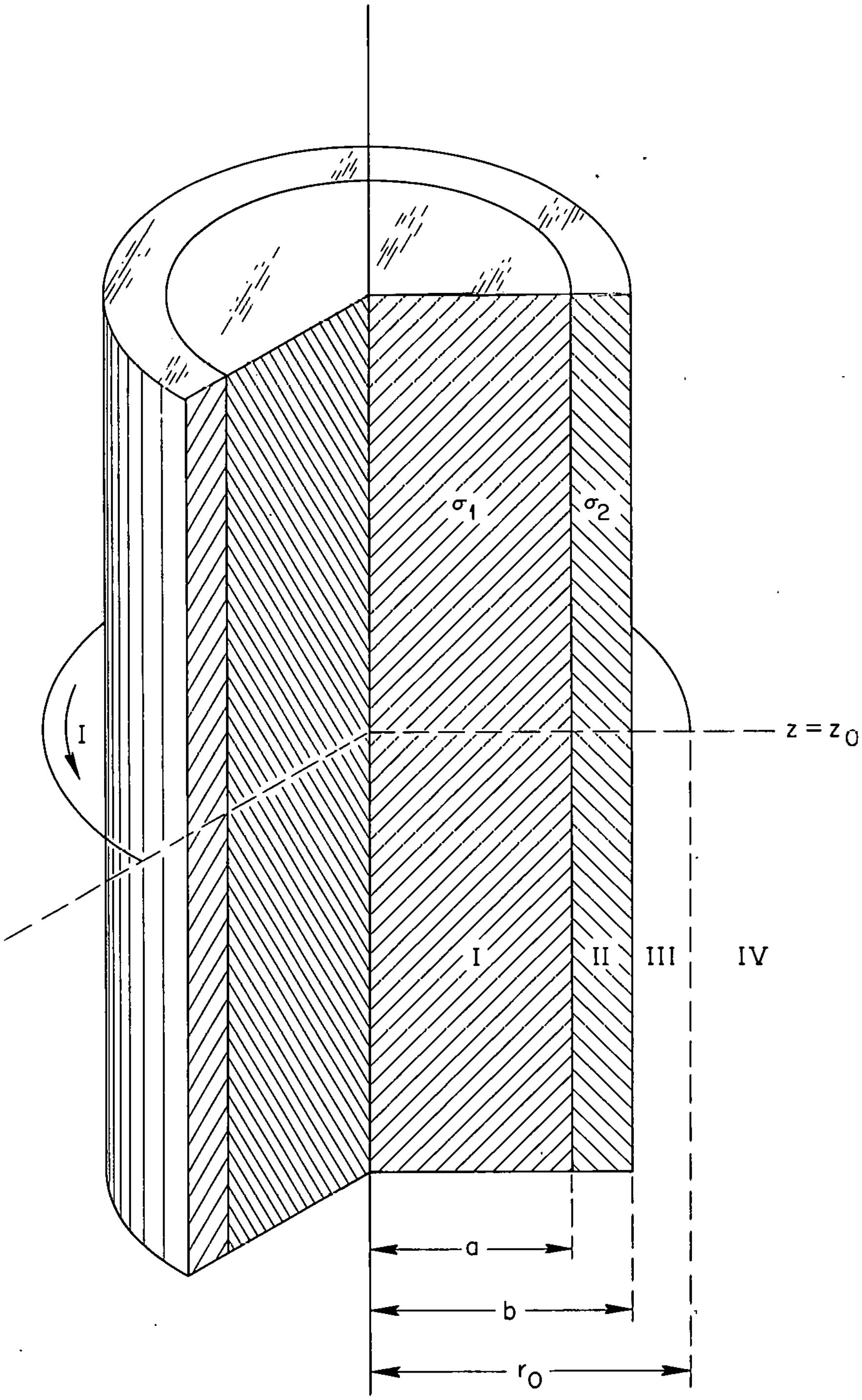

Figure 3. Delta function coil encircling a two conductor rod.

The solution to equation (3.68, page 23) in terms of modified Bessel functions is:

$$R(r) = CI_1\{(\alpha^2 + j\omega\mu\sigma)^{\frac{1}{2}}r\} + DK_1\{(\alpha^2 + j\omega\mu\sigma)^{\frac{1}{2}}r\} \quad . \tag{3.69}$$

We can now write the vector potential in each region. We shall use the fact that it is symmetric (with respect to $z-z_o$) to throw out the sine terms and the fact that $K_1(o)$ and $I_1(\infty)$ both diverge to eliminate their coefficients in regions I and IV, respectively. Thus we have:

$$A^{(1)}(r,z-z_o) = \int_o^\infty C_1(\alpha)I_1\{(\alpha^2 + j\omega\mu\sigma_1)^{\frac{1}{2}}r\} \cos\alpha(z-z_o)\, d\alpha \quad , \tag{3.70}$$

$$A^{(2)}(r,z-z_o) = \int_o^\infty \Big[C_2(\alpha)I_1\{(\alpha^2 + j\omega\mu\sigma_2)^{\frac{1}{2}}r\} + D_2(\alpha)K_1\{(\alpha^2 + j\omega\mu\sigma_2)^{\frac{1}{2}}r\}\Big] \cos\alpha(z-z_o)\, d\alpha \quad , \tag{3.71}$$

$$A^{(3)}(r,z-z_o) = \int_o^\infty [C_3(\alpha)I_1(\alpha r) + D_3(\alpha)K_1(\alpha r)] \cos\alpha(z-z_o)\, d\alpha \quad , \tag{3.72}$$

$$A^{(4)}(r,z-z_o) = \int_o^\infty D_4(\alpha)K_1(\alpha r) \cos\alpha(z-z_o)\, d\alpha \quad . \tag{3.73}$$

The boundary conditions between the different regions are:

$$A^{(1)}(a,z-z_o) = A^{(2)}(a,z-z_o) \quad , \tag{3.74}$$

$$\frac{\partial}{\partial r} A^{(1)}(r,z-z_o)\Big]_{r=a} = \frac{\partial}{\partial r} A^{(2)}(r,z-z_o)\Big]_{r=a} \quad ; \tag{3.75}$$

$$A^{(2)}(b,z-z_o) = A^{(3)}(b,z-z_o) \quad , \tag{3.76}$$

$$\frac{\partial}{\partial r} A^{(2)}(r,z-z_o)\Big]_{r=b} = \frac{\partial}{\partial r} A^{(3)}(r,z-z_o)\Big]_{r=b} \quad , \tag{3.77}$$

$$A^{(3)}(r_o,z-z_o) = A^{(4)}(r_o,z-z_o) \quad , \tag{3.78}$$

$$\frac{\partial}{\partial r} A^{(3)}(r,z-z_o)\Big]_{r=r_o} = \frac{\partial}{\partial r} A^{(4)}(r,z-z_o)\Big]_{r=r_o} + \mu I\delta(z-z_o) \quad . \tag{3.79}$$

If we multiply both sides of equation (3.74) by $\cos\alpha'(z-z_o)$ and integrate from zero to infinity, we obtain:

$$\int_0^\infty\int_0^\infty C_1(\alpha)I_1\{(\alpha^2 + j\omega\mu\sigma_1)^{\frac{1}{2}}r\} \cos\alpha(z-z_o) \cos\alpha'(z-z_o)\, d\alpha$$

$$= \int_0^\infty\int_0^\infty[C_2(\alpha)I_1\{\alpha^2 + j\omega\mu\sigma_2)^{\frac{1}{2}}r\} + D_2(\alpha)K_1\{(\alpha^2 + j\omega\mu\sigma_2)^{\frac{1}{2}}r\}]$$

$$\times [\cos\alpha(z-z_o) \cos\alpha'(z-z_o)]\, d\alpha\, d(z-z_o) \quad . \tag{3.80}$$

We can reverse the order of integration and use the orthogonal properties of the cosine integral or use the Fourier integral theorem:

$$\frac{1}{\pi}\int_0^\infty f(\alpha)\{\int_0^\infty \cos\alpha(z-z_o) \cos\alpha'(z-z_o)\, d(z-z_o)\}\, d\alpha = f(\alpha') \tag{3.81}$$

Thus, we can solve the integral equations (3.74) through (3.79). We shall use α_1 and α_2 to designate $(\alpha^2 + j\omega\mu\sigma_1)^{\frac{1}{2}}$ and $(\alpha^2 + j\omega\mu\sigma_2)^{\frac{1}{2}}$. We shall use primes to designate derivatives with respect to the argument. We get from the integral equations (3.74) through (3.79):

$$C_1I_1(\alpha_1 a) = C_2I_1(\alpha_2 a) + D_2K_1(\alpha_2 a) \quad , \tag{3.82}$$

$$C_1\alpha_1I_1'(\alpha_1 a) = C_2\alpha_2I_1'(\alpha_2 a) + D_2\alpha_2K_1'(\alpha_2 a) \quad , \tag{3.83}$$

$$C_2I_1(\alpha_2 b) + D_2K_1(\alpha_2 b) = C_3I_1(\alpha b) + D_3K_1(\alpha b) \quad , \tag{3.84}$$

$$C_2\alpha_2I_1'(\alpha_2 b) + D_2\alpha_2K_1'(\alpha_2 b) = C_3\alpha I_1'(\alpha b) + D_3\alpha K_1'(\alpha b) \quad , \tag{3.85}$$

$$C_3 I_1(\alpha r_o) + D_3 K_1(\alpha r_o) = D_4 K_1(\alpha r_o) \quad , \tag{3.86}$$

$$C_3 \alpha I_1'(\alpha r_o) + D_3 \alpha K_1'(\alpha r_o) = D_4 \alpha K_1'(\alpha r_o) + \frac{\mu I}{\pi} \quad . \tag{3.87}$$

Now we have six equations with six unknown constants. The equations may be solved to give the constants. We shall define:

$$D \equiv [\alpha_2 K_o(\alpha_2 b) K_1(\alpha b) - \alpha K_o(\alpha b) K_1(\alpha_2 b)][\alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a) - \alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a)]$$
$$+ [\alpha_2 K_o(\alpha_2 a) I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a) I_o(\alpha_1 a)][\alpha I_1(\alpha_2 b) K_o(\alpha b) + \alpha_2 I_o(\alpha_2 b) K_1(\alpha b)] \quad . \tag{3.88}$$

The constants are:

$$C_1 = \frac{\mu \; r_o \; I \; K_1(\alpha r_o)}{a \; b \; \pi \; D} \tag{3.89}$$

$$D_2 = \frac{\mu \; r_o \; I \; K_1(\alpha r_o)}{b \; \pi \; D} \left[(\alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a) \right] \quad , \tag{3.90}$$

$$C_2 = \frac{\mu \; I \; r_o \; K_1(\alpha r_o)}{b \; \pi \; D} \left[\alpha_2 K_o(\alpha_2 a) I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a) I_o(\alpha_1 a) \right] \quad , \tag{3.91}$$

$$C_3 = \frac{\mu \; I \; r_o \; K_1(\alpha r_o)}{\pi} \quad , \tag{3.92}$$

$$D_3 = - \frac{\mu \; r_o \; I}{\pi} \frac{K_1(\alpha r_o)}{K_1(\alpha b)} \left\{ \frac{K_1(\alpha_2 b)}{bD} \left[\alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a) - \alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a) \right] \right.$$
$$\left. - \frac{I_1(\alpha_2 b)}{bD} \left[\alpha_2 K_o(\alpha_2 a) I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a) I_o(\alpha_1 a) \right] + I_1(\alpha b) \right\} \quad . \tag{3.93}$$

$$D_4 = \frac{\mu \; I \; r_o \; K_1(\alpha r_o)}{\pi} \left\{ \frac{K_1(\alpha_2 b)[\alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a)]}{K_1(\alpha b) \; bD} \right.$$
$$\left. + \frac{I_1(\alpha_2 b)}{K_1(\alpha b) \; bD} \left[\alpha_2 I_1(\alpha_1 a) K_o(\alpha_2 a) + \alpha_1 K_1(\alpha_2 a) I_o(\alpha_1 a) \right] - \frac{I_1(\alpha b)}{K_1(\alpha b)} + \frac{I_1(\alpha r_o)}{K_1(\alpha r_o)} \right\} \quad . \tag{3.94}$$

We can now write for the vector potential in each region:

$$A^{(1)}(r,z-z_o) = \frac{\mu I}{\pi} \int_o^\infty \frac{r_o}{ab} \frac{K_1(\alpha r_o)}{D} I_1(\alpha_1 r) \cos\alpha(z-z_o)\, d\alpha \quad . \tag{3.95}$$

$$A^{(2)}(r,z-z_o) = \frac{\mu I}{\pi} \int_o^\infty \frac{r_o K_1(\alpha r_o)}{bD} \left\{\left[(\alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a)\right] K_1(\alpha_2 r)\right.$$

$$\left. + \left[\alpha_2 K_o(\alpha_2 a) I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a) I_o(\alpha_1 a)\right] I_1(\alpha_2 r)\right\} \cos\alpha(z-z_o)\, d\alpha \quad . \tag{3.96}$$

$$A^{(3)}(r,z-z_o) = \frac{\mu I}{\pi} \int_o^\infty r_o K_1(\alpha r_o) \left\{I_1(\alpha r) - \left[\frac{K_1(\alpha_2 b)}{bD\, K_1(\alpha b)} \Big(\alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a)\right.\right.$$

$$- \alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a)\Big) - \frac{I_1(\alpha_2 b)}{bD\, K_1(\alpha b)} \Big(\alpha_2 K_o(\alpha_2 a) I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a) I_o(\alpha_1 a)\Big)$$

$$\left.\left. + \frac{I_1(\alpha b)}{K_1(\alpha b)}\right] K_1(\alpha r)\right\} \cos\alpha(z-z_o)\, d\alpha \quad . \tag{3.97}$$

$$A^{(4)} = \frac{\mu I}{\pi} \int_o^\infty r_o K_1(\alpha r_o) K_1(\alpha r) \left\{\frac{K_1(\alpha_2 b)[\alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a)]}{K_1(\alpha b)\, bD}\right.$$

$$+ \frac{I_1(\alpha_2 b)}{K_1(\alpha b)\, bD} \left[\alpha_2 I_1(\alpha_1 a) K_o(\alpha_2 a) + \alpha_1 K_1(\alpha_2 a) I_o(\alpha_1 a)\right]$$

$$\left. - \frac{I_1(\alpha b)}{K_1(\alpha b)} + \frac{I_1(\alpha r_o)}{K_1(\alpha r_o)}\right\} \cos\alpha(z-z_o)\, d\alpha \quad . \tag{3.98}$$

Equations (3.95) through (3.98) are the equations for the vector potential of a delta function coil located at $r=r_o$ and $z=z_o$. If we make the assumption of a rectangular cross-section coil (Figure 4) with a uniform current distribution, we can write for the superimposed solutions:

$$A(r,z)(\text{total}) \approx \int_{\ell_1}^{\ell_2} \int_{r_1}^{r_2} A(r,\ z,\ r_o,\ z_o)\, dr_o\, dz_o \quad . \tag{3.99}$$

