



Published in final edited form as:

*IEEE Trans Control Syst Technol.* 2015 March ; 23(2): 770–777. doi:10.1109/TCST.2014.2343935.

## Solutions to the Inverse LQR Problem with Application to Biological Systems Analysis

**M Cody Priess,**

Michigan State University Dept. of Mechanical Engineering and the MSU Center for Orthopedic Research (MSUCOR), East Lansing, MI 48824

**Richard Conway,**

Western Digital Inc

**Jongeeun Choi [Member IEEE],**

MSU Dept. of Mechanical Engineering, Dept. of Electrical and Computer Engineering, and MSUCOR

**John M Popovich Jr., and**

MSU Dept. of Surgical Specialties and MSUCOR

**Clark Radcliffe**

Michigan State University Dept. of Mechanical Engineering and the MSU Center for Orthopedic Research (MSUCOR), East Lansing, MI 48824

M Cody Priess: priessma@msu.edu; Richard Conway: rcon-way345@gmail.com; Jongeeun Choi: jchoi@egr.msu.edu; John M Popovich: popovi16@msu.edu; Clark Radcliffe: radcliff@egr.msu.edu

### Abstract

In this paper, we present a set of techniques for finding a cost function to the time-invariant Linear Quadratic Regulator (LQR) problem in both continuous- and discrete-time cases. Our methodology is based on the solution to the inverse LQR problem, which can be stated as: does a given controller  $K$  describe the solution to a time-invariant LQR problem, and if so, what weights  $Q$  and  $R$  produce  $K$  as the optimal solution? Our motivation for investigating this problem is the analysis of motion goals in biological systems. We first describe an efficient Linear Matrix Inequality (LMI) method for determining a solution to the general case of this inverse LQR problem when both the weighting matrices  $Q$  and  $R$  are unknown. Our first LMI-based formulation provides a unique solution when it is feasible. Additionally, we propose a gradient-based, least-squares minimization method that can be applied to approximate a solution in cases when the LMIs are infeasible. This new method is very useful in practice since the estimated gain matrix  $K$  from the noisy experimental data could be perturbed by the estimation error, which may result in the infeasibility of the LMIs. We also provide an LMI minimization problem to find a good initial point for the minimization using the proposed gradient descent algorithm. We then provide a set of examples to illustrate how to apply our approaches to several different types of problems. An important result is the application of the technique to human subject posture control when seated on a moving robot. Results show that we can recover a cost function which may provide a useful insight on the human motor control goal.

## I. Introduction

Systems science has produced a number of useful tools for the analysis of biological systems. The application of systems identification techniques to human motor control problems has allowed researchers to determine physiological parameters, control gains, and control bandwidth for a variety of motion tasks [1, 2, 3]. The goal of biomechanical researchers is often to determine differences which may exist between healthy subjects and those suffering from pain or disease [4]. Even though these differences may be visible in the standard characteristics accessible through system identification (plant parameters, feedback gains, etc), if one assumes that the motion under analysis results from an optimal control method, it may be possible to additionally determine a cost function that would generate that control in some optimal sense [5]. These cost functions can offer additional relevant information about the system- for example, how much weight does the controller put on the various states as compared to the control effort? Several prior studies have attempted to determine optimality criteria from human motion data in an effort to explain human motion goals [5, 6, 7, 8]. In contrast to the more general potential cost functions used in these studies, we propose the use of a control theoretic method using the Linear Quadratic Regulator (LQR) framework.

In optimal control theory, the LQR problem [9] is to find the optimal infinite-horizon full state-feedback control law for the continuous-time LTI system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \quad (1)$$

with respect to the cost function

$$J = \int_0^{\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt. \quad (2)$$

where  $Q = M^T M \succeq 0$  and  $R = R^T \succ 0$ . Assume, for the moment, that  $S = 0$ . Then assuming that  $(A, B)$  is controllable and  $(A, M)$  is detectable, the optimal stabilizing control minimizing  $J$  is found as

$$u(t) = -Kx(t), \quad (3)$$

where

$$K = R^{-1} B^T P, \quad (4)$$

with  $P$  being the unique positive-semidefinite solution to the Algebraic Riccati Equation (ARE)

$$A^T P + PA - PBR^{-1} B^T P + Q = 0. \quad (5)$$

The typical use of the LQR problem in (1)–(5) is the forward result, i.e., to determine the optimal control law  $K$  from a given set of weight matrices  $Q$  and  $R$ . However, the inverse LQR problem has received some attention as well. In general, the inverse problem has been defined by two sub-problems [10]. Given a stabilizing control law (3),

- **P. 1** Determine what necessary and sufficient conditions exist on  $(K, A, B)$  such that  $K$  is an optimal control law for a cost of the form in (2) with  $Q \succeq 0$  and  $R = I$ .
- **P. 2** Determine all  $Q$  for some  $(K, A, B)$  that satisfy the conditions found in P. 1.

According to Fujii and Narazaki [10], Problem 1 was first addressed with Kalman's investigation of the single-input case [11], which was later extended to the multi-input case by Anderson [12]. A necessary and sufficient condition when  $K$  is not necessarily stabilizing and  $R$  unknown was determined by Jameson and Kreindler [13], who also show an analytic solution for recovering  $R \succ 0$  and  $Q = Q^T$ . However, a feasible  $Q$  recovered using this method is not guaranteed to be positive-semidefinite even when the closed-loop system is stable [13].

