SOLUTIONS TO SECOND-ORDER THREE-POINT PROBLEMS ON TIME SCALES

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Abstract. In the first part of the paper we establish the existence of multiple positive solutions to the nonlinear second-order three-point boundary value problem on time scales,

$$u^{\Delta\nabla}(t) + f(t, u(t)) = 0, \quad u(0) = 0,\ \alpha u(\eta) = u(T)$$

for $t \in [0, T] \subset \mathbb{T}$, where $\mathbb{T}$ is a time scale, $\alpha > 0$, $\eta \in (0, \rho(T)) \subset \mathbb{T}$, and $\alpha\eta < T$. We employ the Leggett-Williams fixed-point theorem in an appropriate cone to guarantee the existence of at least three positive solutions to this nonlinear problem. In the second part we establish the existence of at least one positive solution to the related problem

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \quad u(0) = 0,\ \alpha u(\eta) = u(T),$$

using Krasnoselskii's fixed-point theorem of cone expansion and compression of norm type.

1. Preliminaries about time scales

The following definitions, that can be found in Atici and Guseinov [4] and Bohner and Peterson [7], lay out the terms and notation needed later in the discussion. A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$

$$\sigma(t) = \inf\{s \in \mathbb{T} :\ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} :\ s < t\}$$

(supplemented by $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. The forward graininess is $\mu(t) := \sigma(t) - t$. Similarly, the backward graininess is $\nu(t) := t - \rho(t)$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative [7] of f at t, denoted $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|$$

for all $s \in U$. For $\mathbb{T} = \mathbb{R}$, we have $f^\Delta = f'$, the usual derivative, and for $\mathbb{T} = \mathbb{Z}$ we have the forward difference operator, $f^\Delta(t) = f(t+1) - f(t)$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative [4] of f at t, denoted $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \le \varepsilon|\rho(t) - s|$$

for all $s \in U$. For $\mathbb{T} = \mathbb{R}$, we have $f^\nabla = f'$, the usual derivative, and for $\mathbb{T} = \mathbb{Z}$ we have the backward difference operator, $f^\nabla(t) = f(t) - f(t-1)$.

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A function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous or ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous if and only if f is continuous. If $\mathbb{T} = \mathbb{Z}$, then any function is ld-continuous. It is known [7] that if f is ld-continuous, then there is a function $F(t)$ such that $F^{\nabla}(t) = f(t)$. In this case, we define

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

2. Introduction to the boundary value problem

We will be concerned with proving the existence of solutions to the second-order three-point nonlinear boundary value problem on a time scale $\mathbb{T}$ given by

$$u^{\Delta\nabla}(t) + f(t, u(t)) = 0 \quad t \in (0, T) \subset \mathbb{T} \tag{1}$$

$$u(0) = 0, \ \alpha u(\eta) = u(T), \tag{2}$$

where Δ is the delta derivative and ∇ is the nabla derivative. Throughout the paper we assume $\eta \in (0, \rho(T)) \subset \mathbb{T}$ for $0 \in \mathbb{T}_\kappa$, $T \in \mathbb{T}^\kappa$, $\alpha > 0$, and $\alpha\eta < T$. We likewise assume that $f : [0, T] \times [0, \infty) \to [0, \infty)$ is left-dense continuous, and $f(t, \cdot)$ does not vanish identically on any subset of $[0, T] \subset \mathbb{R}$ of positive measure. This boundary value problem (1), (2) was studied by He and Ge [11] and Ma [14],[15],[16] in the case of $\mathbb{T} = \mathbb{R}$, on the unit interval; consequently these results are new for difference equations as well as for the general time scale that contains 0. We seek to show the existence of at least three positive solutions for (1), (2). Some papers in this area include [1],[2],[3],[5],[6],[8],[9]. In this paper we will apply the existence theorem of Leggett and Williams, given below, that is an application of fixed-point index theory.

3. Leggett-Williams theorem

In this brief section we introduce the main terminology needed for discussion of fixed points for operators on cones in a Banach space; the theorem below is the Leggett-Williams fixed point theorem, whose proof can be found in Guo and Lakshmikantham [10], or Leggett and Williams [13].