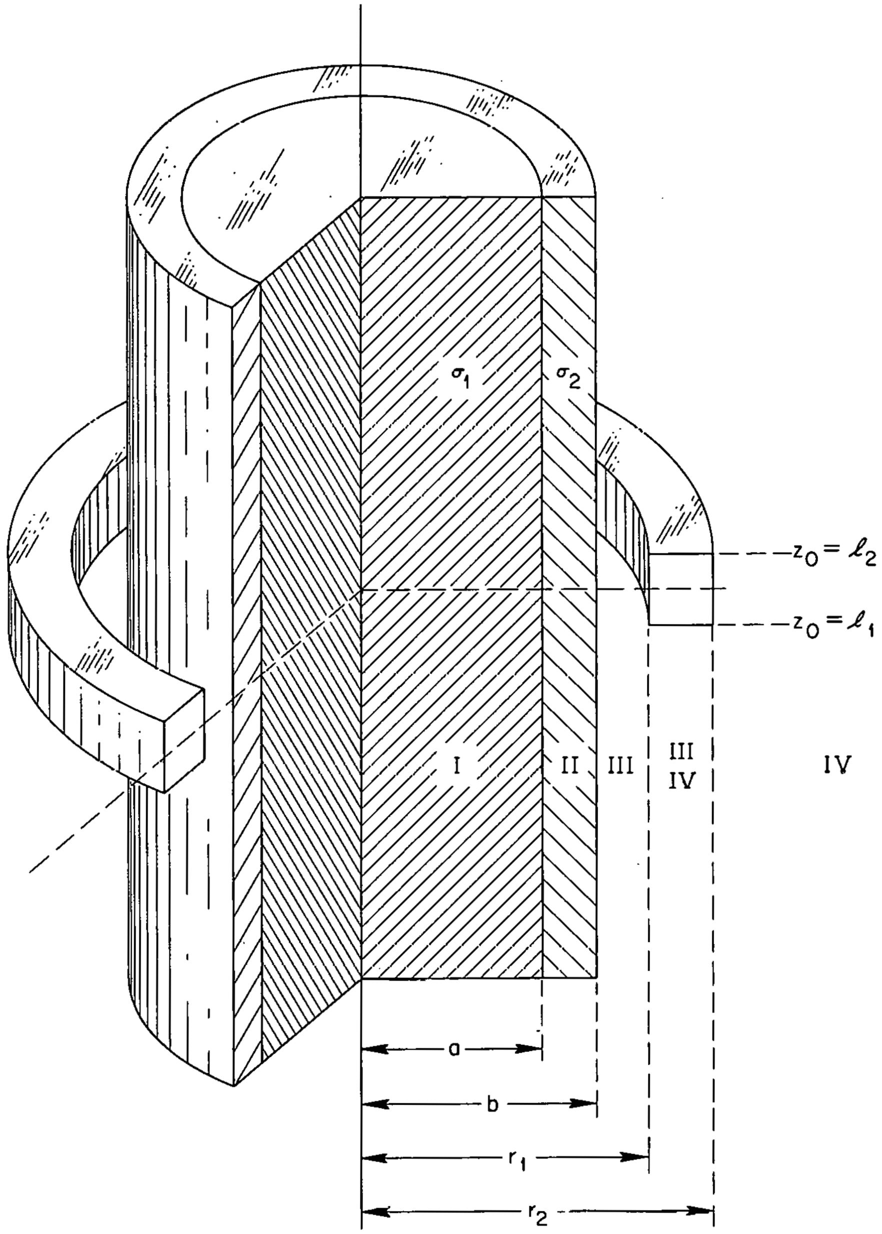

Figure 4. Rectangular cross-section coil encircling a two conductor rod.

Since the integrals of $I_1(\alpha r_o)$ and $K_1(\alpha r_o)$ do not reduce to simple forms, we shall not write equations (3.95 through 3.98, page 28) after the integrations over the coil have been made.

In order for the closed form solutions to be really useful, the final integral equations must be evaluated, either by a computer integration or preferably by an approximation in terms of simple functions. If the latter can be found, it will allow these problems to be solved readily with manual calculations. The evaluation of these integrals is left to future work.

We can effect some reductions in equations (3.95 through 3.99, page 28) if we specialize to the case of a single conductor. If we let the conductivities in the two metals be the same we get:

$$A^{(1)} = A^{(2)} = \frac{\mu I}{\pi} \int_0^{\infty} \frac{r_o}{b} \frac{K_1(\alpha r_o) I_1(\alpha_2 r) \cos\alpha(z-z_o)}{[\alpha I_1(\alpha_2 b) K_o(\alpha b) + \alpha_2 I_o(\alpha_2 b) K_1(\alpha b)]} d\alpha \,, \tag{3.100}$$

$$A^{(3)} = \frac{\mu I}{\pi} \int_0^{\infty} r_o K_1(\alpha r_o) \Bigg\{ I_1(\alpha r) + \left[\frac{I_1(\alpha_2 b)}{bK_1(\alpha b)[\alpha I_1(\alpha_2 b) K_o(\alpha b) + \alpha_2 I_o(\alpha_2 b) K_1(\alpha b)]} - \frac{I_1(\alpha b)}{K_1(\alpha b)} \right] K_1(\alpha r) \Bigg\} \cos\alpha(z-z_o) d\alpha \,, \tag{3.101}$$

$$A^{(4)} = \frac{\mu I}{\pi} \int_0^{\infty} r_o K_1(\alpha r_o) K_1(\alpha r) \left\{ \frac{I_1(\alpha_2 b)}{bK_1(\alpha b)[\alpha I_1(\alpha_2 b) K_o(\alpha b) + \alpha_2 I_o(\alpha_2 b) K_1(\alpha b)]} - \frac{I_1(\alpha b)}{K_1(\alpha b)} + \frac{I_1(\alpha r_o)}{K_1(\alpha r_o)} \right\} \cos\alpha(z-z_o) \, d\alpha \,. \tag{3.102}$$

We get the same equations if we let a=b, with the exception of $A^{(2)}$, which is the vector potential for a region which no longer exists. We also get the same equations if we let a approach zero.

We can also let the conductivity approach zero, or $\alpha_2 = \alpha_1 = \alpha$. Then

$$A^{(1)} = A^{(2)} = A^{(3)} = \frac{\mu I}{\pi} \int_0^\infty r_o K_1(\alpha r_o) I_1(\alpha r) \cos\alpha(z-z_o)\, d\alpha \quad , \tag{3.103}$$

$$A^{(4)} = \frac{\mu I}{\pi} \int_0^\infty r_o I_1(\alpha r_o) K_1(\alpha r) \cos\alpha(z-z_o)\, d\alpha \quad . \tag{3.104}$$

These equations are the same as those given in equations (8-51) of Panofsky and Phillips (1956) for the vector potential of a delta function coil in a nonconducting medium. This integral is equivalent to the earlier form of the vector potential for the coil in air [equation (3.62, page 22)] with ℓ=o. Both of these equations can be evaluated in terms of elliptic integrals according to equation (13.4.1.17) in integrals of Bessel functions by Y. L. Luke (1962):

$$\frac{\mu I r_o}{2} \int_0^\infty J_1(\alpha r_o) J_1(\alpha r)\, e^{-\alpha z}\, d\alpha = \frac{\mu I r_o}{\pi} \int_0^\infty K_1(\alpha r_o) I_1(\alpha r) \cos\alpha z\, d\alpha$$

$$= \frac{\mu I r_o}{\pi \sqrt{\dfrac{4r^2 r_o^2}{z^2 + (r+r_o)^2}}} \left\{ \left(1 - \frac{2r\, r_o}{z^2 + (r_o+r)^2}\right) K\left(\sqrt{\frac{4r\, r_o}{z^2 + (r+r_o)^2}}\right) - E\left(\sqrt{\frac{4r\, r_o}{z^2 + (r+r_o)^2}}\right) \right\} , \tag{3.105}$$

where K(k) and E(k) are the complete elliptic integrals of the first and second kinds, respectively.

CHAPTER IV

RELAXATION SOLUTION OF THE VECTOR POTENTIAL

We shall now solve the differential equation for the vector potential [equation (2.19, page 9)] by a relaxation technique. We shall first assume a linear, isotropic, inhomogeneous medium with a sinusoidal driving current [Dodd (1965)].

Linear Medium and Sinusoidal Driving Current

The individual terms in equation (2.19) may be written in finite difference terms. The first derivative of a function f(x) at a point x may be approximated by values of the function on either side of x, as shown in Figure 5(a).

$$\left(\frac{\partial f}{\partial x}\right)_x = \frac{f(x+a) - f(x-a)}{2a} \quad . \tag{4.1}$$

Figure 5(b) shows how the second derivative may be calculated by first obtaining the derivative at points $x + \frac{a}{2}$ and $x - \frac{a}{2}$. From Figure 5(b):

$$\left(\frac{\partial f}{\partial x}\right)_{x+\frac{a}{2}} = \frac{f(x+a) - f(x)}{a} \quad , \tag{4.2}$$

$$\left(\frac{\partial f}{\partial x}\right)_{x-\frac{a}{2}} = \frac{f(x) - f(x-a)}{a} \quad . \tag{4.3}$$

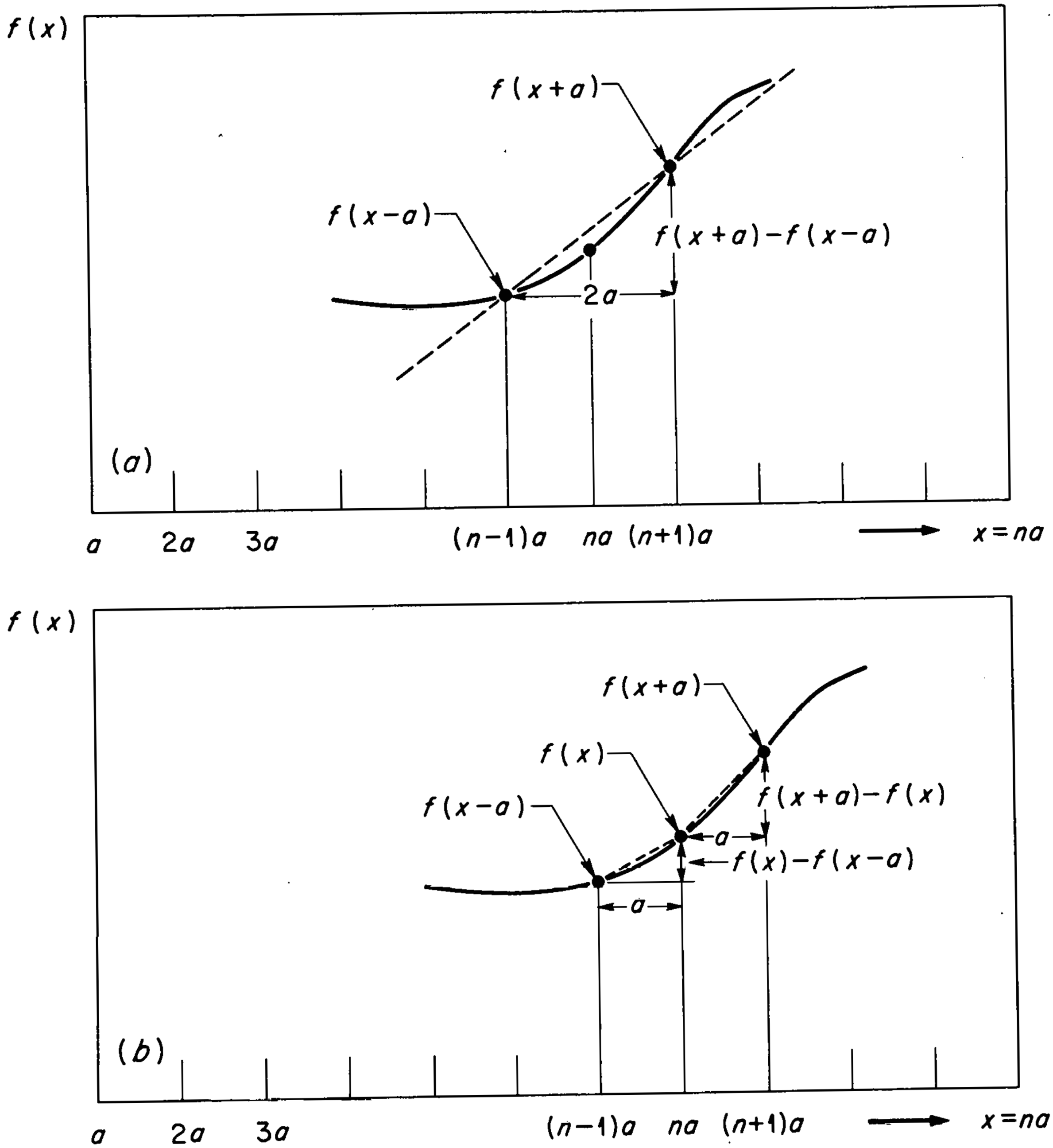

Figure 5. Derivative Approximations. (a) The first derivative approximation and (b) the second derivative approximation.