While further results have been determined for potentially destabilizing controllers [10], only stabilizing controllers are of interest when investigating engineered or biological systems. Additionally, it has been found that when the cross-term  $S$  is included in the LQR cost function, then trivially any controller  $K$  is optimal for some cost function [14]. However, we choose to exclude  $S$  from this work for a number of reasons- it is rarely used in the design of LQR controllers for practical systems, and inverse results that include the cross-term provide less salient information about the control goals than results which separate control and state costs in a straightforward manner (e.g., principle of parsimony). As a result, we will be restricting our focus in the rest of this paper to the stable LQR problem described in (1)–(5) (and later, its discrete-time counterpart) with  $S = 0$ . Molinari [15] found a necessary and sufficient condition to Problem 1 for some  $Q \succeq 0$ ,  $R = I$  when the admissible controls are in the class of control  $u(t)$  such that the corresponding state  $x(t)$  satisfies  $\lim_{t \rightarrow \infty} x(t) = 0$ . This result from [15] is stated as follows.

### Theorem 1

Assume that  $(A, B)$  from (1) is controllable. Then  $K$  will be optimal for some  $Q \succeq 0$  if and only if

1.  $A - BK$  is Hurwitz, and
2.  $T^*(j\omega)T(j\omega) - I \succeq 0$ , where  $T(s) = I + K(Is - A)^{-1}B$ .

For a stabilizing controller  $K$ , a necessary and sufficient condition utilizing coprime matrix fraction descriptions was derived in [16]. An approach utilizing convex optimization to find a maximally diagonal  $Q \succeq 0$  describing a given stabilizing control law  $K$  was proposed in [17]. While  $R = I$  in all of these results, if  $R \succ 0$  is known, then any result found for  $R = I$  can be determined by a simple coordinate transformation of  $B$  and  $K$ .

However, a more general case of the inverse LQR problem is still unanswered- if it exists, what set  $(Q \succeq 0, R \succ 0)$  generates a given stabilizing feedback law  $K$ ? So far as we are aware, there is no analytical solution to this more general problem, although for  $K$  stabilizing, at least one convex optimization formulation exists for determining a feasible solution [18]. This problem is an interesting extension to conventional system identification theory, and has potential uses in both the the reverse-engineering of black-box control systems as well as in the analysis of biological control systems. For example, it may be possible to determine underlying motor control goals from analysis of a human subject's feedback gains, which would give clinical researchers an additional way to quantify and evaluate patients. The objective of this paper is to provide a methodology for determining an inverse LQR solution in both continuous- and discrete-time cases, and, as an example, to apply this method to recover a cost function from a human motor control task.

The contribution of this paper is as follows. We present an LMI-based formulation similar to that in [18] to determine whether or not for a given stabilizing feedback law  $K$  that has been estimated from a set of experimental time-series data, there exists some set  $(Q, R)$  for which  $K$  is the optimal feedback gain. If such a solution exists, then the LMIs are solved for  $(Q, R)$  directly. Our first LMI formulation provides a unique solution when it is feasible, which can be viewed as a regularization of the feasibility formulation given in [18]. If the exact solution does not exist due to the infeasibility of the LMIs, we show how to formulate a gradient descent algorithm based on the derivative of the ARE in order to minimize the difference between the resulting best-fit and experimental feedback gains. This new method is very useful in practice since the estimated gain matrix  $K$  from the noisy experimental data could be perturbed by the estimation error, which may result in the infeasibility of the LMIs. Since this minimization using the gradient descent algorithm guarantees only the local optimality of the solution, finding a good initial starting point (or initial guess) for the gradient descent algorithm becomes important. Hence, we also provide an LMI minimization problem to find a good initial point for the minimization using the gradient approach. We then provide examples to illustrate how to apply our approaches to several different types of problems. One important contribution is to apply our proposed technique to the biological data obtained from a seated balance test using a commercial robot with a human subject. This test is designed to investigate the control mechanism of the human subject on an actuated seat. A practical experimental result obtained in this paper shows a proof of concept in human cost function recovery for future clinical research activities. Previous work [19] has appeared without human test data and analysis. The current work also augments previous work with an LMI formulation that provides a good initial point for the gradient descent method.

The paper is organized as follows. In Section II, we formulate the LMI problem describing the inverse LQR solution in both continuous- and discrete-time. In Section II, we formulate a gradient descent algorithm which can be applied to cases where the LMI problem is infeasible, and show an LMI method for determining a good starting point for this algorithm. In Section III, we demonstrate the use of the LMI method for several feasible example problems. In Section IV, we demonstrate the use of the gradient descent algorithm to solve a problem where the LMI method is infeasible. In Section V, we apply the method

to experimentally determine an LQR-type cost function in a human subject. Finally, in Section VI, we offer some conclusions on the described inverse LQR methods.

## II. Inverse LQR Problem

The inverse LQR problem has both the continuous-time formulation (1)–(5) and a formulation for the discrete-time LTI system

$$x_{k+1} = Ax_k + Bu_k \quad (6)$$

which minimizes the value of the cost function

$$J = \sum_{k=0}^{\infty} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}. \quad (7)$$

Assuming, again, that  $S = 0$ , the optimal feedback control is

$$u_k = -Kx_k,$$

where

$$K = (B^T PB + R)^{-1} B^T PA, \quad (8)$$

and  $P$  is the unique positive-semidefinite solution to the Discrete-time Algebraic Riccati Equation (DARE)

$$A^T PA - P - (A^T PB)(B^T PB + R)^{-1} B^T PA + Q = 0. \quad (9)$$

We additionally define an auxiliary notation for the solution  $K$  to the discrete-time LQR problem as

$$K = \text{DLQR}(A, B, Q, R),$$

and one for the continuous-time LQR problem as

$$K = \text{CLQR}(A, B, Q, R).$$

In formal terms, for the continuous-time (respectively, discrete-time) case, our problem is to determine the weighting matrices

$$\begin{aligned}
& (\hat{Q}, \hat{R}) \text{ such that} \\
& \hat{Q} \succeq 0, \hat{R} \succ 0, \hat{K} = K_e, \\
& \hat{K} = \text{CLQR}(A, B, \hat{Q}, \hat{R}), \\
& [\text{respectively, } \hat{K} = \text{DLQR}(A, B, \hat{Q}, \hat{R})].
\end{aligned} \tag{10}$$

where  $K_e$  is the full-state feedback gain matrix determined via a system identification method from the experimental data.