A nonempty closed convex set P contained in a real Banach space E is called a cone if it satisfies the following two conditions:

(i) if $x \in P$ and $\lambda \geq 0$ then $\lambda x \in P$;
(ii) if $x \in P$ and $-x \in P$ then $x = 0$.

The cone P induces an ordering $\leq$ on E by $x \leq y$ if and only if $y - x \in P$. An operator A is said to be completely continuous if it is continuous and compact (maps bounded sets into relatively compact sets). A map ψ is a nonnegative continuous concave functional on P if it satisfies the following conditions:

(i) $\psi : P \to [0, \infty)$ is continuous;
(ii) $\psi(tx + (1 - t)y) \geq t\psi(x) + (1 - t)\psi(y)$ for all $x, y \in P$ and $0 \leq t \leq 1$.

Let

$$P_c := \{x \in P : \|x\| \ < c\}$$

and

$$P(\psi, a, b) := \{x \in P : a \le \psi(x),\ \|x\| \le b\}.$$

Theorem 1. *Let P be a cone in the real Banach space E, $A : \overline{P_c} \to \overline{P_c}$ be completely continuous and ψ be a nonnegative continuous concave functional on P with $\psi(x) \le \|x\|$ for all $x \in \overline{P_c}$. Suppose there exists $0 < a < b < d \le c$ such that the following conditions hold:*

(i) $\{x \in P(\psi, b, d) : \psi(x) > b\} \neq \emptyset$ *and* $\psi(Ax) > b$ *for all* $x \in P(\psi, b, d)$*;*
(ii) $\|Ax\| < a$ *for* $\|x\| \le a$*;*
(iii) $\psi(Ax) > b$ *for* $x \in P(\psi, b, c)$ *with* $\|Ax\| > d$*.*

Then A has at least three fixed points x_1, x_2, and x_3 in $\overline{P_c}$ satisfying:

$$\|x_1\| < a, \quad \psi(x_2) > b, \quad a < \|x_3\| \ \text{with} \ \psi(x_3) < b.$$

4. BACKGROUND LEMMAS

To prove the main existence result we will employ several straightforward lemmas. These lemmas are based on the linear boundary value problem

$$u^{\Delta\nabla}(t) + y(t) = 0, \quad t \in (0, T) \subset \mathbb{T} \tag{3}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(T). \tag{4}$$

Lemma 2. *If $\alpha\eta \neq T$, then for $y \in C_{ld}[0, T]$ the boundary value problem* (3), (4) *has the unique solution*

$$u(t) := -\int_0^t (t-s)y(s)\nabla s - \frac{\alpha t}{T-\alpha\eta}\int_0^\eta (\eta - s)y(s)\nabla s + \frac{t}{T-\alpha\eta}\int_0^T (T-s)y(s)\nabla s. \tag{5}$$

Proof. Let u be as in (5). Routine calculations verify that u satisfies the boundary conditions in (4). By Theorem 2.10 (*iii*) in [4] or Theorem 8.50 (iii) in [7, p333],

$$\left(\int_a^t f(t,s)\nabla s\right)^\Delta = f(\sigma(t), \sigma(t)) + \int_a^t f^\Delta(t,s)\nabla s$$

if f, f^Δ are continuous. Using this theorem to take the delta derivative of (5) we have

$$u^\Delta(t) = -(\sigma(t) - \sigma(t))y^\sigma(t) - \int_0^t y(s)\nabla s - \frac{\alpha}{T-\alpha\eta}\int_0^\eta (\eta - s)y(s)\nabla s + \frac{1}{T-\alpha\eta}\int_0^T (T-s)y(s)\nabla s.$$

Taking the nabla derivative of this expression yields $u^{\Delta\nabla}(t) = -y(t)$, so that u given in (5) is a solution of (3), (4).