The second derivative may be calculated:

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_x = \frac{\left(\frac{\partial f}{\partial x}\right)_{x+\frac{a}{2}} - \left(\frac{\partial f}{\partial x}\right)_{x-\frac{a}{2}}}{a} = \frac{f(x+a) - f(x) - f(x) + f(x-a)}{a^2}$$

$$= \frac{f(x+a) + f(x-a) - 2f(x)}{a^2} \quad . \qquad (4.4)$$

These are only approximations and are good for "a" so small that the change in f(x) is small from x to x±a. This condition is fulfilled with the exception of permeability variations, which require special treatment. Figure 6 shows how μ, $\partial A/\partial x$, and $\partial/\partial x(1/\mu)$, and A vary in one direction. For this type of function, it is more accurate to represent the derivatives by a one-sided difference equation. The difference equations are:

$$\left(\frac{\partial A}{\partial x}\right)_x = \frac{A_{x+a} - A_x}{a} \quad , \qquad (4.5)$$

$$\left(\frac{1}{r}\frac{\partial rA}{\partial r}\right)_r = \frac{(r+a)\ A_{r+a} - rA_r}{ra} \quad , \qquad (4.6)$$

$$\left[\frac{\partial}{\partial x}\left(\frac{1}{\mu}\right)\right]_x = \frac{\frac{1}{\mu_{x+a}} - \frac{1}{\mu_x}}{a} \quad . \qquad (4.7)$$

The function at any point may be represented:

$$(f)_x = f_x \quad . \qquad (4.8)$$

In this manner the various terms in equation (2.19, page 9) may be approximated, using equations (4.5), (4.6), and (4.7) for the first derivatives and equation (4.4) for the second derivatives. Solving

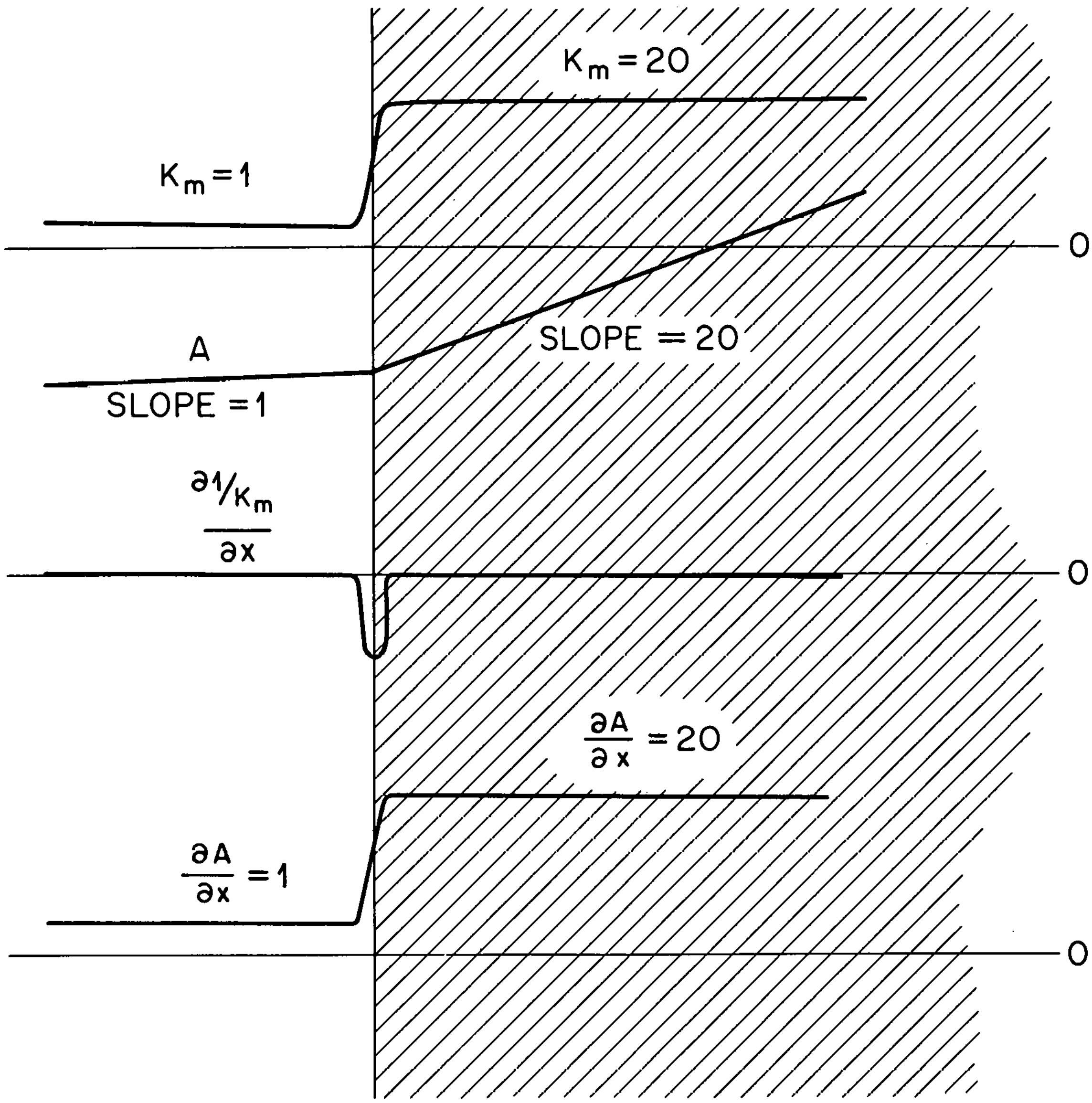

Figure 6. Parameter variations across a boundary of a magnetic material.

equation (2.19, page 9) for the vector potential at any point $A_{r,z}$, in terms of the vector potential at the four nearest neighbors, we obtain:

$$A_{r,z} = \left\{ (1+a/r)\left(\frac{\mu_{r,z}}{\mu_{r+a,z}}\right) A_{r+a,z} + A_{r-a,z} + \frac{\mu_{r,z}}{\mu_{r,z+a}} A_{r,z+a} + A_{r,z-a} + a^2\mu_{r,z}J_{r,z} \right\} \div \left\{ 2 + \frac{a}{r} + \frac{a^2}{r^2} + \frac{\mu_{r,z}}{\mu_{r+a,z}} + \frac{\mu_{r,z}}{\mu_{r,z+a}} + ja^2\omega\mu_{r,z}\sigma_{r,z} \right\} \quad , \quad (4.9)$$

where $J_{r,z}$ is the applied current density at the point (r,z) in the coil.

Application of Technique

In solving for this potential, the problem is first laid out in a two-dimensional mesh of points, which have a specified value of $J_{r,z}$, $\mu_{r,z}$, $\sigma_{r,z}$ at each point, r and z. It is sufficient to work the problem in one-half plane only, due to the axial symmetry, as shown in Figure 7.

Equation (4.9) will simplify somewhat, depending on the location of the particular point. For example, $\sigma_{r,z} = 0$ everywhere except in the conductor and in some instances in the coil; $J_{r,z} = 0$ everywhere except in the coil, and $\mu_{r,z}/\mu_{r+a,z} = 1$ for all nonmagnetic materials. With the help of a large digital computer, the value of $A_{r,z}$ can be calculated at every point. Along the boundaries of the mesh, the values of $A_{r,z}$ are held to zero. This is exact along the coil axis and the remaining boundary should be far enough away to approximate infinity. A distance of two coil diameters is adequate for most cases. The computer starts at a point in the mesh and works through point by point, using the proper values for $\sigma_{r,z}$, $\mu_{r,z}$, and $J_{r,z}$ in equation (4.9). After going through the entire mesh many times (iterations), the vector

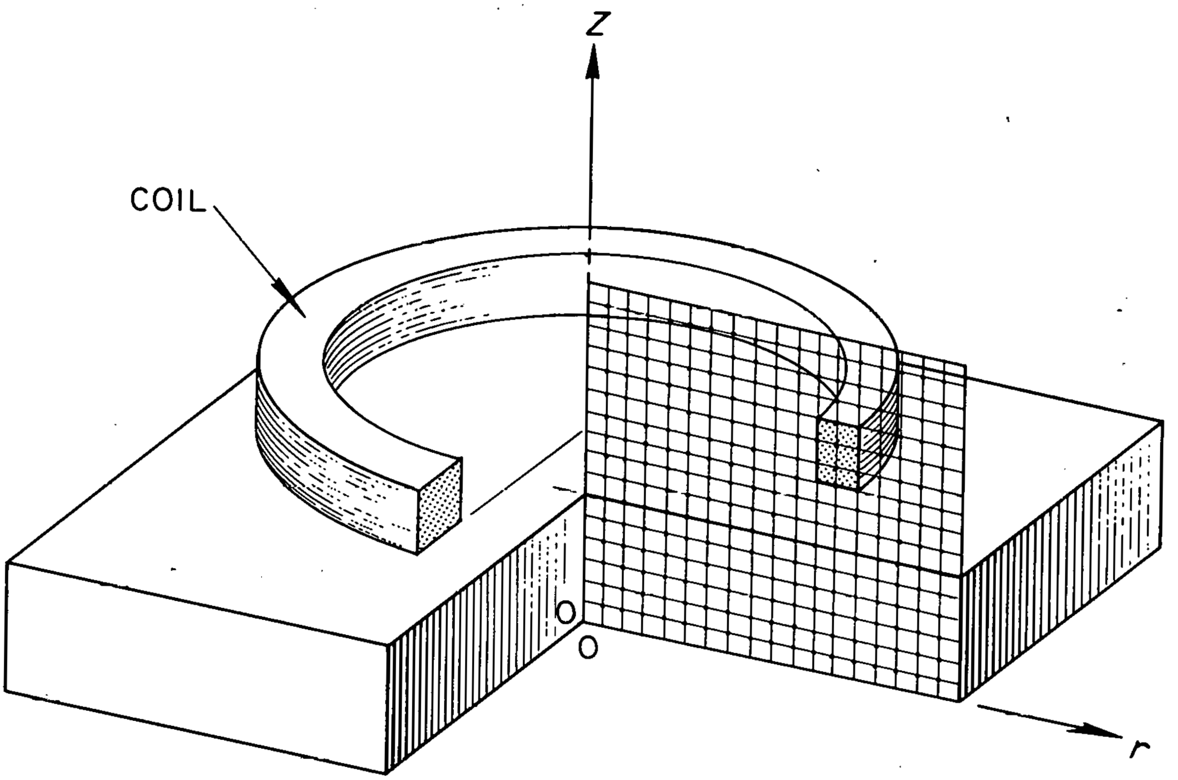

Figure 7. Layout of problem on a lattice of points.

potential will converge to a value determined by Maxwell's equations [Binns and Lawreson (1963)]. The finer the mesh, the greater the accuracy and the longer and more expensive the problem. Typically, for a 70 by 70 mesh, it takes 500 iterations to converge within one per cent at a cost of two hundred dollars.

Once a particular shape of coil and conductor has been chosen, the problem is solved for that case only. One main disadvantage of this technique is this inflexibility. A whole series of relaxations must be performed in order to observe the effect of varying only one parameter. However, there still exist two degrees of freedom after the relaxation has been completed. These involve the two products in equation (4.9, page 36), $a^2\mu_{r,z}J_{r,z}$ and $a^2\omega\mu_{r,z}\sigma_{r,z}$. Frequency and conductivity may be varied, in general, as long as their products remain constant. Also in the finite difference equations, a^2 and the product $\omega\mu_{r,z}\sigma_{r,z}$ may be varied, provided $a^2\omega\mu_{r,z}\sigma_{r,z}$ remains constant. Then, however, the driving current density must be varied to keep the product $a^2\mu_{r,z}J_{r,z}$ the same. This means, for example, that the solution for the vector potential at each point for a coil one inch in diameter above a copper plane of conductivity 1.732 μohm-cm with a driving current of one ampere at a frequency of one kilocycle is the same as the vector potential of a one-inch coil above an aluminum plane of conductivity 3.464 μohm-cm with a driving current of one ampere at 500 cycles per second. Also, the solution is the same for a 0.707-inch coil above a 3.464 μohm-cm aluminum plane with two amperes driving current at a frequency of one kilocycle. Thus, equation (4.9) can be used to solve for the vector potential, at any point in space, produced by a sinusoidal driving current. Any physically observable electromagnetic phenomenon can then be calculated from the vector potential.

Linear Medium and Pulsed Currents

In principle, the vector potential produced by any continuous time-varying current pulse could be calculated by a Fourier synthesis of the results of computations for sinusoidal driving currents. However, this would require solutions of a given problem at many different frequencies, particularly for sharp pulses. In fact, a pulse of finite duration theoretically requires an infinite number of Fourier components. For this reason and because of storage problems in the computer and the existence of nonlinear media, we prefer to solve the differential equation for the vector potential directly without assuming sinusoidal time variation. Hence, we return to equation (2.18, page 9).

Now $J(t)$ can be approximated in time for any current wave form, as shown in Figure 8.

Appropriate to a time-sequential current, we will perform a time-sequential relaxation. The term $\partial A/\partial t$ will become:

$$\partial A/\partial t = \frac{A_{r,z,t} - A_{r,z,t-\tau}}{\tau} \quad . \tag{4.10}$$

The solution will have to start where $J(t)$ is constant and $\partial A/\partial t = 0$ and proceed to values of t where $J(t)$ varies. Using the finite difference approximations given in equation (4.10) for the other

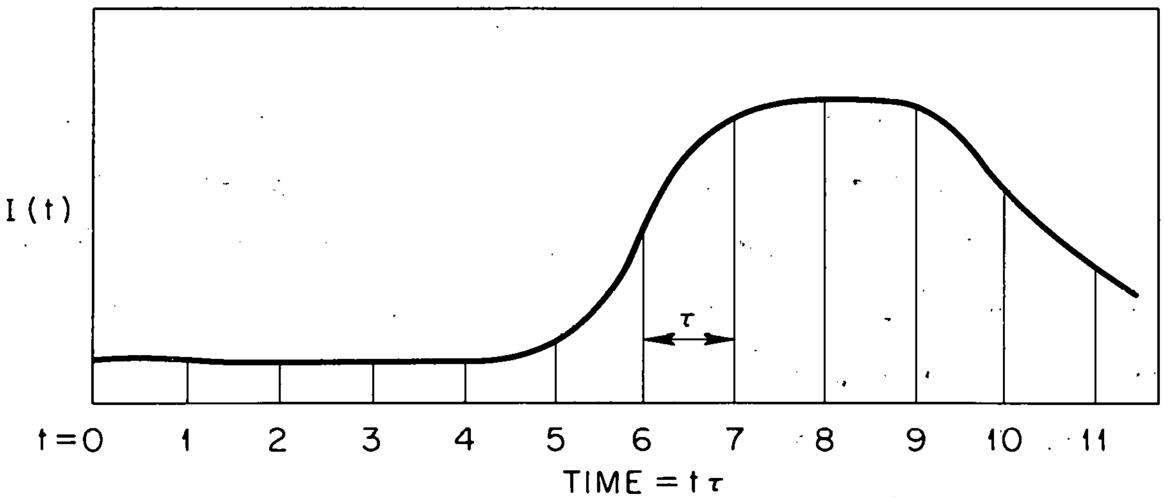

Figure 8. Approximation of a current pulse.

partial derivatives in equation (2.18, page 9) and solving for $A_{r,z,t}$ yield

$$A_{r,z,t} = \left\{(1+a/r)\left(\frac{\mu_{r,z}}{\mu_{r+a,z}}\right) A_{r+a,z,t} + A_{r-a,z,t} + \left(\frac{\mu_{r,z}}{\mu_{r,z+a}}\right) A_{r,z+a,t} + A_{r,z-a,t} + a^2\mu_{r,z}J_{r,z,t} + \frac{a^2}{\tau}\,\mu_{r,z}\sigma_{r,z}\,A_{r,z,t-\tau}\right\} \div \left\{2 + \frac{a}{r} + \left(\frac{a}{r}\right)^2 + \frac{a^2}{\tau}\,\mu_{r,z}\sigma_{r,z} + \frac{\mu_{r,z}}{\mu_{r+a,z}} + \frac{\mu_{r,z}}{\mu_{r,z+a}}\right\} \quad . \tag{4.11}$$

Equation (4.11) is the general relaxation equation and can be used to solve for the vector potential everywhere in space at each value of t. Although equation (4.9, page 36) is, in principle, derivable from this equation if $J_{r,z,t}$ varies sinusoidally with time and with $\tau \to o$, the derivation given earlier avoids some of the difficulties encountered in trying to reduce equation (4.11). For an arbitrary time variation, the solution for the previous value of t, $A_{r,z,t-\tau}$, must be stored in the computer. Also a large number of iterations must be made for each value of t.