### A. Solution via Linear Matrix Inequalities (LMIs)

For the formulated inverse LQR problem in Sections I and II, there is an associated uniqueness issue; for example, if we multiply  $Q$  and  $R$  in (10) by the scalar  $\beta > 0$  and find the LQR solution, then the resulting controller gain matrix  $K$  will be identical no matter what the value of  $\beta$ . Consequently, we expect there to be a manifold of possible solutions  $(Q, R)$  to the inverse problem defined in (10). Therefore, we define the additional criteria that an optimal solution  $(\hat{Q}, \hat{R})$  must minimize the condition number of the weighting matrix

$$\begin{bmatrix} \hat{Q} & 0 \\ 0 & \hat{R} \end{bmatrix}, \text{ which can be defined explicitly as}$$

$$\begin{aligned}
& (\hat{Q}, \hat{R}, \hat{\alpha}) = \arg \min_{(Q, R) \in (\mathcal{Q}, \mathcal{R}), \alpha} \alpha^2 \\
& \text{such that } I \preceq \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \preceq \alpha I.
\end{aligned} \tag{11}$$

Minimizing the condition number ensures the numerical stability for operations involving  $(Q, R)$  [20]. This will also lead us to obtain the unique solution to our problem (See Remark 1). Note, however, that (11) will force  $Q \succ 0$ , which is more restrictive than the  $Q \succeq 0$  requirement in (10) and is, in general, not a necessary condition for the defined LQR problem. Additionally, forcing  $Q \succ 0$  means that  $(A, M)$  (with  $Q = M^T M$ ) will trivially satisfy the detectability requirement.

The problems defined in (10)–(11) can be written together as a convex optimization problem subject to LMI constraints. For the continuous-time LQR from (1)–(5), the LMI optimization can be written as follows.

$$\begin{aligned}
& (\hat{Q}, \hat{R}, \hat{P}, \hat{\alpha}) = \arg \min_{Q, R, P, \alpha} \alpha^2, \text{ such that} \\
& P \succeq 0, \\
& A^T P + PA - PBK_e + Q = 0, \\
& B^T P - RK_e = 0, \\
& I \preceq \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \preceq \alpha I.
\end{aligned} \tag{12}$$

For the discrete-time LQR defined in (6)–(9), the problem becomes

$$\begin{aligned}
(\hat{Q}, \hat{R}, \hat{P}, \hat{\alpha}) &= \arg \min_{Q, R, P, \alpha} \alpha^2, \text{ such that} \\
&P \succeq 0, \\
A^T P A - P - A^T P B K_e + Q &= 0, \\
B^T P A - (B^T P B + R) K_e &= 0, \\
I &\preceq \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \preceq \alpha I.
\end{aligned} \tag{13}$$

Since one of the major applications of the inverse LQR solution presented here is to use the recovered cost matrices to draw some broader conclusions from a control system, it is important that any solution be unique. If multiple solutions to the inverse LQR problem exist for a given system, then multiple cost functions give equivalent descriptions of the controller and no useful conclusions can be found. In that regard, we make the following statement:

**Remark 1**—(12) and (13) are convex optimization problems with strictly convex objectives [21]. Therefore, if a feasible solution exists that minimizes the objective function, it will be unique [21, 22]. Note that strict convexity of the objective is only a sufficient condition for uniqueness of the solution. Our approach used in (12) and (13) can be viewed as a regularization of the feasibility formulation given in [18], providing a great utility to inverse problem applications. Note also that (12) and (13) can be formulated and solved as semi-definite programs (SDP) by adding the LMI constraint

$$\begin{bmatrix} \gamma & \alpha \\ \alpha & 1 \end{bmatrix} \succeq 0$$

and then minimize  $\gamma$  instead of  $\alpha^2$ .

The feasible solution to the problem in (12) (respectively, in (13)), will satisfy both (10) and (11) simultaneously. Infeasibility implies that there is no solution to the LQR problem such that  $K \hat{=} K_e$  while satisfying all constraints.

Previous works [10, 11, 15, 16, 17] include results that a stabilizing  $K$  is optimal relative to some  $Q \succeq 0$  for  $R$  known, while the inverse problem (10) involves both  $Q$  and  $R$  unknown. We therefore make Remark 2.

**Remark 2**—If the LMI problem defined in (12) (respectively, (13)) is feasible, then at least one exact solution to the inverse problem of (10) exists. If the problem is infeasible, then no exact solution exists.

However, if no exact solution exists, then an approximation  $K \hat{=}$  minimizing the residual error between  $K$  and  $K_e$  can be found via a gradient descent law, which is the main idea in the following subsection.