Briefly consider the boundary value problem

$$x^{\Delta\nabla} = 0, \quad x(0) = 0, \quad \alpha x(\eta) = x(T). \tag{6}$$

A solution x must be linear, and the first boundary condition implies $x = mt$. Applying the second boundary condition, we see that $m(\alpha\eta - T) = 0$. Since $\alpha\eta \neq T$, $m = 0$. Therefore (6) has only the trivial solution. Now suppose u and w are solutions of (3), (4); set $x := u - w$. Then x satisfies (6), so that $x \equiv 0$. Thus $u = w$ and u in (5) is the unique solution. □

Lemma 3. *If* $u(0)=0$ *and* $u^{\Delta\nabla}\le 0$, *then* $\frac{u(T)}{T}\le\frac{u(t)}{t}$ *for all* $t\in(0,T]\subset\mathbb{T}$.

Proof. Let $h(t):=u(t)-\frac{tu(T)}{T}$. Then $h(0)=h(T)=0$ and $h^{\Delta\nabla}\le 0$ so that $h(t)\ge 0$ on $[0,T]$. □

Lemma 4. *Let* $0<\alpha<T/\eta$. *If* $y\in C_{ld}[0,T]$ *and* $y\ge 0$, *the unique solution* u *of* (3), (4) *satisfies*

$$u(t)\ge 0,\quad t\in[0,T]\subset\mathbb{T}.$$

Proof. From the fact that $u^{\Delta\nabla}(t)=-y(t)\le 0$, we know that the graph of u is concave down on $(0,T)$. If $u(T)\ge 0$, then the concavity of u and the boundary condition $u(0)=0$ imply that $u(t)\ge 0$ for $t\in[0,T]$. If $u(T)<0$, then we have $u(\eta)<0$ and

$$u(T)/T=\alpha u(\eta)/T>u(\eta)/\eta,$$

a contradiction of Lemma 3. □

Lemma 5. *Let* $\alpha\eta>T$. *If* $y\in C_{ld}[0,T]$ *and* $y\ge 0$, *then* (3), (4) *has no nonnegative solution.*

Proof. Assume (3), (4) has a nonnegative solution u. If $u(T)>0$, then $u(\eta)>0$ and

$$u(T)/T=\alpha u(\eta)/T>u(\eta)/\eta,$$

a contradiction of Lemma 3. If $u(T)=0$ and $u(\tau)>0$ for some $\tau\in(0,T)$, then $u(\eta)=u(T)=0$, where $\tau\ne\eta$. If $\tau\in(0,\eta)$, then $u(\tau)>u(\eta)=u(T)$, a contradiction of the concavity of u. If $\tau\in(\eta,T)$, then $u(0)=u(\eta)<u(\tau)$, another violation of the concavity of u. Therefore $u(T)<0$, so that no nonnegative solution exists. □

Remark 6. *In view of Lemma 5, in the rest of this paper we assume* $\alpha\eta<T$. *The work will be in the Banach space* $C_{ld}[0,T]$ *with the sup norm.*

Lemma 7. *Let* $0<\alpha<T/\eta$. *If* $y\in C_{ld}[0,T]$ *and* $y\ge 0$, *then the unique solution* u *as in* (5) *of* (3), (4) *satisfies*

$$\inf_{t\in[\eta,T]}u(t)\ge r\|u\|, \tag{7}$$

where

$$r:=\min\left\{\frac{\alpha(T-\eta)}{T-\alpha\eta},\frac{\alpha\eta}{T},\frac{\eta}{T}\right\}>0. \tag{8}$$

Proof. First consider the case where $0<\alpha<1$. By the second boundary condition we know that $u(\eta)\ge u(T)$. Pick $t_0\in(0,T)$ such that $u(t_0)=\|u\|$. If $t_0\le\eta<T$, then

$$\min_{t\in[\eta,T]}u(t)=u(T)$$

and

$$\begin{aligned}u(t_0)&\le u(T)+\frac{u(T)-u(\eta)}{T-\eta}(0-T)\\&=\frac{-\eta u(T)+Tu(\eta)}{T-\eta}\\&=\frac{(T-\alpha\eta)u(T)}{\alpha(T-\eta)}.\end{aligned}$$