A Nonlinear Medium and Pulsed Currents

Up to this point, we have been assuming that the permeability was constant at any point in the medium, varying only from point to point. This is an approximation which is good for weak fields. In general, $\vec{B}$ varies with $\vec{H}$ as shown in Figure 9(a).

The incremental permeability is defined as:

$$\mu = \frac{\Delta B}{\Delta H} \quad . \tag{4.12}$$

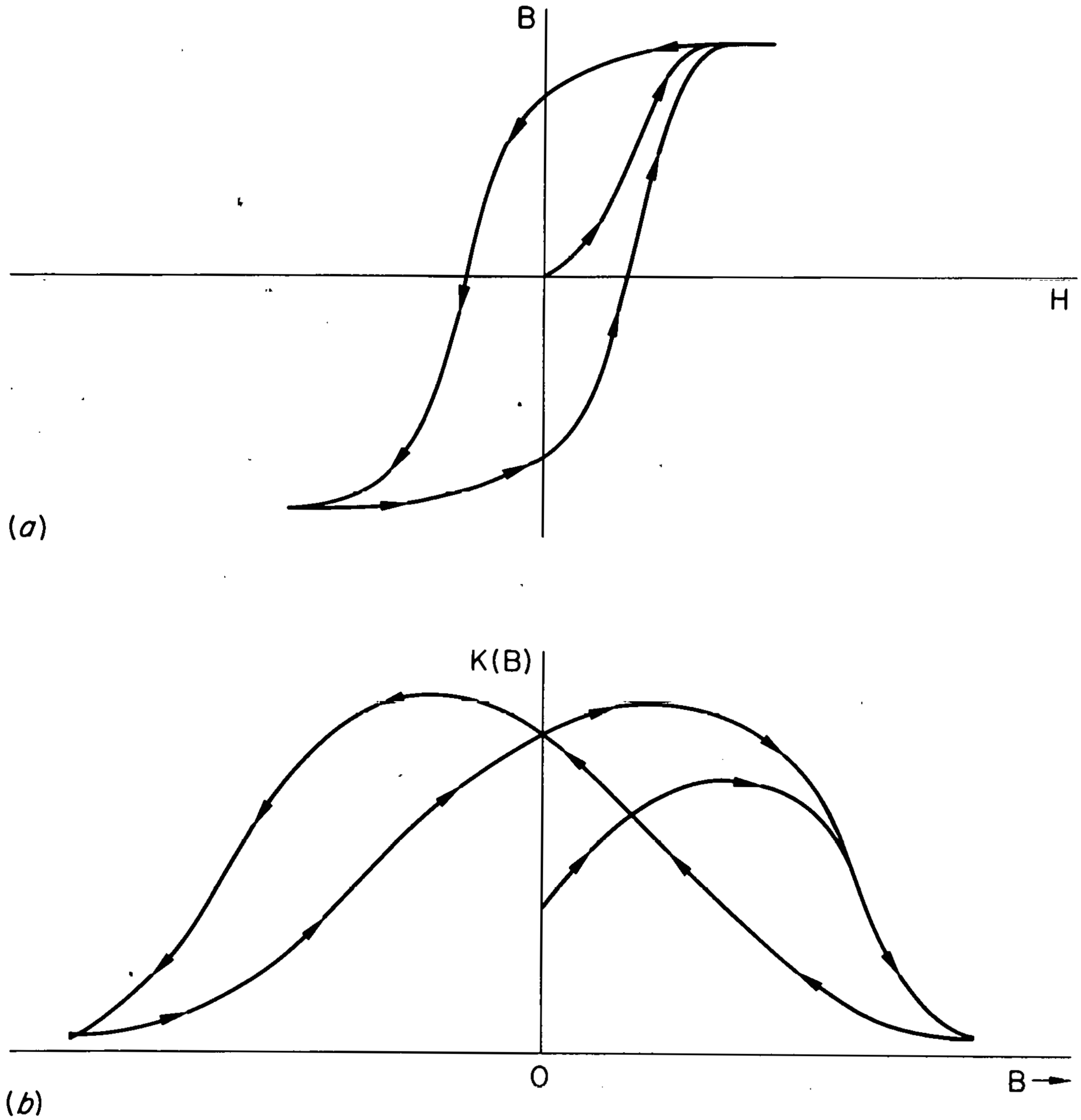

Figure 9. Magnetic parameter variations. (a) Variation of $\vec{B}$ with $\vec{H}$; (b) incremental permeability.

This is shown as a function of B in Figure 9(b). Note that this is a function not only of B, but also of the history of previous magnetization. It should also be noted that $\mu(B)$ is affected by temperature, cold working, alloy content, heat treatment, and a large number of other factors. However, μ can be measured as a function of B and stored in the computer memory, point by point. In our program we use the curl of $\vec{A}$ rather than $\vec{B}$. Expanding the curl of $\vec{A}$ and noting that $\vec{A}$ has only a θ component, we have [Morse and Feshbach (1953)]:

$$\nabla \times \vec{A} = \hat{a}_r \left(-\frac{\partial A}{\partial z}\right) + \hat{a}_z \left(\frac{1}{r}\frac{\partial rA}{\partial r}\right) \quad . \tag{4.13}$$

$$B = \pm \left[\left(\frac{\partial A}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial rA}{\partial r}\right)^2\right]^{\frac{1}{2}} \quad .$$

In finite difference form, this becomes:

$$B_{r,z} = \pm \left\{\frac{\left(A_{r,z+a} - A_{r,z}\right)^2}{a^2} + \frac{\left((r+a)\ A_{r+a,z} - rA_{r,z}\right)^2}{r^2a^2}\right\}^{\frac{1}{2}} \quad . \tag{4.14}$$

This would be computed at each point in the nonlinear medium at the end of each iteration. The proper value of $\mu_{r,z}$ at that point for the next iteration would then be determined from the stored curve and placed in the memory. The ambiguity in sign in equation (4.14) would have to be removed here by knowing the past history. By applying equation (4.14) between each iteration of equation (4.11, page 41), we can calculate $A_{r,z,t}$ at every point in space at any time in the presence of a nonlinear medium.

CHAPTER V

OBSERVABLE PHYSICAL PHENOMENA

Once the vector potential has been determined, either by a "closed form" solution or by a relaxation technique, any physically observable electromagnetic phenomenon can be calculated from it. In this chapter we shall give the equations, both in the differential and the finite difference forms, of some of these phenomena.

Dissipated Power

From the vector potential, the dissipated power density due to the eddy currents can be calculated:

$$P = JE = \sigma E^2 = -\omega^2\sigma A^2 \quad , \tag{5.2}$$

where A is the root mean square vector potential. The negative sign denotes a power loss from the field.

Coil Impedance

Another physical quantity that can be calculated is the impedance of the coil in the presence of a metal. This is of particular importance in the testing of materials. When the vector potential is obtained, it is integrated [Reitz and Milford (1960)] over the coil to obtain the induced voltage:

$$V = j\omega\int \vec{A}\cdot d\vec{s} \quad , \tag{5.3}$$

or in finite difference terms,

$$V = j\omega \sum_{\text{coil}} 2\pi r A_{r,z} \quad . \tag{5.4}$$

The impedance is $Z = \frac{V}{I}$, so equations (5.3) and (5.4) become:

$$Z = \frac{j\omega}{I} \int \vec{A} \cdot d\vec{s} \quad , \tag{5.5}$$

$$Z = \frac{j\omega \sum\limits_{coil} 2\pi r A_{r,z}}{I} \quad . \tag{5.6}$$

This impedance is usually normalized so that many values will fit on a small plot. This is done by dividing by the magnitude of the coil impedance in air:

$$Z_n = j\int_{coil} \vec{A} \cdot d\vec{s}(\text{conductor present}) \div \int_{coil} \vec{A} \cdot d\vec{s}(\text{coil alone}) \quad , \tag{5.7}$$

or, in finite difference terms:

$$Z_n = j \sum_{coil} 2\pi r A_{r,z}(\text{conductor}) \div \sum_{coil} 2\pi r A_{r,z}(\text{air}) \quad . \tag{5.8}$$

We have obtained the impedance of a coil in the presence of a metal without defects. Once the vector potential has been determined for a particular coil and metal, we can use superposition to determine the solution with a defect present (even though the defect violates our assumption of axial symmetry). A defect can be represented as a current equal in magnitude and flowing in the opposite direction to the induced eddy currents. The vector potential of a coil with a defect present is the sum of vector potentials of the coil and conductor alone and the defect alone (provided the current which the defect produces around itself can be solved). The addition of the current of the defect to the induced eddy current gives, of course, zero current flowing through the defect. Although the impedance change due to an actual defect is difficult to calculate in general, we may approximate it. The impedance change due to a small, spherical defect not too close to the surface [Burrows (1964)] is:

$$Z'_n = \frac{3}{2}\,\sigma\,\mathrm{vol}\left(\frac{A_{defect}}{I}\right)^2 \quad , \tag{5.9}$$

where A_{defect} is the vector potential at the defect and "vol" is the volume of the defect.

Electromagnetic Forces

We can also calculate the electromagnetic forces in any conductor which may be present. These forces are given [Stratton (1941)] as:

$$\vec{F} = \rho\vec{E} + \vec{J} \times \vec{B} - \frac{1}{2}\,E^2\,\nabla\epsilon - \frac{1}{2}\,H^2\nabla\mu + \frac{K_m K_c - 1}{c^2}\,\frac{\partial\vec{S}}{\partial t} \quad . \tag{5.10}$$

This is the force exerted by an electromagnetic field on a unit volume of isotropic matter, neglecting electro- and magnetostrictive forces. The neglecting of these latter forces is justifiable since they produce deformation of the material but no net force. The first term vanishes when the charge density ρ is summed over the electrons and ions. The third term is also taken to be zero for the interior of a metal. The last term is due to the light pressure and is negligibly small. Thus, the force reduces to:

$$\vec{F} = \vec{J} \times \vec{B} - \frac{1}{2}\,H^2\nabla\mu \quad ,$$

or:

$$F = \vec{J} \times \vec{B} - \frac{1}{2}\,B^2\nabla\frac{1}{\mu} \quad . \tag{5.11}$$

We shall first consider only nonmagnetic materials, which require only the first term in equation (5.11).

We have, from Ohm's law:

$$\vec{J} = \sigma\vec{E} = -\sigma \frac{\partial\vec{A}}{\partial t} \quad . \tag{5.12}$$

Using this, the force becomes:

$$\vec{F} = -\sigma \frac{\partial\vec{A}}{\partial t} \times (\nabla \times \vec{A}) \quad . \tag{5.13}$$

Expanding the curl of $\vec{A}$ in cylindrical coordinates:

$$\nabla \times \vec{A} = \hat{a}_r \left(-\frac{\partial A}{\partial z}\right) + \hat{a}_z \frac{1}{r}\frac{\partial rA}{\partial r} \tag{5.14}$$

so that

$$\vec{F} = -\sigma\hat{a}_\theta \frac{\partial A}{\partial t} \times \left[\hat{a}_r\left(-\frac{\partial A}{\partial z}\right) + \hat{a}_z \frac{1}{r}\frac{\partial rA}{\partial r}\right] \tag{5.15}$$

$$\vec{F} = -\hat{a}_z\sigma \frac{\partial A}{\partial t}\frac{\partial A}{\partial z} - \left(\hat{a}_r\sigma \frac{\partial A}{\partial t}\frac{1}{r}\frac{\partial rA}{\partial r}\right) \quad . \tag{5.16}$$

In the z direction, force density will always be away from the coil since $\partial A/\partial z$ will be negative, making F_z positive. In the r direction, it can be either positive or negative, depending upon the sign of $\partial rA/\partial r$. The z component of force of any volume $dv = 2\pi r dr dz \cong 2\pi r a^2$ is:

$$F_{z_{r,z}} = -2\pi r a \sigma_{r,z} \frac{\partial A_{r,z}}{\partial t}\left[A_{r,z+a} - A_{r,z}\right] \quad . \tag{5.17}$$

The r component of force is:

$$F_{r_{r,z}} = -2\pi a \sigma_{r,z} \frac{\partial A_{r,z}}{\partial t}\left[(r+a)\, A_{r+a,z} - rA_{r,z}\right] \quad . \tag{5.18}$$

<u>Sinusoidal forces</u>. If we assume a sinusoidal driving current,

$$\frac{\partial A_{r,z}}{\partial t} = j\omega A_{r,z} \quad , \tag{5.19}$$

then the forces become:

$$F_{r_{r,z}} = -2j\pi a\omega\sigma_{r,z}\, A_{r,z}\left[(r+a)\, A_{r+a,z} - rA_{r,z}\right] \quad , \tag{5.20}$$

$$F_{z_{r,z}} = -2j\pi ra\omega\sigma_{r,z}\, A_{r,z}\left[A_{r,z+a} - A_{r,z}\right] \quad . \tag{5.21}$$

Taking the real parts of the force we have:

$$F_{r_{r,z}} = \pi\, \frac{r}{a}\, a^2\omega\sigma_{r,z}|A_{r,z}|\ \left|(r+a/r)A_{r+a,z} - A_{r,z}\right|$$

$$\times\left[\sin(2\omega t+\phi + \phi') + \sin(\phi - \phi')\right] \quad . \tag{5.22}$$

$$F_{z_{r,z}} = \pi\, \frac{r}{a}\, a^2\omega\sigma_{r,z}|A_{r,z}|\ \left|A_{r,z+a} - A_{r,z}\right|$$

$$\times\left[\sin(2\omega t+\phi + \phi') + \sin(\phi - \phi')\right] \quad . \tag{5.23}$$

The angle ϕ represents the phase shift of $|A_{r,z}|$ from zero and the angle ϕ' represents the phase shift of $|A_{r,z+a} - A_{r,z}|$. When equation (5.23) is expanded in terms of the real and imaginary parts of the vector potential, we get:

$$F_{z_{r,z}} = \pi\frac{r}{a}\, a^2\omega\sigma_{r,z}\left\{\left(\mathrm{Rl}\ A_{r,z}\ \mathrm{Im}\ A_{r,z-a} - \mathrm{Im}\ A_{r,z}\ \mathrm{Rl}\ A_{r,z-a}\right) + \sin 2\omega t\right.$$

$$\times\left[\mathrm{Rl}\ A_{r,z}\left(\mathrm{Rl}\ A_{r,z} - \mathrm{Rl}\ A_{r,z-a}\right) + \mathrm{Im}\ A_{r,z}\left(\mathrm{Im}\ A_{r,z} - \mathrm{Im}\ A_{r,z-a}\right)\right] + \cos 2\omega t$$

$$\left.\times\left[\mathrm{Im}\ A_{r,z}\left(\mathrm{Rl}\ A_{r,z} - \mathrm{Rl}\ A_{r,z-a}\right) + \mathrm{Rl}\ A_{r,z}\left(\mathrm{Rl}\ A_{r,z} - \mathrm{Rl}\ A_{r,z-a}\right)\right]\right\} \quad . \tag{5.24}$$