## B. Solution via Gradient Descent Algorithm

In accordance with Remark 2, if the solution to the LMI problem in (12) (respectively, (13)) is infeasible, then we consider the following minimization problem

$$\begin{aligned} \hat{\theta} &= \arg \min_{\theta} \|K(\theta) - K_e\|_F^2, \\ K(\theta) &= \text{CLQR}(A, B, Q(\theta), R(\theta)), \\ &[\text{respectively, } K(\theta) = \text{DLQR}(A, B, Q(\theta), R(\theta))]. \end{aligned} \quad (14)$$

where  $\|\cdot\|_F$  denotes the Frobenius norm, i.e.,  $\|A\|_F := \sqrt{\text{trace}(A^T A)}$  (for  $A \in \mathbb{R}^{a \times b}$ ), and  $\theta$  defines the upper-triangular entries of the symmetric weighting matrices  $Q \in \mathbb{R}^{n \times n}$ ,  $R \in \mathbb{R}^{m \times m}$  as

$$\theta := [Q_{11}, Q_{12}, \dots, Q_{nn}, R_{11}, R_{12}, \dots, R_{mm}]^T.$$

We can find a local minimum to (14) by using the analytical gradient of the cost in (14). For a concise presentation, we first vectorize  $K(\theta)$  and  $K_e$  such that

$$K^v(\theta) := \text{vec}(K(\theta)), \quad K_e^v := \text{vec}(K_e),$$

where the  $\text{vec}(\cdot)$  operator converts an arbitrary matrix  $A \in \mathbb{C}^{m \times n}$  into a column vector such that

$$\text{vec}(A) = [a_{11}, \dots, a_{m1}, a_{12}, \dots, a_{m2}, \dots, a_{1n}, \dots, a_{mn}]^T,$$

where  $a_{ij}$  is the  $(i, j)$ <sup>th</sup> element of  $A$ .

Let us define  $e := K^v(\theta) - K_e^v$ , then we have  $e^T e = \|K^v(\theta) - K_e^v\|^2 = \|K(\theta) - K_e\|_F^2$ . We now present a gradient descent law that drives each element  $\theta_i$  of  $\theta$  such that the error norm in (14) is decreased as follows.

$$\begin{aligned} \frac{d\theta_i(t)}{dt} &= -\lambda \frac{\partial (e^T(t)e(t))}{\partial \theta_i(t)} \\ &= -\lambda \left[ \frac{\partial e^T(t)}{\partial \theta_i(t)} e(t) + e^T(t) \frac{\partial e(t)}{\partial \theta_i(t)} \right], \end{aligned} \quad (15)$$

where  $\lambda$  is a positive constant that controls the convergence rate. Note that we need to compute

$$\frac{\partial e(t)}{\partial \theta_i(t)} = \frac{\partial K^v(\theta(t))}{\partial \theta_i(t)}. \quad (16)$$



To obtain the value in (16) analytically, we first introduce the notation

$$\bullet' := \frac{\partial \bullet}{\partial \theta_i(t)}.$$

As it is clear that  $(K^v, Q, R)$  are functions of  $\theta$  and  $t$ , we will drop the explicit dependencies as in  $(K^v(\theta(t)), Q(\theta(t)), R(\theta(t)))$ .

Consider the continuous-time case for a representative presentation. The discrete-time case follows similar steps. Now we have the following.

$$\frac{\partial e}{\partial \theta_i} = (K^v)' = \text{vec}((R^{-1})' B^T P + R^{-1} B^T P'). \quad (17)$$

The stabilizing solution  $P$  to the ARE in (5) is analytic in  $A, B, M$  [23], and can therefore be differentiated implicitly with respect to  $\theta$ . If we take the derivative of the ARE from (5) with respect to  $\theta_i$ , we arrive at

$$\begin{aligned} 0 &= A^T P' + P' A - [P' B R^{-1} B^T P + P B (R^{-1})' B^T P + P B R^{-1} B^T P'] + Q' \\ &= \bar{A}^T P' + P' \bar{A} + [-P B (R^{-1})' B^T P + Q'], \end{aligned} \quad (18)$$

where  $\bar{A} = A - B R^{-1} B^T P$  and  $P = P^T \succeq 0$  is the solution to the original ARE in (5). Equation (18) defines a Lyapunov equation that can be solved for  $P'$ . By determining (15) for all elements  $i$  of  $\theta_i(t)$ , we find a directional derivative that drives the elements in  $\theta(t)$  such that the norm of the error in (14) is minimized. For actual computation of  $\theta(t)$ , we apply the preceding result in a discrete sense, i.e., we replace the continuous-time  $\theta_i(t)$  in (15) with the discrete-time equivalent  $\theta_i(k) = \theta_i^k$ ,

$$\theta_i^{k+1} = \theta_i^k - \lambda \frac{\partial (e^T e)}{\partial \theta_i^k}, \quad (19)$$

which is iterated for a desired number of iterations. Additionally, a projection rule for  $\theta$  is applied because  $Q$  must remain positive-semidefinite and  $R$  must remain positive-definite for the LQR solution to exist.

The success of any gradient descent algorithm depends on the quality of the initial starting point (or initial guess). We can exploit the fact that there always exists an exact solution to the inverse LQR problem when  $S \neq 0$  [14] to determine a ‘‘close’’ approximation for the case when  $S = 0$ . To this end, we consider the following LMI problem.

$$\begin{aligned}
(Q_s, S_s, R_s, P_s) = \arg \min_{Q, S, R, P} \|S\|_F, \text{ such that} \\
P \succeq 0, \\
B^T P + S^T - R K_e = 0, \\
A^T P + P A - (P B + S) K_e + Q = 0, \\
\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succ 0.
\end{aligned} \tag{20}$$

Additional constraints on  $Q$  and  $R$  may be included in (20) to improve the quality of the initial point.

Solving this set of LMIs for  $Q_s, R_s, S_s$  yields an initial point  $(Q_s, R_s)$  which is “nearly” optimal in the sense that it is based on an exact solution minimizing  $S$ . Note that by introducing an additional LMI constraint and decision variable, it would be possible to reformulate (20) as a SDP which can be solved efficiently. Complete details of the algorithm are given in Table I.