Therefore

$$\min_{t\in[\eta,T]} u(t) \geq \frac{\alpha(T-\eta)}{T-\alpha\eta}\|u\|.$$

If $\eta \leq t_0 < T$, again we have $u(T) = \min_{t\in[\eta,T]} u(t)$. As in Lemma 3, $u(\eta)/\eta \geq u(t_0)/t_0$. Using the boundary condition $\alpha u(\eta) = u(T)$, we find that $u(T) > \alpha\eta u(t_0)/T$, so that

$$\min_{t\in[\eta,T]} u(t) > \frac{\alpha\eta}{T}\|u\|.$$

Now consider the case $1 \leq \alpha < T/\eta$. The boundary condition this time implies $u(\eta) \leq u(T)$. Set $u(t_0) = \|u\|$. Note that by the concavity of u we have $t_0 \in [\eta, T]$ and $\min_{t\in[\eta,T]} u(t) = u(\eta)$. Once again by Lemma 3 it follows that $u(\eta)/\eta \geq u(t_0)/t_0$, so that

$$\min_{t\in[\eta,T]} u(t) \geq \frac{\eta}{T}\|u\|.$$

$\square$

Remark 8. *Below we will make use of the constants*

$$m := \left(\frac{2T-\alpha\eta}{T-\alpha\eta}\int_0^T (T-s)\nabla s + \frac{\alpha T}{T-\alpha\eta}\int_0^\eta (\eta-s)\nabla s\right)^{-1} \tag{9}$$

and

$$\delta := \min\left\{\frac{\eta}{T-\alpha\eta}\int_\eta^T (T-s)\nabla s,\ \frac{\alpha\eta}{T-\alpha\eta}\int_\eta^T (T-s)\nabla s\right\}, \tag{10}$$

where $\delta > 0$ since $0 < \eta < \rho(T)$ and $\alpha\eta < T$. (If $\eta = \rho(T)$, then the nabla integral would be zero.) For example, if $\mathbb{T} = \mathbb{R}$, then m is given by

$$m = \left(\frac{2T-\alpha\eta}{T-\alpha\eta}\int_0^T (T-s)ds + \frac{\alpha T}{T-\alpha\eta}\int_0^\eta (\eta-s)ds\right)^{-1} = \frac{2(T-\alpha\eta)}{T(2T^2-\alpha\eta T+\alpha\eta^2)}$$

and

$$\delta = \min\left\{\frac{\eta(T-\eta)^2}{2(T-\alpha\eta)},\ \frac{\alpha\eta(T-\eta)^2}{2(T-\alpha\eta)}\right\}.$$

If $\mathbb{T} = \mathbb{Z}$, then m is given by

$$m = \left(\frac{2T-\alpha\eta}{T-\alpha\eta}\sum_{s=1}^{T}(T-s) + \frac{\alpha T}{T-\alpha\eta}\sum_{s=1}^{\eta}(\eta-s)\right)^{-1} = \frac{2(T-\alpha\eta)}{T(2T^2-2T-\alpha\eta T+\alpha\eta^2)}$$

and

$$\delta = \min\left\{\frac{\eta(T-\eta)(T-\eta-1)}{2(T-\alpha\eta)},\ \frac{\alpha\eta(T-\eta)(T-\eta-1)}{2(T-\alpha\eta)}\right\}.$$

For a general time scale, however, these constants are difficult to calculate; see Bohner and Peterson [7] for a thorough introduction to the calculus on time scales and its computational limitations.

5. TRIPLE POSITIVE SOLUTIONS

Let the Banach space $E = C_{ld}[0,T]$ be endowed with the sup norm, and define the cone $P \subset E$ by

$$P = \{u \in E : u \text{ concave and nonnegative valued on } [0,T]\}.$$

Let the nonnegative continuous concave functional $\psi : P \to [0,\infty)$ by defined by

$$\psi(u) = \min_{t\in[\eta,T]} u(t), \quad u \in P. \tag{11}$$

Note that for $u \in P$, $\psi(u) \le \|u\|$, and by Lemma 2 u is a solution of the boundary value problem (1), (2) if and only if u has the form given in (5).