γ

The net force in the z direction is:

$$F_z = \pi a^2 \omega \sum_{r,z}^{\text{metal}} \sigma_{r,z}(r/a)[\text{Rl } A_{r,z} \text{ Im } A_{r,z-a} - \text{Im } A_{r,z} \text{ Rl } A_{r,z-a}] \quad . \tag{5.25}$$

Time sequential forces. If we assume a pulse of current and perform a time sequential relaxation we have for the time dependence:

$$\frac{\partial A_{r,z}}{\partial t} = \frac{A_{r,z,t} - A_{r,z,t-\tau}}{\tau} \quad . \tag{5.26}$$

The z and r components of force then become:

$$F_{z_{r,z,t}} = -\, 2\pi r a \sigma_{r,z} \frac{[A_{r,z,t} - A_{r,z,t-\tau}]}{\tau} [A_{r,z+a,t} - A_{r,z,t}] \quad , \tag{5.27}$$

$$F_{r_{r,z,t}} = -(2\pi a \sigma_{r,z}/\tau)[A_{r,z,t} - A_{r,z,t-\tau}][(r+a)A_{r+a,z,t} - rA_{r,z,t}] \, . \tag{5.28}$$

This gives the value of the force at any time (time=$t\tau$) and at any point. The total impulse at any point would be:

$$I_{z_{r,z}} = \tau \sum_{t=o}^{\infty} F_{z_{r,z,t}} \quad , \tag{5.29}$$

$$I_{r_{r,z}} = \tau \sum_{t=o}^{\infty} F_{r_{r,z,t}} \quad . \tag{5.30}$$

The total force on the metal would be:

$$F_z = \sum_{r,z}^{\text{metal}} F_{z_{r,z}} \quad . \tag{5.31}$$

For the case of a sinusoidal current, if we took the absolute value of F_z after summing, we would get the peak force per cycle. For a time sequential relaxation, the absolute value gives the total force on the metal at any time, t. The total impulse on the metal would be:

$$I_z = \tau \sum_{t=o}^{\infty} F_{z,t} \quad . \tag{5.32}$$

Due to the cylindrical symmetry, there will be no net r component of force.

Forces in magnetic materials. The first term in equation (5.11, page 46) gives the Lorentz force density which we have already calculated; the second term is due to magnetic materials, and we shall now consider it.

$$\vec{F} = -\frac{1}{2} B^2 \nabla \frac{1}{\mu} \quad . \tag{5.33}$$

Substituting the curl of $\vec{A}$ for $\vec{B}$ and expanding:

$$F_z = -\left\{\left(\frac{\partial A}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial rA}{\partial r}\right)^2\right\} \frac{\partial(1/\mu)}{\partial z} \pi r a^2 \quad , \tag{5.34}$$

$$F_r = -\left\{\left(\frac{\partial A}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial rA}{\partial r}\right)^2\right\} \frac{\partial(1/\mu)}{\partial r} \pi r a^2 \quad . \tag{5.35}$$

The finite difference terms are:

$$F_{z_{r,z,t}} = -\frac{\pi r}{a}\left\{(A_{r,z+a,t} - A_{r,z,t})^2 + \frac{[(r+a)A_{r+a,z,t} - rA_{r,z,t}]^2}{r^2}\right\} \times \left\{\frac{1}{\mu_{r,z+a,t}} - \frac{1}{\mu_{r,z,t}}\right\} \quad , \tag{5.36}$$

$$F_{r_{r,z,t}} = -\frac{\pi r}{a}\left\{(A_{r,z+a,t} - A_{r,z,t})^2 + \frac{[(r+a)A_{r+a,z,t} - rA_{r,z,t}]^2}{r^2}\right\} \times \left\{\frac{1}{\mu_{r+a,z,t}} - \frac{1}{\mu_{r,z,t}}\right\} \quad . \tag{5.37}$$

The total magnetic force is the sum over all r and z. Again, there is no net force in the r direction. The total force is the sum of the eddy-current forces and the magnetic forces.

CHAPTER VI

APPLICATIONS

The relaxation technique has been applied to a large number of practical problems. The results of some of these are given in this chapter.

Figure 10 [Dodd (1965)] shows the phase and amplitude contours of the vector potential produced by a long coil. The contours are plotted in a plane containing the coil axis due to the axial symmetry. Since the eddy-current density is directly proportional to the vector potential, in the conductor these are also contours of eddy-current density. Figure 11 [Dodd (1965)] shows the contours of eddy-current heating density for the same coil. Figure 12 shows the vector potential produced by a coil encircling a conducting rod. Figure 13 shows how the vector potential is changed when the conducting rod is ferromagnetic. Note how the vector potential is "attracted" by the rod. Also, the eddy-current density is relatively constant over a large outer portion of the rod and rapidly decreases toward the center of the rod.

Figure 14 shows the phase and amplitude contours of the vector potential produced by a square cross-section coil. A family of four of these coils having the same relative dimensions but different sizes was built. The impedance was measured (see Chapter VII) for the coils at various frequencies and for various spacings (lift-off) between the coil and the conductor. Figure 15 shows how these measured values agree with values calculated by the relaxation technique. The accuracy

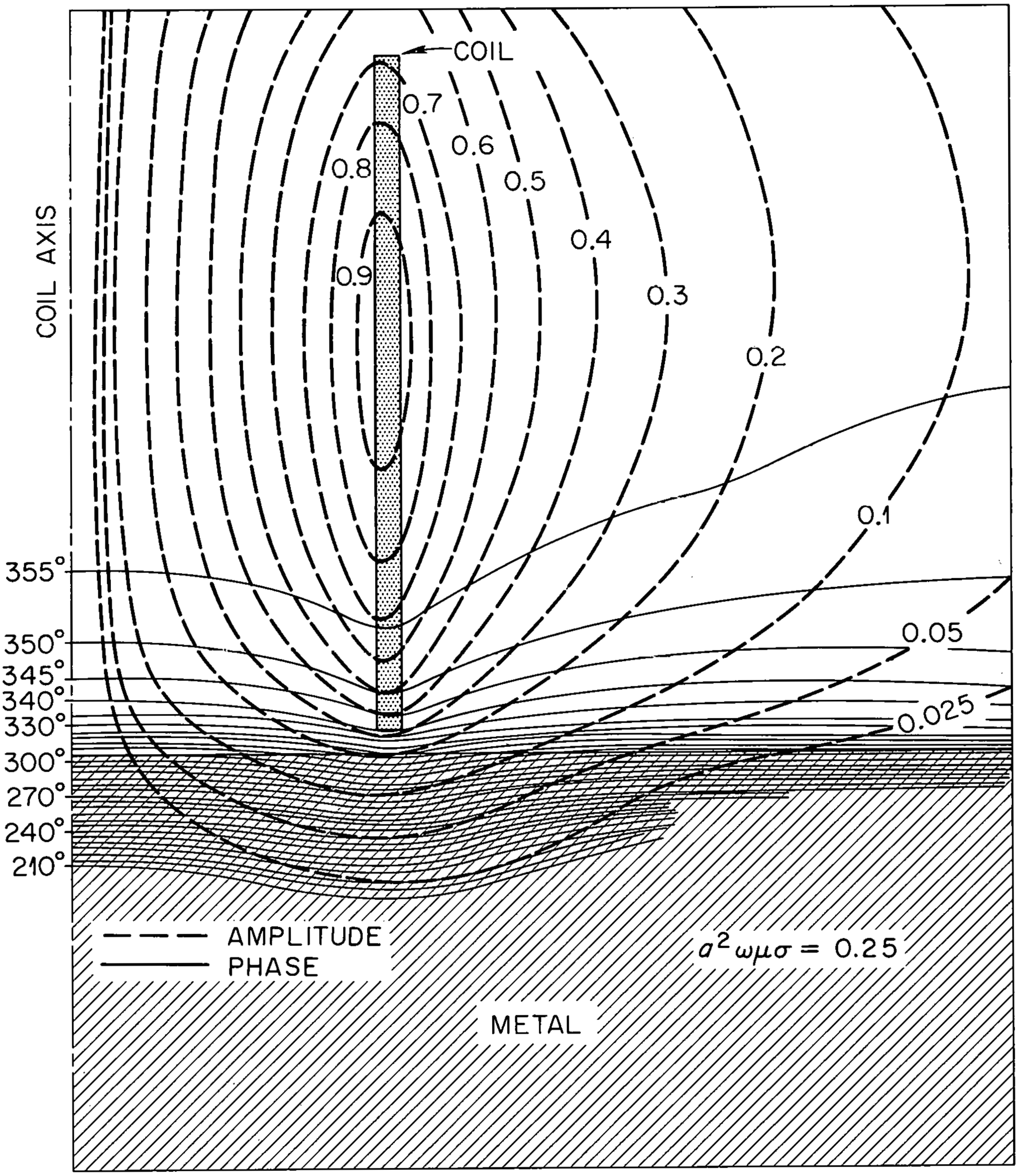

Figure 10. Phase and amplitude of the vector potential of a coil above a metal plane.

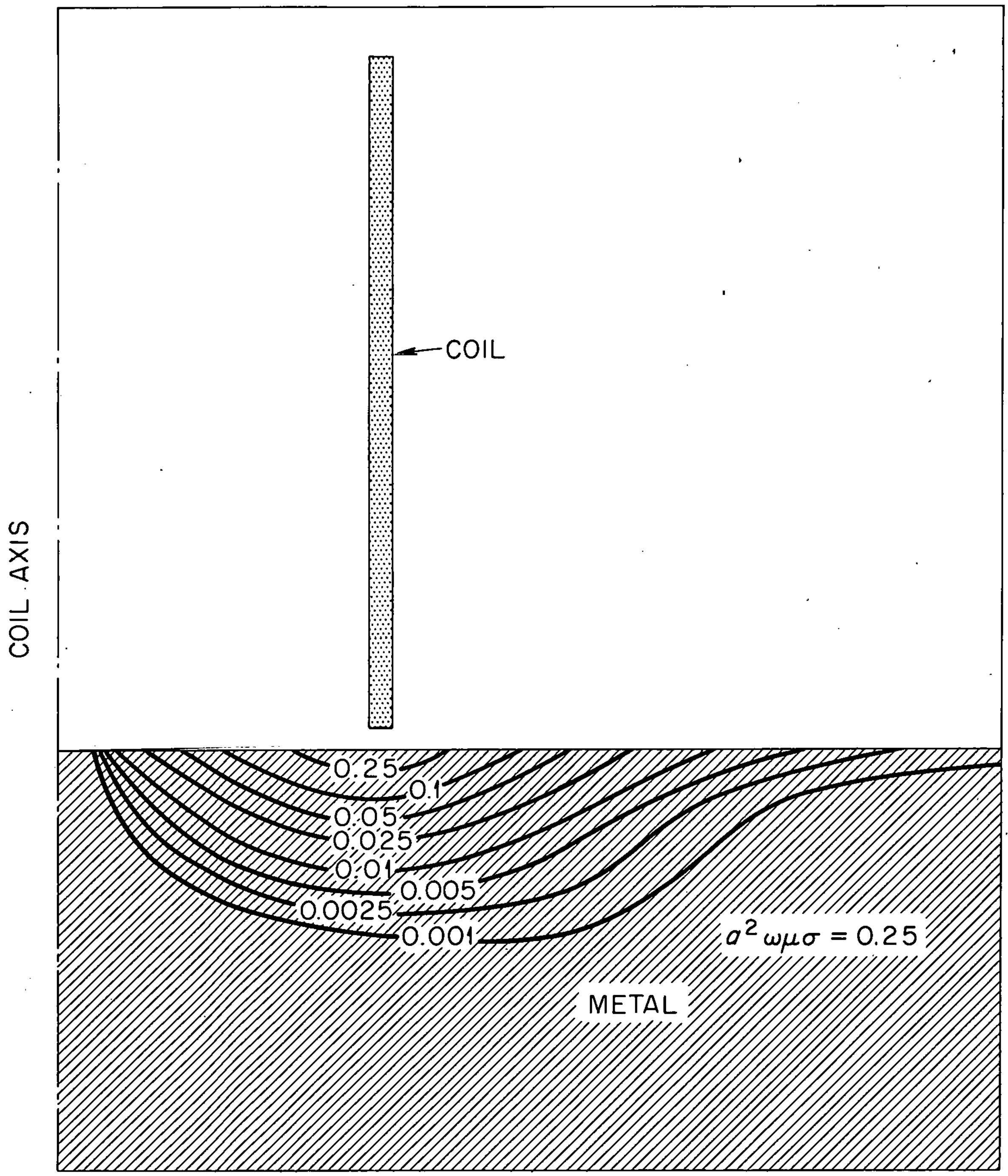

Figure 11. Contours of eddy-current heating density.