### III. Illustrative Example with Feasible Solution

Consider the continuous-time LTI system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad y(t) = Cx(t); \tag{21}$$

where

$$A = \begin{bmatrix} 0 & 1 & 7 & 9 \\ 4 & -8 & -5 & -3 \\ 8 & -7 & 7 & -6 \\ 10 & -5 & -5 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 5 & -9 \\ -1 & 1 & 5 & -9 \\ -3 & 9 & -3 & 1 \\ 7 & -4 & 1 & 6 \end{bmatrix}, \tag{22} \\
C = I_4.$$

In contrast to (1), we include an output equation so that measurement noise can be considered later.

The system in (22) is unstable, with two eigenvalues in the open right-half plane. However, the pair  $(A, B)$  is controllable, allowing us to design an LQR controller with weights

$$\begin{aligned}
Q_0 = \begin{bmatrix} 13.9 & -1.32 & 3.9 & 2.65 \\ -1.32 & 7.35 & 3.72 & -2.27 \\ 3.9 & 3.72 & 5.28 & 0.373 \\ 2.65 & -2.27 & 0.373 & 4.33 \end{bmatrix}, \\
R_0 = \begin{bmatrix} 10.6 & 0.302 & 6.56 & -1.83 \\ 0.302 & 4.69 & 2.06 & 3.87 \\ 6.56 & 2.06 & 7.17 & -0.266 \\ -1.83 & 3.87 & -0.266 & 6.88 \end{bmatrix}.
\end{aligned} \tag{23}$$

The feedback gain matrix  $K_0$  resulting from the solution of the ARE in (4)–(5) with  $(Q, R) = (Q_0, R_0)$  from (23) is

$$K_0 = \begin{bmatrix} 2.362 & -1.32 & 1.186 & 1.836 \\ 4.293 & -0.8472 & 3.789 & 1.538 \\ -2.252 & 1.339 & -2.081 & -1.312 \\ -2.707 & -0.6996 & -2.062 & -0.7263 \end{bmatrix}.$$

$Q_0$  and  $R_0$  in this case were chosen to minimize the condition number of the weighting matrix used to produce  $K_0$ . The closed-loop system is then

$$\frac{dx(t)}{dt} = (A - BK_0)x(t) + Br(t), \quad y(t) = Cx(t), \quad (24)$$

i.e.  $u(t) = -K_0x(t) + r(t)$ . The system in (24) is simulated by computing the response at a finite number of sampling points  $t_k$ . The input  $r(t)$  is then computed as a zero-order hold of a sampled input  $r(t_k)$ , which we realize as an identically distributed (i.i.d.) Gaussian white noise process with  $r(t_k) \sim \mathcal{N}(0, 10I_4)$ .

Suppose that a noisy version of the sampled state response of the “true” system  $A - BK_0$  is available and denoted by  $\tilde{y}(t_k) = y(t_k) + w(t_k)$ , where  $w(t_k)$  is the measurement noise that is realized by the i.i.d. Gaussian white noise process  $w(t_k) \sim \mathcal{N}(0, \Sigma)$  with  $\Sigma = I_4$ . Let  $y^\star(t_k)$  be the sampled state response of the estimated system  $A - BK^\star$ , and  $T$  be the total number of samples  $k$  in the measured responses. Then, if the system matrices  $(A, B, C)$  are known, the feedback gain matrix can be estimated by a maximum-likelihood estimator (MLE) via minimizing the least-squares error.

$$K_e = \arg \min_{K^\star} J(K^\star),$$

$$J(K^\star) = \sum_{k=0}^{T-1} (y^\star(t_k) - \tilde{y}(t_k))^T \Sigma^{-1} (y^\star(t_k) - \tilde{y}(t_k)). \quad (25)$$

We apply MATLAB’s numerical optimization algorithm *fmin-search* to the problem of (25), with  $A, B$  known and recover

$$K_e = \begin{bmatrix} 2.376 & -1.328 & 1.188 & 1.847 \\ 4.294 & -0.8621 & 3.77 & 1.509 \\ -2.279 & 1.366 & -2.067 & -1.323 \\ -2.7 & -0.06092 & -2.036 & -0.7217 \end{bmatrix} \approx K_0.$$

Measurement noise and other disturbances may perturb the estimation  $K_e$  away from  $K_0$ . However, under the conditions we have specified, we make Remark 3.

**Remark 3**—In general, for a given set of system matrices (A, B, C),  $K_e$  minimizing the cost function in (25) is the MLE of  $K_0$  [24], and under mild conditions (e.g., identifiability [24]), has a limit of  $K_0$  w. p. 1, as the length of the experiment T goes to  $\infty$  [25].

Once we have recovered  $K_e$  we can solve the convex LMI optimization problem defined in (12) efficiently using the SeDuMi [26] package with the YALMIP modeling toolbox [27] in MATLAB. The optimal estimated weights ( $\hat{Q}, \hat{R}$ ) are found to be

$$\hat{Q} = \begin{bmatrix} 13.63 & -1.184 & 4.065 & 2.723 \\ -1.184 & 7.334 & 3.497 & -2.449 \\ 4.065 & 3.497 & 4.924 & 0.1462 \\ 2.723 & -2.449 & 0.1462 & 4.395 \end{bmatrix},$$

$$\hat{R} = \begin{bmatrix} 10.55 & 0.3451 & 6.513 & -1.717 \\ 0.3451 & 4.326 & 1.811 & 3.514 \\ 6.513 & 1.811 & 6.93 & -0.4812 \\ -1.717 & 3.514 & -0.4812 & 6.555 \end{bmatrix}.$$

Because the original weights  $Q_0$  and  $R_0$  were designed to minimize the condition number of the overall weighting matrix, the recovered  $\hat{Q}$  and  $\hat{R}$  match the original weights closely. In general, however, the condition number minimization in (12) and (13) means that the recovered  $\hat{Q}$  and  $\hat{R}$  may not be numerically similar to  $Q_0$  and  $R_0$ , but will produce an equivalent controller in the forward LQR problem (assuming that  $K_e$  estimates  $K_0$  accurately).