Theorem 9. *Suppose that there exist constants $0 < a < b < b/r \le c$ such that*

(D_1) $f(t,u) < ma$ *for* $t \in [0,T]$, $u \in [0,a]$,
(D_2) $f(t,u) \ge b/\delta$ *for* $t \in [\eta,T]$, $u \in [b, b/r]$,
(D_3) $f(t,u) \le mc$ *for* $t \in [0,T]$, $u \in [0,c]$,

where r, m and δ are as defined in (8), (9), *and* (10), *respectively. Then the boundary value problem* (1), (2) *has at least three positive solutions u_1, u_2, u_3 satisfying*

$$\|u_1\| < a, \qquad b < \psi(u_2), \qquad \|u_3\| > a \;\text{ with }\; \psi(u_3) < b.$$

Proof. Define the operator $A : P \to E$ by

$$\begin{aligned} Au(t) &= -\int_0^t (t-s)f(s,u(s))\nabla s - \frac{\alpha t}{T-\alpha\eta}\int_0^\eta (\eta-s)f(s,u(s))\nabla s \\ &\quad + \frac{t}{T-\alpha\eta}\int_0^T (T-s)f(s,u(s))\nabla s. \end{aligned}$$

Note that if $u \in P$, the fact that f is nonnegative and Lemma 4 imply that $Au(t) \ge 0$ for $t \in [0,T]$. Since $(Au)^{\Delta\nabla}(t) = -f(t,u(t))$ for $t \in (0,T)$, we see that $Au \in P$; i.e., $A : P \to P$. Moreover, A is completely continuous.

We now show that all of the conditions of Theorem 1 are satisfied. For all $u \in P$ we have $\psi(u) \le \|u\|$. If $u \in \overline{P_c}$, then $\|u\| \le c$ and assumption (D_3) implies $f(t,u(t)) \le mc$ for $t \in [0,T]$.

As a result,

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,T]} -\int_0^t (t-s)f(s,u(s))\nabla s \\
&\quad -\frac{\alpha t}{T-\alpha\eta}\int_0^\eta (\eta-s)f(s,u(s))\nabla s + \frac{t}{T-\alpha\eta}\int_0^T (T-s)f(s,u(s))\nabla s \\
&\leq \max_{t\in[0,T]} \int_0^t (t-s)f(s,u(s))\nabla s \\
&\quad +\frac{\alpha t}{T-\alpha\eta}\int_0^\eta (\eta-s)f(s,u(s))\nabla s + \frac{t}{T-\alpha\eta}\int_0^T (T-s)f(s,u(s))\nabla s \\
&\leq \max_{t\in[0,T]} \left(\int_0^t (t-s)\nabla s + \frac{\alpha t}{T-\alpha\eta}\int_0^\eta (\eta-s)\nabla s + \frac{t}{T-\alpha\eta}\int_0^T (T-s)\nabla s\right) mc \\
&= \left(\int_0^T (T-s)\nabla s + \frac{\alpha T}{T-\alpha\eta}\int_0^\eta (\eta-s)\nabla s + \frac{T}{T-\alpha\eta}\int_0^T (T-s)\nabla s\right) mc \\
&= c.
\end{aligned}$$

Therefore $A : \overline{P_c} \to \overline{P_c}$. In the same way, if $u \in P_a$, then assumption (D_1) yields $f(t,u(t)) < ma$ for $t \in [0,T]$; as in the argument above, it follows that $A : \overline{P_a} \to P_a$. Hence, condition (ii) of Theorem 1 is satisfied.

To check condition (i) of Theorem 1, choose $u_P(t) \equiv b/r$ for $t \in [0,T]$, where r is given in (8). Then $u_P \in P(\psi, b, b/r)$ and $\psi(u_P) = \psi(b/r) > b$, so that $\{u \in P(\psi, b, b/r) : \psi(u) > b\} \neq \emptyset$. Consequently, if $u \in P(\psi, b, b/r)$, then $b \leq u(s) \leq b/r$ for $s \in [\eta, T]$. From assumption (D_2) we have that

$$f(t,u(t)) \geq b/\delta$$

for $t \in [\eta, T]$; by the definitions of ψ and the cone P, we must distinguish two cases: $\psi(Au(t)) = Au(\eta)$ and $\psi(Au(t)) = Au(T)$.