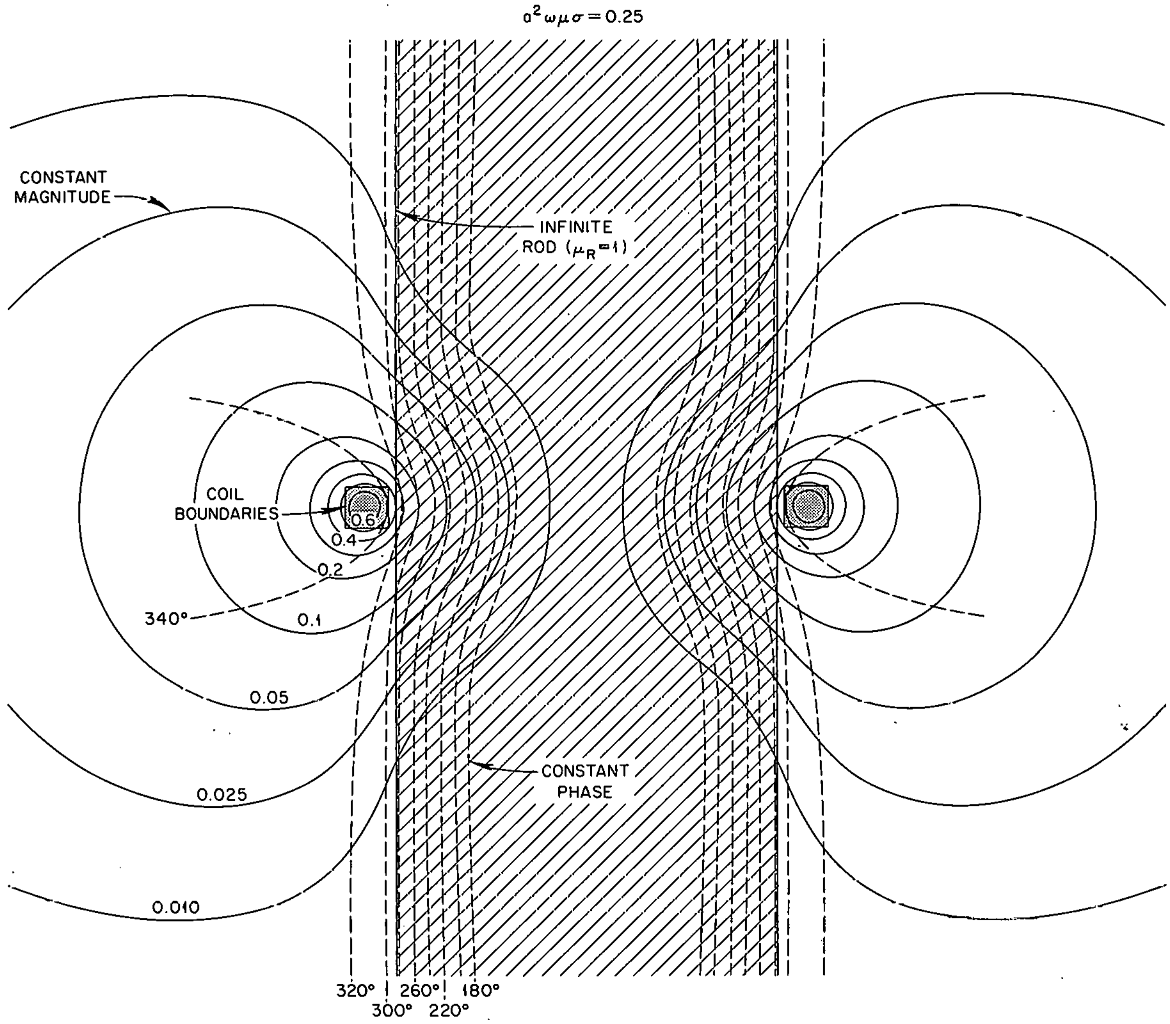

Figure 12. Coil encircling a conducting rod.

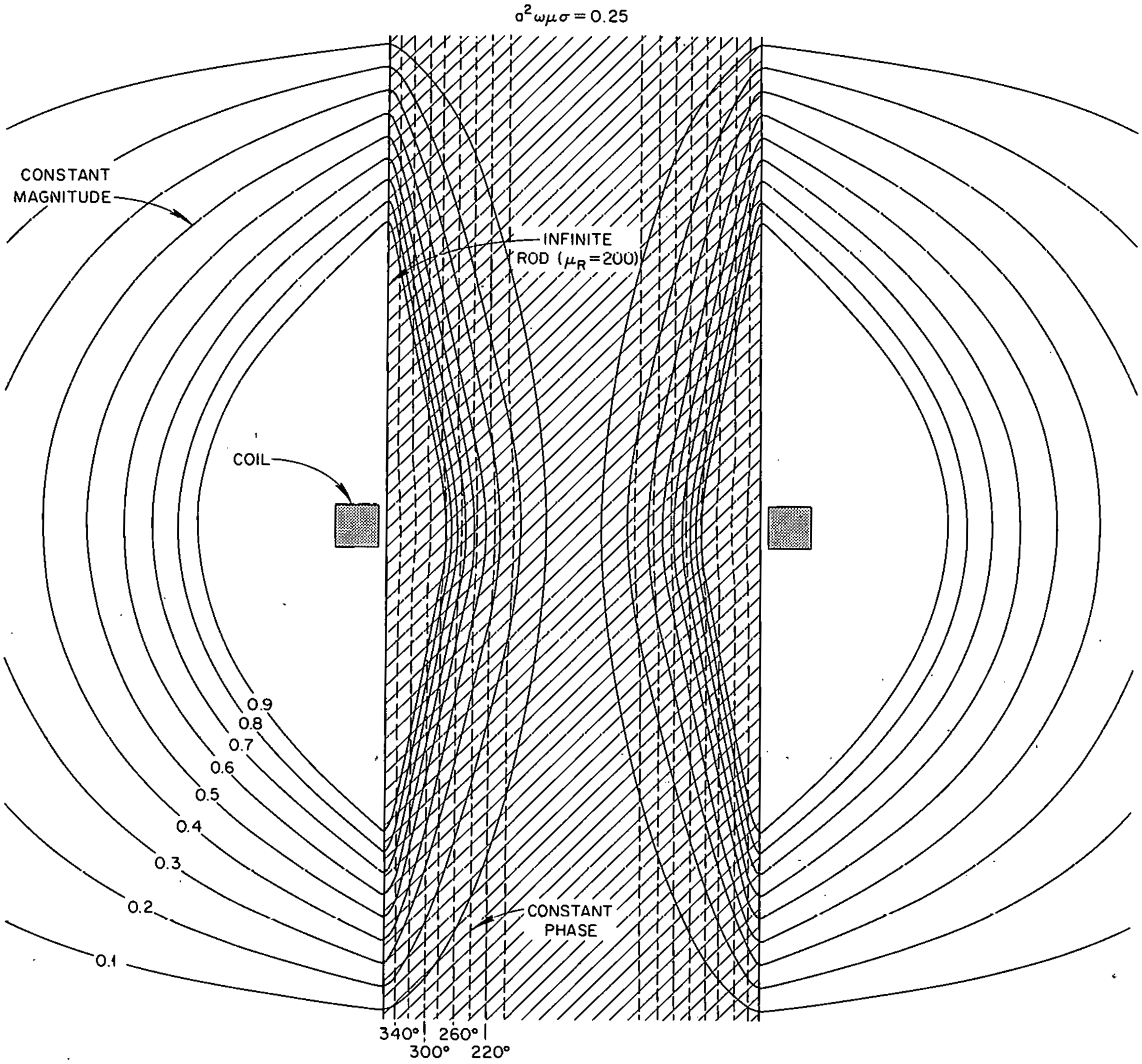

Figure 13. Coil encircling a ferromagnetic rod.

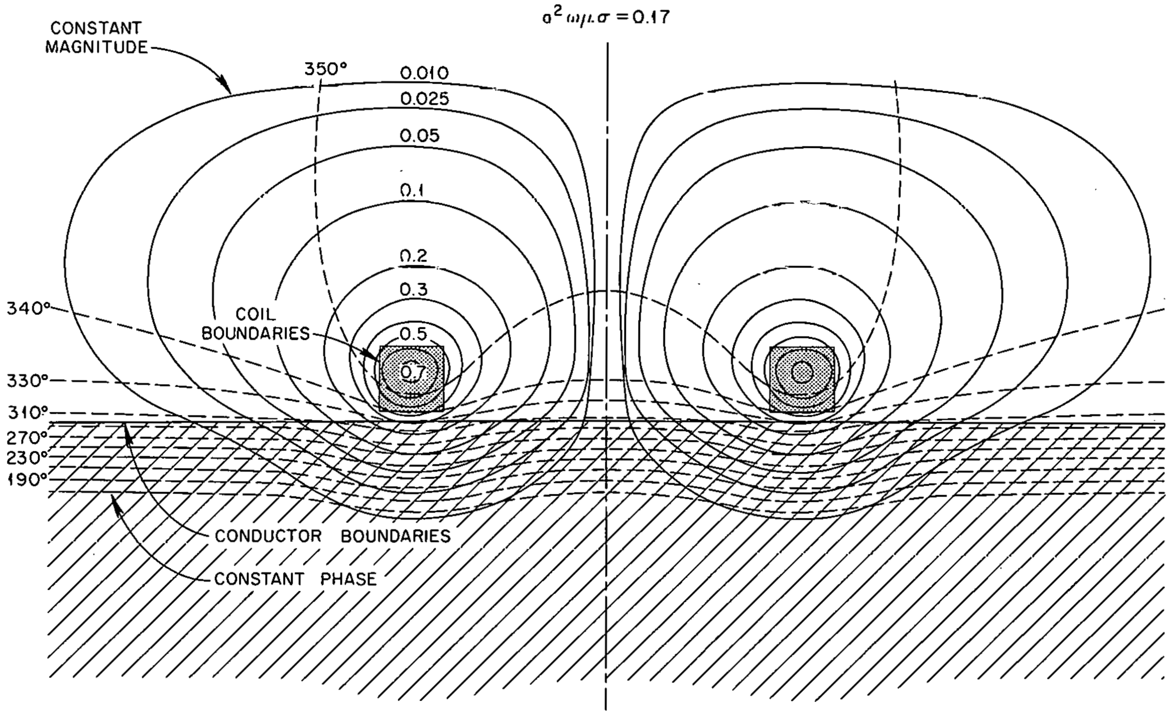

Figure 14. Coil above a conducting plane.

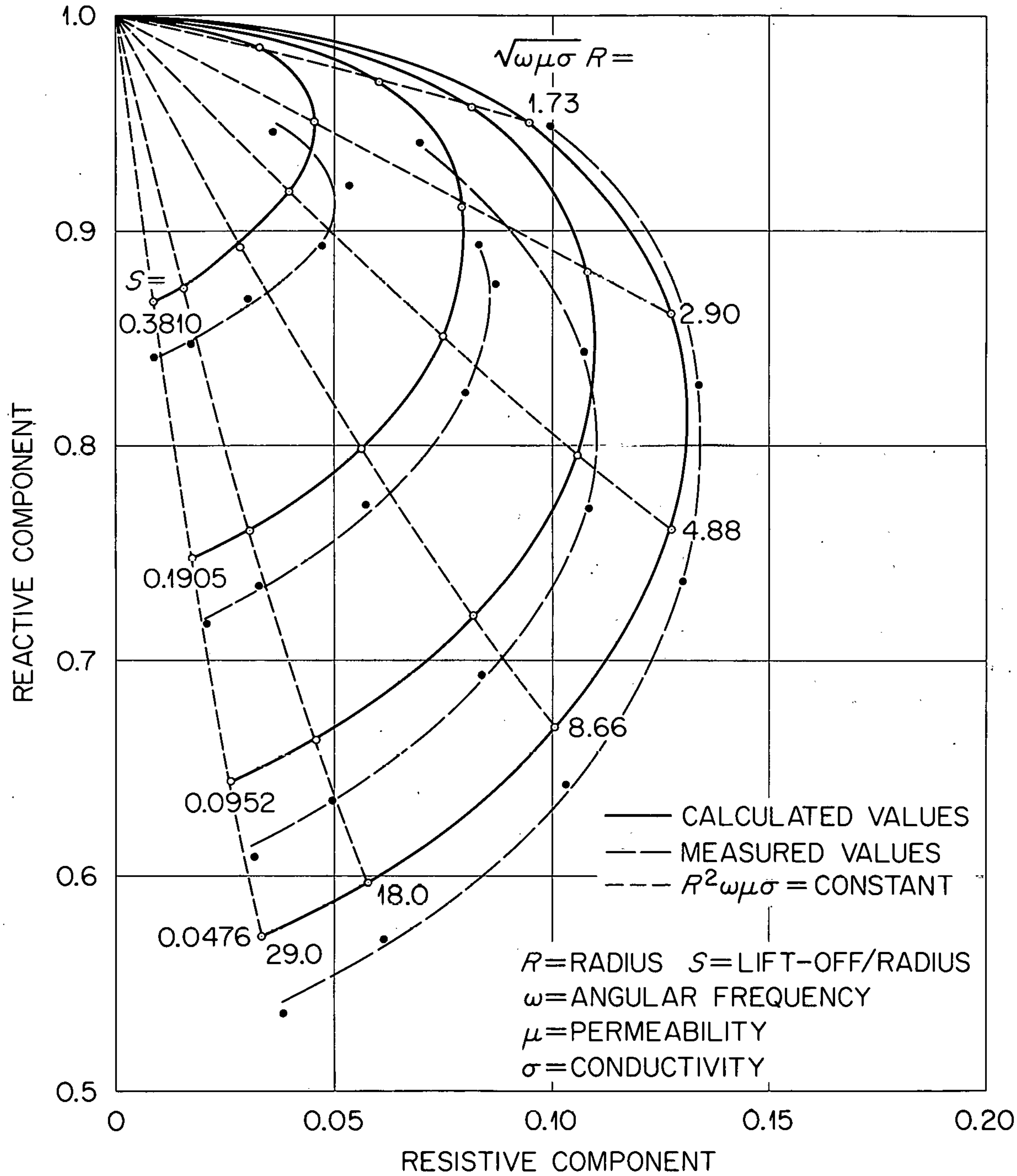

Figure 15. Normalized impedance of a coil above a conducting plane.

of the measured values is rather poor in the low frequency regions and better in the high frequency regions. The calculated values had some inaccuracy along the spacing direction (along lines of constant $R^2\omega\mu\sigma$). This is due to the fact that impedance is affected so strongly by spacing (lift-off), and the relaxation technique does not define the exact location of the coil and the metal. The error is always less than one lattice space. The agreement in values of $R^2\omega\mu\sigma$ is quite good for the higher frequencies.

Another problem of interest in the testing of metals is the shaping of fields by the use of ferrites. Figure 16 shows how the field of the coil in Figure 10, page 53, is "focused" by the addition of a ferrite cup. This tends to concentrate the eddy currents into a smaller volume and make the coil more sensitive to defects.

Figure 17 shows the contour of net downward force produced in a conducting ring.

Since the force is directly proportional to the square of the current, both the calculated and measured (see Chapter VII) forces were normalized by dividing them by the square of the current. The values are compared below.