#### IV. Illustrative Example with Infeasible Solution

In the case when the LMI problem in (12) is infeasible, we apply the gradient descent algorithm outlined in Table I. Consider the system of (21) with

$$A = \begin{bmatrix} 100 & 0 & -1 \\ 0 & 0.1 & 50 \\ 0.333 & 10 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 10 \\ 1 & 1 & 0 \\ 0.1 & -20 & 4 \end{bmatrix},$$

$$C = I_3.$$

Design a controller via pole-placement such that the closed-loop poles are  $\lambda_1 = -90$ ,  $\lambda_2 = -20$ ,  $\lambda_3 = -10$ . This results in the feedback gain matrix

$$K_0 = \begin{bmatrix} -3.69 & 20.1 & 49.3 \\ 3.69 & 0.00244 & 0.712 \\ 18.6 & 2.01 & 4.83 \end{bmatrix}.$$

The closed-loop system is again simulated at a number of sampling points  $t_k$  with an additive i.i.d. measurement noise realization of  $w(t_k) \sim \mathcal{N}(0, I)$  such that  $\tilde{y}(t_k) = y(t_k) +$

$w(t_k)$ . If we again use a sampled input sequence which is a realization of the process  $r(t_k) \sim \mathcal{N}(0, 10I_3)$ , the gain matrix  $K_e$  recovered via system identification with  $(A, B, C)$  known is

$$K_e = \begin{bmatrix} -3.47 & 20.2 & 49.3 \\ 3.7 & 0.0519 & 0.714 \\ 18.7 & 2.21 & 4.83 \end{bmatrix} \approx K_0.$$

However, for this system, numerical computation shows that the solution to the LMI problem in (12) is infeasible. Applying the gradient descent algorithm from (19) using the initial point derived from a solution to (20)

$$Q_s = \begin{bmatrix} 71.4 & 96.4 & 62.8 \\ 96.4 & 302 & 8.62 \\ 62.8 & 8.62 & 1550 \end{bmatrix}, R_s = \begin{bmatrix} 1.33 & -1.49 & -3.19 \\ -1.49 & 387 & -77 \\ -3.19 & -77 & 53.3 \end{bmatrix},$$

yields

$$\hat{K} = \begin{bmatrix} -3.75 & 27.6 & 45.3 \\ 3.93 & -0.276 & -1.7 \\ 19.7 & 2.98 & 4.57 \end{bmatrix},$$

which has a residual cost of  $e^T e = 77.24$  after 5000 iterations (Figure 1). The final optimal weighting matrices formed from  $\hat{\theta}$  were

$$\hat{Q} = \begin{bmatrix} 71.4 & 96.4 & 62.7 \\ 96.4 & 302 & 8.76 \\ 62.7 & 8.76 & 1550 \end{bmatrix}, \hat{R} = \begin{bmatrix} 1 & 7.09 & -5.05 \\ 7.09 & 388 & -76.2 \\ -5.05 & -76.2 & 53.1 \end{bmatrix}.$$

Even though we were not able to recover  $K \hat{=} K_0$ , the gradient descent algorithm has produced a locally-optimal estimate minimizing the residual error  $e^T e$  despite the imperfect initial guess  $\theta_0$ .

## V. Experimental Human Cost Function Recovery

We have developed an experimental setup for identification of the human response during an upright seated balance task (Figure 2), and to which our inverse LQR solution method can be applied. One subject volunteered for this portion of the experiment and the testing was designated as Non-Regulated Research by the MSU Institutional Review Board (IRB). The subject was seated on a hexapod robot (R-3000 Rotopod, Mikrolar Inc., Hampton, NH), which was used to apply rigid position disturbances to the subject's lower body about a lateral bending axis centered on the pelvis. In order to calculate kinematics, LED markers (Visualeyez Motion Capture System, Phoenix Technologies Inc., Burnaby Canada) were

attached to the subject (on the trunk and sacrum) and the robotic platform. Experimental kinematic data were sampled at 100 samples/sec. During the trial, the subject was given the goal of keeping the upper body as close to vertical as possible. We constructed a rigid-body model of this task (Figure 3) similar to the seated balance model developed by Reeves et al. [3], and having parameter values adapted from [3, 28]. In this model, control authority from the spine muscles is represented by the torque,  $\tau$ , acting about the  $L4$  vertebrae. The lower body has mass  $M_1$  at a distance  $l_1$  from the actuator pivot point, and moment of inertia  $J_1$  about the center of mass. The pivot point at the  $L4$  vertebrae is a distance  $l_{12}$  from the actuator pivot. The upper body has mass  $M_2$  at a distance  $l_2$  from the  $L4$  pivot, with moment of inertia  $J_2$  about the center of mass. The spine has some intrinsic stiffness  $k_h$  and intrinsic damping  $c_h$ . Because the actuator provides a rigid disturbance  $\beta_i$ , we have modeled the interaction of the seat and lower body through the soft gluteal tissues which have stiffness  $k_b$  and damping  $c_b$ , which would be fitted to the experimental data later.