First, suppose $\psi(Au(t)) = Au(\eta)$. Then

$$\begin{aligned}
\psi(Au) &= Au(\eta) \\
&= -\int_0^\eta (\eta-s)f(s,u(s))\nabla s - \frac{\alpha\eta}{T-\alpha\eta}\int_0^\eta (\eta-s)f(s,u(s))\nabla s \\
&\quad +\frac{\eta}{T-\alpha\eta}\int_0^T (T-s)f(s,u(s))\nabla s \\
&= -\frac{T}{T-\alpha\eta}\int_0^\eta (\eta-s)f(s,u(s))\nabla s + \frac{\eta}{T-\alpha\eta}\int_0^T (T-s)f(s,u(s))\nabla s \\
&= \frac{\eta T}{T-\alpha\eta}\int_\eta^T f(s,u(s))\nabla s + \frac{T}{T-\alpha\eta}\int_0^\eta sf(s,u(s))\nabla s - \frac{\eta}{T-\alpha\eta}\int_0^T sf(s,u(s))\nabla s \\
&> \frac{\eta}{T-\alpha\eta}\int_\eta^T Tf(s,u(s))\nabla s - \frac{\eta}{T-\alpha\eta}\int_\eta^T sf(s,u(s))\nabla s \\
&\geq \frac{b\eta}{\delta(T-\alpha\eta)}\int_\eta^T (T-s)\nabla s \\
&\geq b,
\end{aligned}$$

for δ as in (10).

Next, suppose $\psi(Au(t)) = Au(T)$. Then

$$\begin{aligned}
\psi(Au) &= Au(T) \\
&= -\int_0^T (T-s)f(s,u(s))\nabla s - \frac{\alpha T}{T-\alpha\eta}\int_0^\eta (\eta-s)f(s,u(s))\nabla s \\
&\quad + \frac{T}{T-\alpha\eta}\int_0^T (T-s)f(s,u(s))\nabla s \\
&= \frac{\alpha\eta}{T-\alpha\eta}\int_0^T (T-s)f(s,u(s))\nabla s - \frac{\alpha T}{T-\alpha\eta}\int_0^\eta (\eta-s)f(s,u(s))\nabla s \\
&= \frac{\alpha\eta T}{T-\alpha\eta}\int_\eta^T f(s,u(s))\nabla s + \frac{\alpha T}{T-\alpha\eta}\int_0^\eta sf(s,u(s))\nabla s - \frac{\alpha\eta}{T-\alpha\eta}\int_0^T sf(s,u(s))\nabla s \\
&> \frac{\alpha\eta}{T-\alpha\eta}\int_\eta^T Tf(s,u(s))\nabla s - \frac{\alpha\eta}{T-\alpha\eta}\int_\eta^T sf(s,u(s))\nabla s \\
&\geq \frac{b\alpha\eta}{\delta(T-\alpha\eta)}\int_\eta^T (T-s)\nabla s \\
&\geq b,
\end{aligned}$$

again for δ as in (10). In either case we have

$$\psi(Au) > b, \quad u \in P(\psi, b, b/r),$$

so that condition (i) of Theorem 1 holds.

Lastly we consider Theorem 1 (iii). Suppose $u \in P(\psi, b, c)$ with $\|Au\| > b/r$. Then, using the definition of ψ and Lemma 7, we see that

$$\begin{aligned}
\psi(Au) &= \min_{t\in[\eta,T]} Au(t) \\
&\geq r\|Au\| \\
&> rb/r \\
&= b.
\end{aligned}$$

$\square$

6. ONE POSITIVE SOLUTION

Now we study the related boundary value problem

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0 \quad t \in (0,T) \subset \mathbb{T} \tag{12}$$

$$u(0) = 0, \ \alpha u(\eta) = u(T), \tag{13}$$

where again $\eta \in (0, \rho(T)) \subset \mathbb{T}$ for $0 \in \mathbb{T}_\kappa$, $T \in \mathbb{T}^\kappa$, $\alpha > 0$, and $\alpha\eta < T$. Here