Frequency (Hertz)	Force, g/amp^2, RMS		Per cent Error
	Measured	Calculated	
138	2.65	2.98	12.5
414	4.32	5.14	19
1656	5.81	6.18	6.5

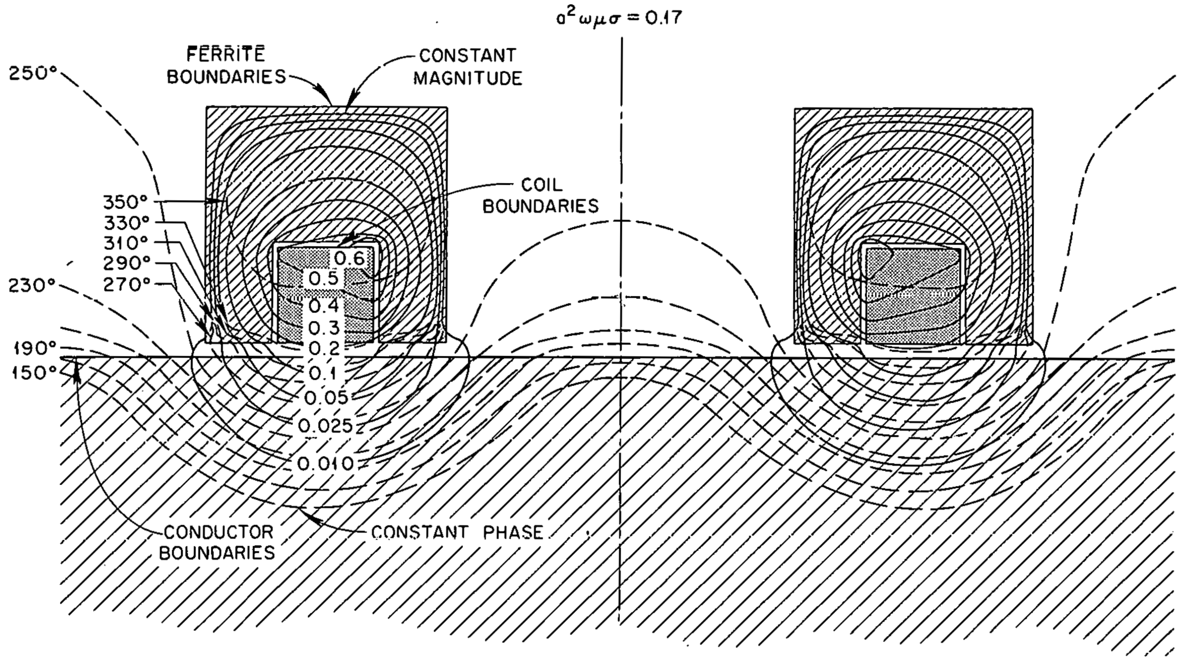

Figure 16. Coil with a ferrite cup above a conducting plane.

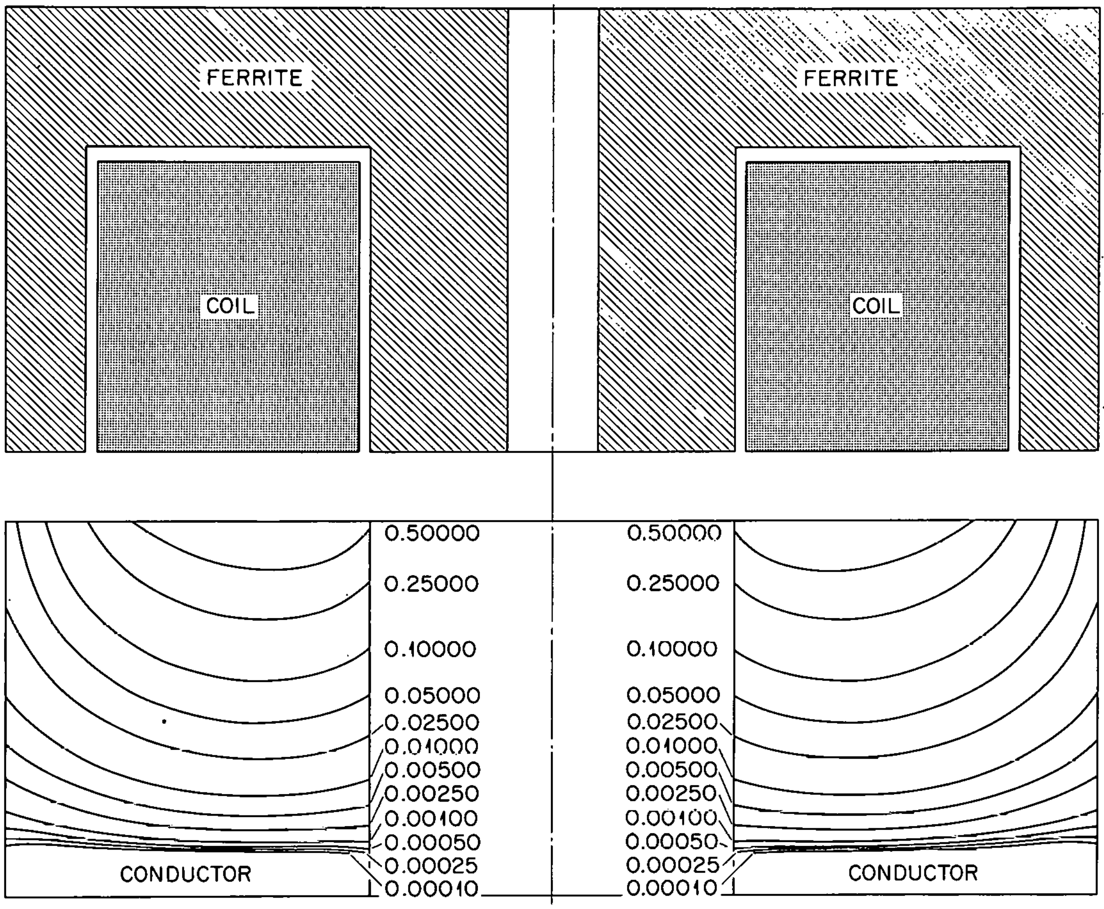

Figure 17. Eddy-current force contours.

The large percentage error is probably due to the value of the relative permeability chosen for the ferrite and the size of the mesh used to represent the problem. The value of the relative permeability used in the calculations was 1000 while the actual value was about fifteen per cent lower. The largest part of the error is probably due to the fact that the force is a very sensitive function of the spacing between the coil and metal, as indicated in Table IV of Chapter VII. The relaxation technique only defines the position of the coil and metal to within one lattice spacing. Also, a relatively coarse 70 by 70 mesh was used. A finer mesh would have given more accuracy, but at an increased cost.

CHAPTER VII

EXPERIMENTAL RESULTS

A number of coils were constructed as accurately as possible, and measurements have been made of the coil impedances and the forces generated in the presence of a large semi-infinite conductor of known conductivity. These measurements are described in this chapter.

Impedance Measurements

A family of four coils, having the same relative dimensions but different sizes, was constructed. The dimensions and other parameters of the coils are given in Table I.

The dimensions of all these coils can be expressed in terms of a mean coil radius, $\bar{r}$. If this is done, all the coils have a square cross section of $\bar{r}/3$, and have values of $\bar{r}$ of 0.300, 0.600, 0.900, and 1.200 inches. The impedance of the coils was measured at various distances above an aluminum disk 2 inches thick and 12 inches in diameter, having a measured resistivity of 4.2 micro-ohm cm. The system used to measure the impedance is shown in Figure 18. Voltage readings were made on either side of the precision low inductance resistor. Then the coil and the resistor were electrically interchanged, and the voltage between the precision resistor and ground was measured, giving the current through the resistor. From the first two voltages and the current, we can calculate the coil impedance as follows.

The voltages may be written:

$$\vec{V}_1 = I \vec{z} \ , \tag{7.1}$$

TABLE I

COIL PARAMETERS

Coil	Inner Diameter r_1 (inches)	Outer Diameter r_2 (inches)	Length $\ell_2-\ell_1$ (inches)	Wire Size (A.W.G.)	Number of Turns
A	0.500 min	0.706 min	0.100 min	No. 40	622
	0.503 max	0.710 max	0.103 max		
B	1.000 min	1.417 min	0.200 min	No. 34	718
	1.003 max	1.430 max	0.204 max		
C	1.500 min	2.100 min	0.306 min	No. 32	925
	1.503 max	2.115 max	0.322 max		
D	2.000 min	2.820 min	0.402 min	No. 30	1392
	2.002 max	2.830 max	0.405 max		

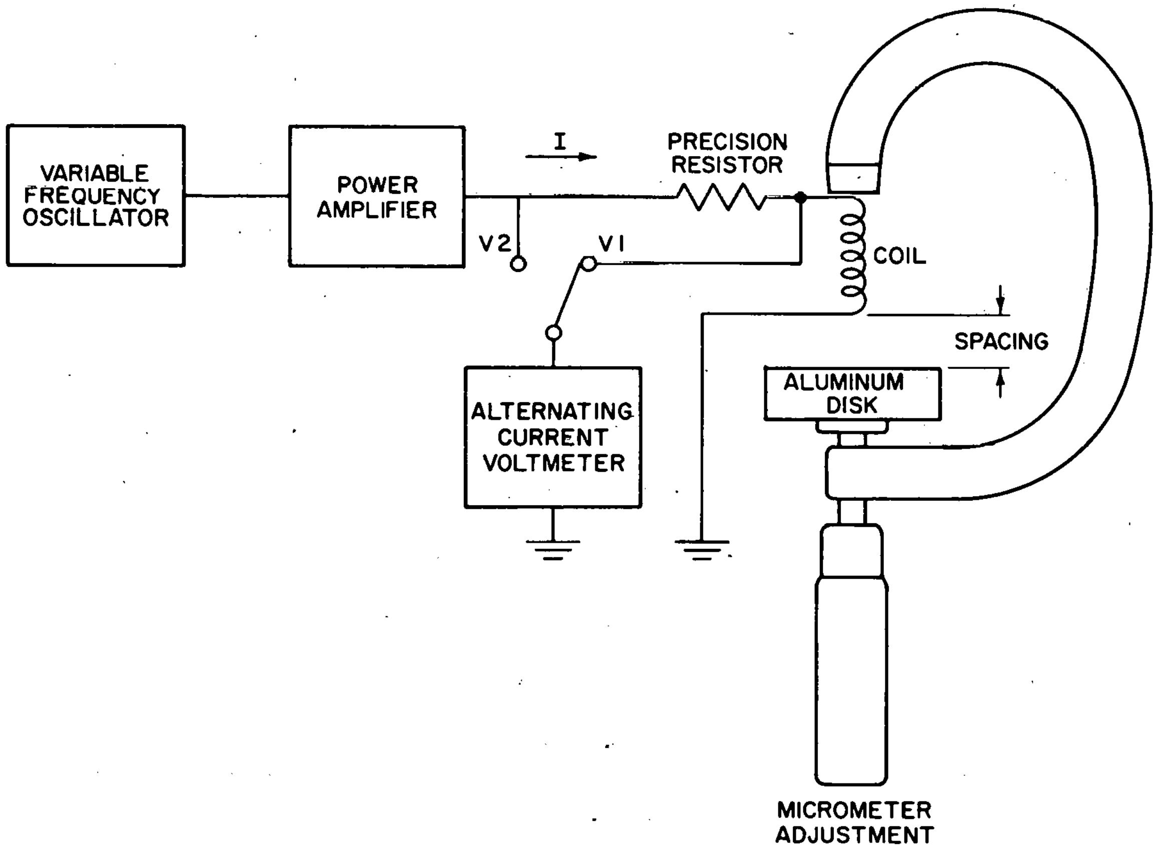

Figure 18. Diagram of impedance measurement apparatus.

$$\vec{V}_2 = I(R+\vec{z}) \quad . \tag{7.2}$$

Now, since $|\vec{V}_1|$, $|\vec{V}_2|$, and I are known, we can calculate the real and imaginary parts of $\vec{z}$:

$$R\ell\ \vec{z} = \frac{V_2^2 - V_1^2 - (RI)^2}{2RI^2} \tag{7.3}$$

$$\mathrm{Im}\ \vec{z} = \sqrt{\left(\frac{V_1}{I}\right)^2 - (R\ell\ \vec{z})^2} \tag{7.4}$$

The impedance was then normalized by subtracting off the direct-current coil resistance and dividing by the magnitude of the coil reactance in air. Subtracting the direct-current coil resistance eliminates the part of the coil that is not affected by vector potential, and dividing by the coil reactance allows the impedance of the entire family of coils to be plotted on a common graph. Table II shows the resistance and inductance of the various coils in air at different frequencies. The frequencies are chosen for common values of $\bar{r}\sqrt{\omega\mu\sigma}$. Table III shows the coil resistive component and inductance for the coils at various spacings above the aluminum disk and at the same frequencies. The spacings for the different coils are chosen to have the same value of distance divided by mean coil radius.

Each value in Tables II and III represents the average of ten readings at different current levels, and the standard deviation of these readings is given after each (average) value in the tables. In some instances there was wave distortion due to an impedance mismatch for some of the values, and these values have been omitted.

TABLE II

MEASUREMENTS OF COIL RESISTANCE AND INDUCTANCE IN AIR AS A FUNCTION OF FREQUENCY

Coil	Frequency (hertz)	Resistive Component (ohms)	Standard Deviation	Inductive Component (milli-henries)	Standard Deviation
A	273.42	108.29	0.256	7.08	0.821
A	765.6	107.71	0.172	7.38	0.105
A	2187.4	107.09	0.27	7.46	0.017
A	6835.5	107.17	0.72	7.47	0.01
A	29,280	110.70	1.546	7.60	0.004
A	76,600			7.67	0.003
B	68.4	58.43	0.10		
B	191.5	58.24	0.053	18.52	0.098
B	547	58.34	0.157	18.21	0.049
B	1710	57.23	0.30	18.26	0.009
B	7320	58.01	0.45	18.20	0.008
B	19,150	44.10	1.015	18.53	0.006
C	30.4	72.51	0.104		
C	85.1	72.58	0.061	52.16	0.319
C	243.1	72.32	0.089	49.38	0.063
C	760	73.87	0.161	48.51	0.020
C	3252			48.42	0.014
C	8510	67.02	0.867	48.91	0.017
D	17.1	88.73	0.190	142.55	9.537
D	47.9	89.04	0.076	134.31	0.442
D	137	88.67	0.107	134.68	0.066
D	427.5				
D	1830	92.95	0.555	133.84	0.044
D	4787.5	91.19	2.119	134.41	0.053

TABLE III

MEASUREMENTS OF COIL RESISTANCE AND INDUCTANCE
AS A FUNCTION OF FREQUENCY AND LIFT-OFF

Coil	Frequency (hertz)	Lift-Off (inches)	Resistive Component (ohms)	Standard Deviation	Inductive Component (milli-henries)	Standard Deviation
A	273.42	0.0143	108.93	0.153	7.29	0.742
A	765.6	0.0143	112.15	0.267	6.23	0.168
A	2187.4	0.0143	120.35	0.248	5.50	0.023
A	6835.5	0.0143	139.96	0.193	4.82	< 0.001
A	29,280	0.0143	196.06	0.446	4.34	< 0.001
A	76,600	0.0143	217.24	5.789	4.07	0.004
A	273.42	0.0286	109.21	0.161	6.42	0.906
A	765.6	0.0286	111.33	0.154	6.33	0.120
A	2187.4	0.0286	118.28	0.153	5.73	0.014
A	6835.5	0.0286	134.13	0.155	5.17	0.004
A	29,280	0.0286	178.21	0.563	4.81	0.003
A	76,600	0.0286	184.57	5.550	4.62	0.007
A	273.42	0.0571	108.71	0.363	6.29	1.709
A	765.6	0.0571	110.80	0.113	6.49	0.101
A	2187.4	0.0571	115.55	0.155	6.15	0.019
A	6835.5	0.0571	126.42	0.300	5.75	0.003
A	29,280	0.0571	156.26	0.590	5.56	0.004
A	76,600	0.0571	151.94	15.678	5.42	0.011
A	273.42	0.1143	108.46	0.129	6.96	2.118
A	765.6	0.1143	109.88	0.213	6.79	0.119
A	2187.4	0.1143	112.23	0.143	6.65	0.009
A	6835.5	0.1143	117.73	0.365	6.45	0.004
A	29,280	0.1143	133.79	0.413	6.41	0.003
A	76,600	0.1143	89.22	14.482	6.40	0.007
B	68.4	0.0286	59.37	0.16	18.11	1.20
B	191.5	0.0286	61.44	0.06	15.38	0.12
B	547	0.0286	66.56	0.12	13.42	0.05
B	1710	0.0286	78.05	0.09	11.73	0.008
B	7320	0.0286	109.77	0.38	10.35	< 0.01
B	19,150	0.0286	145.70	2.08	9.96	< 0.01
B	68.4	0.0571	58.91	0.056	17.58	0.769
B	191.5	0.0571	60.74	0.057	15.68	0.120
B	547	0.0571	65.05	0.182	14.09	0.038
B	1710	0.0571	74.57	0.081	12.60	0.009
B	7320	0.0571	99.77	0.297	11.51	0.008
B	19,150	0.0571	127.07	0.956	11.23	< 0.001