The nonlinear equations of motion were derived using Lagrange's equation using a state-space representation  $x = \begin{bmatrix} \beta_1 & \dot{\beta}_1 & \beta_2 & \dot{\beta}_2 \end{bmatrix}^T$ , and linearized about the operating point  $x = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$  to form the plant model  $G$ . We assume that the sampled system outputs are a noisy direct measure of the states, i.e.  $\tilde{y}(t_k) = Cy(t_k) + w(t_k)$ , with  $C = I_4$  and  $w(t_k)$  a realization of the i.i.d. process  $w(t_k) \sim \mathcal{N}(0, \Sigma)$  with  $\Sigma = \sigma^2 I_4$ . We presumed the existence of a full-state feedback controller  $K = \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \end{bmatrix}$  which would produce the voluntary input torque  $\tau$  via  $\tau = -Ky$ .

We fit the 6 unknown model parameters  $\zeta := [K_1, \dots, K_4, k_b, c_b]^T$  by using MATLAB's *fminsearchbnd* function to perform the minimization

$$\zeta_e = \arg \min_{\zeta^*} \sum_{t=0}^{T-1} (y^*(t_k) - \tilde{y}(t_k))^T W (y^*(k) - \tilde{y}(t_k)),$$

where  $W = \text{diag}(1, 0, 1, 0)$ ,  $k$  is the sample index,  $T$  is the number of samples in the experiment,  $y^*(t_k)$  is the sampled state response of the estimated system using  $\zeta^*$ , and  $\tilde{y}(t_k)$  is the sampled response of the experimentally measurable system states. The resulting best-fit and experimental responses for  $\beta_1$  and  $\beta_2$  are shown in Figure 4. The estimated parameters  $\zeta_e$  are shown in Table II. The estimated feedback controller  $K_e$  was formed as  $K_e = [K_{1,e}, \dots, K_{4,e}]$ .

The values  $k_{b,e}$  and  $c_{b,e}$  were incorporated into the  $A$  matrix of the plant  $G$  to form  $A_e$ , and the the columns of  $B$  associated with  $\beta_i$  and  $\beta_i$  were removed to form  $B_e$ . The inverse LQR procedure of (12) was then applied to determine  $\hat{Q}$  and  $\hat{R}$  such that  $K_e = \text{CLQR}(A_e, B_e, \hat{Q}, \hat{R})$ . In this case, a feasible solution to the LMI problem could not be found. The gradient descent method of Algorithm I was applied, recovering

$$\hat{Q}=10^{10} \times \begin{bmatrix} 1.693 & 0.2328 & -0.1532 & 0.09889 \\ 0.2328 & 1.794 & 0.1507 & 0.571 \\ -0.1532 & 0.1507 & 5.247 & -0.0426 \\ 0.09889 & 0.571 & -0.0426 & 0.5411 \end{bmatrix}, \hat{R}=1.00.$$

with residual cost  $\|\hat{K}-K_e\|_p^2=24.514$ .

This  $(\hat{Q}, \hat{R})$  meets all the conditions defined in (12). Notice that the diagonal element associated with the upper body angle ( $Q_{33}$ ) provides the largest single contribution to the cost. Further, the relative weights of  $Q$  and  $R$  suggest that the system states are penalized much more heavily than the control effort in the cost function. However, it is not immediately clear from  $\hat{Q}$  whether linear combinations of the states may offer a more salient picture of how the cost is distributed. Therefore, we apply a similarity transform to  $x$  and  $\hat{Q}$  such that  $\tilde{Q}$  is diagonal and operates on  $\tilde{x}$ , which is a vector of linear combinations of the elements in  $x$ . If we let  $V$  be an orthogonal matrix whose columns are the eigenvectors of  $\hat{Q}$ , and  $\Lambda$  the square diagonal matrix whose diagonal elements are the corresponding eigenvalues of  $\hat{Q}$ , then  $\tilde{Q} = \Lambda$ , and  $\tilde{x} = V^T x$ . This similarity transformation will satisfy the equality  $x^T \hat{Q} x = \tilde{x}^T \tilde{Q} \tilde{x}$ . For the experimental  $\hat{Q}$  found above,

$$V = \begin{bmatrix} -0.003621 & -0.874 & -0.4842 & -0.04044 \\ -0.362 & 0.4511 & -0.8148 & 0.03988 \\ 0.01901 & -0.05254 & 0.01132 & 0.9984 \\ 0.932 & 0.1729 & -0.3186 & -0.005035 \end{bmatrix},$$

$$\tilde{Q} = 10^{10} \times \text{diag}(0.3181, 1.544, 2.153, 5.259).$$

Note that the last column in  $V$ , which is the eigenvector of  $\hat{Q}$  that corresponds to the largest eigenvalue in  $\tilde{Q}$ , will produce a coordinate in  $x$  that is a linear combination of the body angles and rates (i.e.  $x_4 = -0.04044\beta_1 + 0.03988\dot{\beta}_1 + 0.9984\beta_2 - 0.005035\dot{\beta}_2$ ), with a highest weight on the upper body angle  $\beta_2$ . If we consider only the largest eigenvalue,  $x_4$  is reasonably consistent with the motion goal given to the subject (minimize  $|\beta_2|$ ). However, a quantitative clinical study would have to be performed to draw any scientific conclusions.

## VI. Conclusions

In this paper, we described a comprehensive methodology for determining a cost function to the time-invariant LQR problem in both continuous- and discrete-time cases. Our results have potential application not only to the determination of human control cost, but also to the reverse-engineering of black-box controllers, and offer a new dimension of information (control design cost function) beyond that available using traditional system identification techniques. A set of several numerical problems and an experimental result with a human subject on a seated balance testing apparatus successfully demonstrate that our proposed method is able to determine a salient measure of control performance weights from experimental data. We plan to use this methodology in the future to more comprehensively

evaluate human postural control and determine if consistent features or control goals can be extracted from the resulting cost functions.