$(A1)$ $a \in C_{ld}[0,T]$ is nonnegative such that $a(t_0) > 0$ for at least one $t_0 \in [\eta, T)$

$(A2)$ $f : [0, \infty) \to [0, \infty)$ is continuous such that

$$f_0 := \lim_{u \to 0^+} \frac{f(u)}{u} \quad \text{and} \quad f_\infty := \lim_{u \to \infty} \frac{f(u)}{u}$$

both exist.

To establish the existence of at least one positive solution we will employ the following fixed-point theorem due to Krasnoselskii [12], that can also be found in the book by Guo [10].

Theorem 10. *Let E be a Banach space, $K \subseteq E$ be a cone, and suppose that Ω_1, Ω_2 are bounded open balls of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator such that either*

(i) $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$, *or*
(ii) $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_2$

holds. Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The boundary value problem (12), (13) has a solution u if and only if u is a fixed point of the operator

$$\begin{aligned} Au(t) &= -\int_0^t (t-s)a(s)f(u(s))\nabla s - \frac{\alpha t}{T-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(u(s))\nabla s \\ &\quad + \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\nabla s. \end{aligned}$$

Let $\mathcal{B}$ denote the Banach space $C_{ld}[0,T]$ with the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$. Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{x \in \mathcal{B} : x(t) \ge 0, \inf_{t \in [\eta, T]} x(t) \ge r\|x\|\}$$

for r given in (8). By Lemma 7, $A\mathcal{P} \subseteq \mathcal{P}$, and $A : \mathcal{P} \to \mathcal{P}$ is completely continuous.

Theorem 11. *Assume $(A1)$ and $(A2)$ hold. If either*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(f is superlinear), or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(f is sublinear),*

then (12), (13) has at least one positive solution.

Proof. First suppose f is superlinear. Since $f_0 = 0$, there exists an $H_1 > 0$ such that $f(u) \le \varepsilon u$ for $0 < u < H_1$, where ε is such that

$$\frac{\varepsilon}{T-\alpha\eta}\int_0^T (T-s)a(s)\nabla s \le \frac{1}{T}.$$

If $u \in \mathcal{P}$ with $\|u\| = H_1$, then

$$\begin{aligned} Au(t) &\le \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\nabla s \\ &\le \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)\varepsilon u(s)\nabla s \\ &\le \frac{\varepsilon\|u\|T}{T-\alpha\eta}\int_0^T (T-s)a(s)\nabla s \\ &\le H_1. \end{aligned}$$

It follows that if

$$\Omega_1 := \{u \in C_{ld}[0,T] : \|u\| < H_1\},$$

then $\|Au\| \le \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$. Since $f_\infty = \infty$, there exists an $\hat{H}_2 > 0$ such that $f(u) \ge ku$ for $u \ge \hat{H}_2$, where $k > 0$ is chosen such that