TABLE III (continued)

Coil	Frequency (hertz)	Lift-Off (inches)	Resistive Component (ohms)	Standard Deviation	Inductive Component (milli-henries)	Standard Deviation
B	68.4	0.1143	58.99	0.097	17.25	1.129
B	191.5	0.1143	60.31	0.060	16.29	0.150
B	547	0.1143	63.43	0.058	15.07	0.038
B	1710	0.1143	69.34	0.101	14.08	< 0.001
B	7320	0.1143	85.81	0.170	13.34	0.004
B	19,150	0.1143	97.71	0.589	13.24	0.006
B	68.4	0.2286	58.84	0.085	17.72	0.616
B	191.5	0.2286	59.53	0.067	17.17	0.106
B	547	0.2286	61.37	0.072	16.33	0.021
B	1710	0.2286	64.17	0.103	15.80	0.004
B	7320	0.2286	72.08	0.289	15.40	0.008
B	19,150	0.2286	72.00	0.647	15.50	0.006
C	30.4	0.0429	73.50	0.101	47.14	2.831
C	85.1	0.0429	76.00	0.091	41.19	0.484
C	243.1	0.0429	82.10	0.098	36.15	0.088
C	760	0.0429	97.13	0.155	31.07	0.013
C	3252	0.0429	136.21	0.455	27.65	0.009
C	8510	0.0429	181.81	0.326	26.18	0.007
C	30.4	0.0857	73.28	0.062	49.17	1.595
C	85.1	0.0857	75.43	0.046	42.15	0.336
C	243.1	0.0857	80.62	0.074	37.96	0.069
C	760	0.0857	92.69	0.115	33.63	0.014
C	3252	0.0857	122.17	0.270	30.94	0.008
C	8510	0.0857	157.62	0.393	27.79	0.009
C	30.4	0.1715	73.57	0.092	45.34	2.427
C	85.1	0.1715	74.96	0.047	43.22	0.325
C	243.1	0.1715	78.52	0.076	40.68	0.064
C	760	0.1715	86.89	0.156	37.41	0.018
C	3252	0.1715	105.39	0.156	35.80	0.009
C	8510	0.1715	126.49	0.815	34.97	0.010
C	30.4	0.3429	72.82	0.109	46.96	3.067
C	85.1	0.3429	73.76	0.097	45.50	0.522
C	243.1	0.3429	75.70	0.091	44.02	0.054
C	760	0.3429	80.42	0.272	41.96	0.026
C	3252	0.3429	88.55	0.263	41.34	0.015
C	8510	0.3429	95.74	0.555	40.98	0.009
D	17.1	0.0571	90.27	0.212	137.94	8.268
D	47.9	0.0571	94.44	0.083	112.34	0.641
D	137	0.0571	103.68	0.066	98.45	0.123

TABLE III (continued)

Coil	Frequency (hertz)	Lift-Off (inches)	Resistive Component (ohms)	Standard Deviation	Inductive Component (milli-henries)	Standard Deviation
D	1830	0.0571	185.36	0.289	76.24	0.022
D	4787	0.0571	259.82	0.709	72.88	0.021
D	17.1	0.1143	89.69	0.241	132.35	13.851
D	47.9	0.1143	93.49	0.063	114.32	0.452
D	137	0.1143	101.19	0.164	103.24	0.186
D	427.5	0.1143	119.66	0.284	93.23	0.103
D	1830	0.1143	167.28	0.380	84.52	0.022
D	4787.5	0.1143	226.04	1.273	82.04	0.019
D	17.1	0.2286	89.67	0.319	136.48	15.649
D	47.9	0.2286	92.51	0.063	119.39	0.523
D	137	0.2286	97.76	0.081	110.74	0.055
D	427.5	0.2286	110.31	0.573	103.80	0.095
D	1830	0.2286	142.33	0.338	97.74	0.034
D	4787.5	0.2286	180.32	0.753	96.75	0.025
D	17.1	0.4572	89.41	0.351	141.31	15.444
D	47.9	0.4572	91.21	0.065	124.41	0.588
D	427.5	0.4572	100.42	0.274	116.57	0.102
D	1830	0.4572	117.45	0.484	113.11	0.036
D	4787.5	0.4572	135.72	1.146	113.45	0.032

A summary of the results listed in the tables is plotted in Figure 15, page 58, where it is compared with data calculated by the relaxation method. Each experimental point represents the average of all four coils, weighted according to their standard deviations.

Discussion of Errors in Impedance Measurements

One limitation in the accuracy was the accuracy of the alternating current digital voltmeter (which reads to four places) used. This was specified to be plus or minus two digits in the last place, plus or minus 0.1 per cent of the reading from 50 hertz to 20 kilohertz, plus or minus 0.1 per cent of full scale from 20 kilohertz to 50 kilohertz, and plus or minus 0.3 per cent of full scale from 50 kilohertz to 100 kilohertz. The voltmeter had an input impedance of 10 megohms shunted by 20 picofarads. The maximum decrease in the reading in the worst possible case is less than 0.1 per cent, so the effects of the voltmeter loading were neglected. Thus, the basic accuracy on most of the measurements was on the order of plus or minus 0.1 per cent.

The interwinding capacitance and the coil-to-metal capacitance probably contributed some error to the measurements, but these were generally quite small. The first-order effect of the interwinding capacitance should have been canceled out due to the fact that we normalized the data at a particular frequency using the values measured in air at the same frequency.

The heating of the coils caused a resistance change of about 0.35 per cent per degree centigrade. Also it caused some small increase (approximately 20×10^{-6} inches per inch per degree centigrade) in the dimensions of the coils. The maximum temperature change was less than 10°C.

Perhaps the largest error was due in some cases to harmonic distortion. The specifications on the oscillator and the power amplifier were for less than one per cent total harmonic distortion from 50 hertz to 20 kilohertz. However, for certain frequencies outside this band, this distortion level was exceeded.

The dimensional tolerances on the coils were fairly close, in most cases less than one per cent. The error in spacing was generally less than 0.001 inch. Also the variations in coil dimensions will probably produce second-order errors in the results and cancel when the average of the four coils is taken.

The results are rather inaccurate when the frequency is low, for, in the normalization, one large number is subtracted from another, giving a number with a relatively large standard deviation. The accuracy of the resistive component decreases at higher frequencies. However, it is divided by a large number in the normalization, which reduces the error to a small value relative to the inductive component. As the frequency is increased, the accuracy improves, becoming quite good for the last four frequencies.

Force Measurements

The net eddy-current force on a conductor was measured as outlined in Figure 19. The force that a coil encased in a ferrite cup exerted

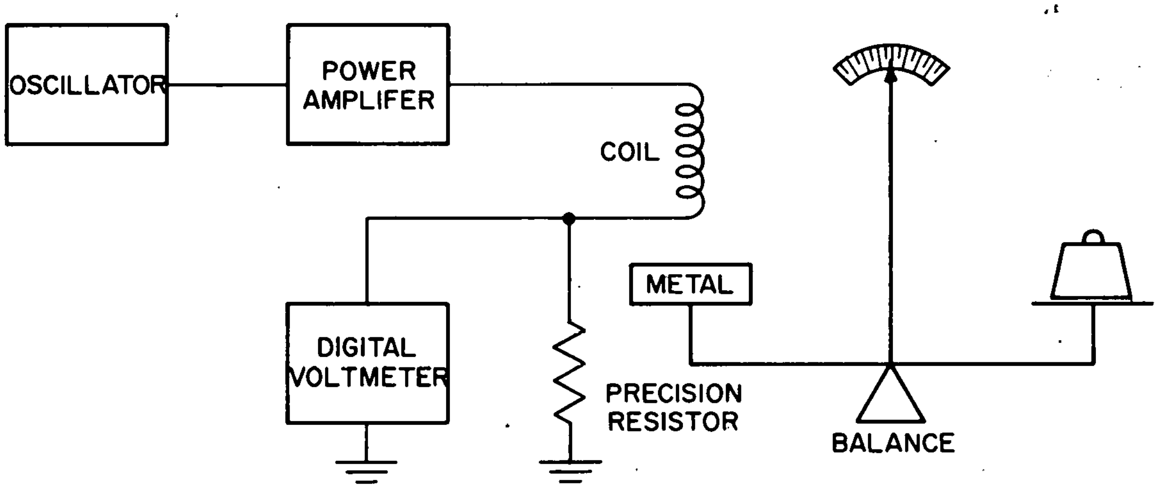

Figure 19. Force measurement apparatus.

on a copper ring was measured at three frequencies and various current levels.

The weight of the copper disk was exactly balanced by a counterweight. Then a small additional weight was added, and the current needed to rebalance the weight was recorded. A small weight was run up and down the pointer, adjusting the sensitivity of the balance by moving the center of gravity. The system could be adjusted from stable to bi-stable by moving the center of gravity from below the balance point to above the balance point.

Since the force is directly proportional to the square of the applied current, the force divided by the current squared should be a constant. Table IV gives a summary of the force measurements.

Discussion of Errors in Force Measurements

Some of the same errors exist in the force measurement as occurred in the impedance measurements. The digital voltmeter again limited the overall accuracy to plus or minus 0.1 per cent plus or minus two digits. The error due to heating the coil was considerably smaller, due to the higher Q of the coil. This was offset, however, by more heating of the copper disk, due to the higher powers used. The harmonic distortion was less than one per cent.

The balance used to measure the force had a sensitivity of about plus or minus 0.001 gram.

The overall accuracy of the force measurements was on the order of one per cent. The measured forces are compared with the results of a relaxation calculation at the end of Chapter VI.

TABLE IV

FORCE ON A COPPER RING

Lift-Off (inches)	Frequency (hertz)	Force (grams)	I (root mean square) (amps)	f/I^2 (root mean square) (grams/amps^2)
1/8	138.5	1.000	0.732	1.867
1/8	138.5	4.000	1.450	1.902
1/8	414	1.000	0.564	3.14
1/8	414	4.000	1.121	3.18
1/8	1656	1.000	0.480	4.34
1/8	1656	4.000	0.963	4.31
1/16	138.2	1.000	0.612	2.67
1/16	138.2	2.000	0.874	2.62
1/16	138.9	2.000	0.868	2.65
1/16	414	1.000	0.474	4.45
1/16	414	2.000	0.678	4.35
1/16	414	3.000	0.836	4.29
1/16	414	4.000	0.964	4.30
1/16	414	5.000	1.086	4.24
1/16	1656	1.000	0.414	5.83
1/16	1656	2.000	0.580	5.94
1/16	1656	3.000	0.722	5.76
1/16	1656	4.000	0.828	5.83
1/16	1656	5.000	0.940	5.66

CHAPTER VIII

RECOMMENDATIONS AND CONCLUSIONS

This thesis has presented two different methods of determining the vector potential of a cylindrical coil in the presence of conductors. From this vector potential, any electromagnetic induction phenomenon can be determined. The relaxation solution has the advantage that it is very versatile. It can be solved for any size and shape coil and conductor (so long as axial symmetry is retained) with any type of driving current in an inhomogeneous, nonlinear medium. However, it does have the disadvantage that it is an expensive process. The closed form solution, although more restricted in its use, should be more accurate and cheaper to apply. If a simple and accurate approximation could be made for the closed form integral equations, these calculations could be made quickly and cheaply with only a pencil and paper.

We can use superposition of solutions to overcome some of the restrictions of axial symmetry in both cases.

These two methods should allow us to solve a very large number of difficult electromagnetic induction problems.

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LIST OF SYMBOLS

In the first column the symbol used is given, and in the second column the name. In the third column the meter-kilogram-second (MKS) units are given. In the last column the dimensions are given in terms of mass (M), length (L), time (T), and electric charge (Q).

Symbol	Name	MKS Units	Dimensions
A	vector potential	$\frac{\text{webers}}{\text{meter}}$	$\frac{ML}{TQ}$
a	distance between mesh points	meters	L
B	magnetic induction	$\frac{\text{webers}}{\text{meter}^2}$	$\frac{M}{TQ}$
D	electric displacement	$\frac{\text{coulomb}}{\text{meter}^2}$	$\frac{Q}{L^2}$
E	electric intensity	$\frac{\text{volt}}{\text{meter}}$	$\frac{ML}{T^2Q}$
F	force	newtons	$\frac{ML}{T^2}$
H	magnetic intensity	$\frac{\text{ampere}}{\text{meter}}$	$\frac{Q}{TL}$
I	applied current	ampere	$\frac{Q}{T}$
I	impulse	newton-sec	$\frac{ML}{T}$
J_o	applied current density	$\frac{\text{ampere}}{\text{meter}^2}$	$\frac{Q}{TL^2}$
J	current density	$\frac{\text{ampere}}{\text{meter}^2}$	$\frac{Q}{TL^2}$
j	square root of minus one		

Symbol	Name	MKS Units	Dimensions
t	time	seconds	T
ϵ	dielectric constant	$\frac{\text{farad}}{\text{meter}}$	$\frac{T^2Q^2}{ML^3}$
K_e	relative permittivity (ϵ/ϵ_o)		
μ	permeability	$\frac{\text{henry}}{\text{meter}}$	$\frac{ML}{Q^2}$
K_m	relative permeability (μ/μ_o)		
ρ	charge density	$\frac{\text{coulomb}}{\text{meter}^3}$	$\frac{Q}{L^3}$
S	poynting vector $(\vec{E} \times \vec{H})$	$\frac{\text{watts}}{\text{meter}^2}$	$\frac{M}{T^3}$
σ	conductivity	$\frac{\text{mho}}{\text{meter}}$	$\frac{TQ^2}{ML^3}$
τ	time interval	seconds	T
ω	angular frequency	$\frac{\text{radians}}{\text{second}}$	$\frac{1}{T}$

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