$$\frac{k\eta r}{T - \alpha\eta} \int_\eta^T (T-s)a(s)\nabla s \ge 1.$$

Set $H_2 = \max\{2H_1, \frac{\hat{H}_2}{r}\}$ and

$$\Omega_2 := \{u \in C_{ld}[0,T] : \|u\| < H_2\}.$$

If $u \in \mathcal{P}$ with $\|u\| = H_2$, then

$$\min_{t\in[\eta,T]} u(t) \ge r\|u\| \ge \hat{H}_2,$$

so that

$$\begin{aligned}
Au(\eta) &= -\int_0^\eta (\eta - s)a(s)f(u(s))\nabla s - \frac{\alpha\eta}{T-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(u(s))\nabla s \\
&\quad + \frac{\eta}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\nabla s \\
&= -\frac{T}{T-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(u(s))\nabla s + \frac{\eta}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\nabla s \\
&= \frac{\eta T}{T-\alpha\eta}\int_\eta^T a(s)f(u(s))\nabla s + \frac{T}{T-\alpha\eta}\int_0^\eta sa(s)f(u(s))\nabla s \\
&\quad - \frac{\eta}{T-\alpha\eta}\int_0^T sa(s)f(u(s))\nabla s \\
&> \frac{\eta}{T-\alpha\eta}\int_\eta^T Ta(s)f(u(s))\nabla s - \frac{\eta}{T-\alpha\eta}\int_\eta^T sa(s)f(u(s))\nabla s \\
&\ge \frac{k\eta r\|u\|}{T-\alpha\eta}\int_\eta^T (T-s)a(s)\nabla s \\
&\ge \|u\|.
\end{aligned}$$

In other words, if $u \in \mathcal{P} \cap \partial\Omega_2$, then $\|Au\| \ge \|u\|$. Thus by the first part of Theorem 10, it follows that A has a fixed point u in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$ with $H_1 \le \|u\| \le H_2$.

Now suppose f is sublinear. Since $f_0 = \infty$, there exists an $H_3 > 0$ such that $f(u) \ge mu$ for $0 < u < H_3$, where m is such that

$$\frac{\eta r m}{T - \alpha\eta} \int_\eta^T (T-s)a(s)\nabla s \ge 1.$$

Then as above,

$$\begin{aligned} Au(\eta) &\geq \frac{\eta}{T-\alpha\eta}\int_\eta^T (T-s)a(s)f(u(s))\nabla s \\ &\geq \frac{\eta r\|u\|m}{T-\alpha\eta}\int_\eta^T (T-s)a(s)\nabla s \\ &\geq \|u\| \\ &\geq H_3. \end{aligned}$$

Thus we let

$$\Omega_3 := \{u \in C_{ld}[0,T] : \|u\| < H_3\}$$

so that $\|Au\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_3$.

Next consider $f_\infty = 0$. By definition there exists $\hat{H}_4 > 0$ such that $f(u) \leq \lambda u$ for $u \geq \hat{H}_4$, where $\lambda > 0$ satisfies

$$\frac{\lambda}{T-\alpha\eta}\int_0^T (T-s)a(s)\nabla s \leq \frac{1}{T}. \tag{14}$$

Suppose f is bounded. Then $f(u) \leq M$ for all $u \in [0,\infty)$ for some constant $M > 0$. Pick

$$H_4 := \max\left\{2H_3,\ \frac{TM}{T-\alpha\eta}\int_0^T (T-s)a(s)\nabla s\right\}.$$

If $u \in \mathcal{P}$ with $\|u\| = H_4$, then

$$\begin{aligned} Au(t) &\leq \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\nabla s \\ &\leq \frac{TM}{T-\alpha\eta}\int_0^T (T-s)a(s)\nabla s \\ &\leq H_4, \end{aligned}$$

and $\|Au\| \leq \|u\|$.

Now suppose f is unbounded. From $(A2)$ there exists $H_4 \geq \max\{2H_3, \frac{\hat{H}_4}{r}\}$ such that $f(u) \leq f(H_4)$ for $0 < u \leq H_4$. If $u \in \mathcal{P}$ with $\|u\| = H_4$, then

$$\begin{aligned} Au(t) &\leq \frac{t}{T-\alpha\eta}\int_0^T (T-s)a(s)f(u(s))\nabla s \\ &\leq \frac{T}{T-\alpha\eta}\int_0^T (T-s)a(s)f(H_4)\nabla s \\ &\leq \frac{\lambda H_4 T}{T-\alpha\eta}\int_0^T (T-s)a(s)\nabla s \\ &\leq H_4 \end{aligned}$$

using (14).

Consequently, in either case we take

$$\Omega_4 := \{u \in C_{ld}[0,T] : \|u\| < H_4\}$$

so that for $u \in \mathcal{P} \cap \partial\Omega_4$ we have $\|Au\| \leq \|u\|$. Thus by the second part of Theorem 10, it follows that A has a fixed point u in $\mathcal{P} \cap (\overline{\Omega}_4 \backslash \Omega_3)$ with $H_3 \leq \|u\| \leq H_4$. □

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