## Acknowledgments

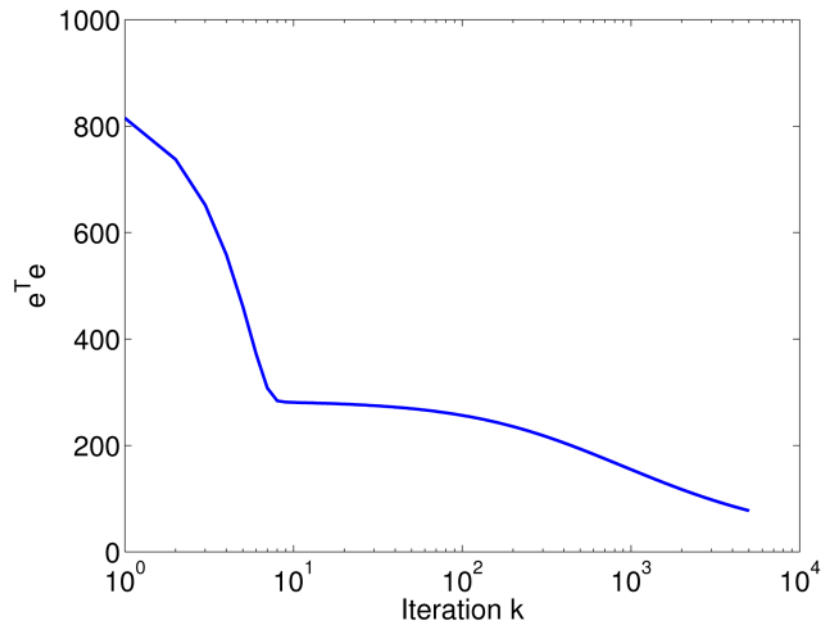
This work has been supported in part by grant number U19 AT006057 from the National Center for Complementary and Alternative Medicine (NCCAM) at the National Institutes of Health.

## References

1. Peng G, Hain T, Peterson B. A dynamical model for reflex activated head movements in the horizontal plane. *Biological cybernetics*. 1996; 75(4):309–319. [PubMed: 8953740]
2. Goodworth AD, Peterka RJ. Contribution of sensorimotor integration to spinal stabilization in humans. *Journal of neurophysiology*. 2009; 102(1):496–512. [PubMed: 19403751]
3. Reeves NP, Cholewicki J, Narendra KS. Effects of reflex delays on postural control during unstable seated balance. *Journal of biomechanics*. 2009; 42(2):164–170. [PubMed: 19121523]
4. Abaid N, Cappa P, Palermo E, Petrarca M, Porfiri M. Gait detection in children with and without hemiplegia using single-axis wearable gyroscopes. *PLoS one*. 2013; 8(9):e73152. [PubMed: 24023825]
5. Mombaur K, Truong A, Laumond JP. From human to humanoid locomotion- an inverse optimal control approach. *Autonomous robots*. 2010; 28(3):369–383.
6. Arechavaleta G, Laumond JP, Hicheur H, Berthoz A. An optimality principle governing human walking. *IEEE Transactions on Robotics*. 2008; 24(1):5–14.
7. Berret B, Chiovetto E, Nori F, Pozzo T. Evidence for composite cost functions in arm movement planning: an inverse optimal control approach. *PLoS computational biology*. 2011; 7(10):e1002183. [PubMed: 22022242]
8. Niu X, Latash ML, Zatsiorsky VM. Reproducibility and variability of the cost functions reconstructed from experimental recordings in multifinger prehension. *Journal of Motor Behavior*. 2012; 44(2):69–85. [PubMed: 22364441]
9. Antsaklis, PJ.; Michel, AN. *Linear systems*. Birkhäuser; Boston: 2005.
10. Fujii T, Narazaki M. A complete optimality condition in the inverse problem of optimal control. *SIAM journal on control and optimization*. 1984; 22(2):327–341.
11. Kalman R. When is a linear control system optimal? *Journal of Basic Engineering*. 1964; 86:51.
12. Anderson B. The inverse problem of optimal control.” DTIC Document. Tech Rep. 1966
13. Jameson A, Kreindler E. Inverse problem of linear optimal control. *SIAM Journal on Control*. 1973; 11(1):1–19.
14. Kreindler E, Jameson A. Optimality of linear control systems. *IEEE Transactions on Automatic Control*. 1972; 17(3):349–351.
15. Molinari B. The stable regulator problem and its inverse. *IEEE Transactions on Automatic Control*. 1973; 18(5):454–459.
16. Kong H, Goodwin G, Seron M. A revisit to inverse optimality of linear systems. *International Journal of Control*. 2012; 85(10):1506–1514.
17. Aioun, F.; Heniche, A.; Bourles, H. Maximally-diagonal solution to the inverse lqr problem. *Proceedings of the 33rd IEEE Conference on Decision and Control*; 1994; IEEE; 1994. p. 1445-1446.
18. Boyd, S.; El Ghaoul, L.; Feron, E.; Balakrishnan, V. *Linear matrix inequalities in system and control theory*. Vol. 15. Society for Industrial Mathematics; 1987.
19. Priess, MC.; Choi, J.; Radcliffe, C. Determining human control intent using inverse LQR solutions. *ASME Dynamic System and Control Conference*; October 2013.;
20. Cheney, EW.; Kincaid, DR. *Numerical mathematics and computing*. Brooks/Cole Publishing Company; 2012.
21. Boyd, SP.; Vandenberghe, L. *Convex optimization*. Cambridge university press; 2004.



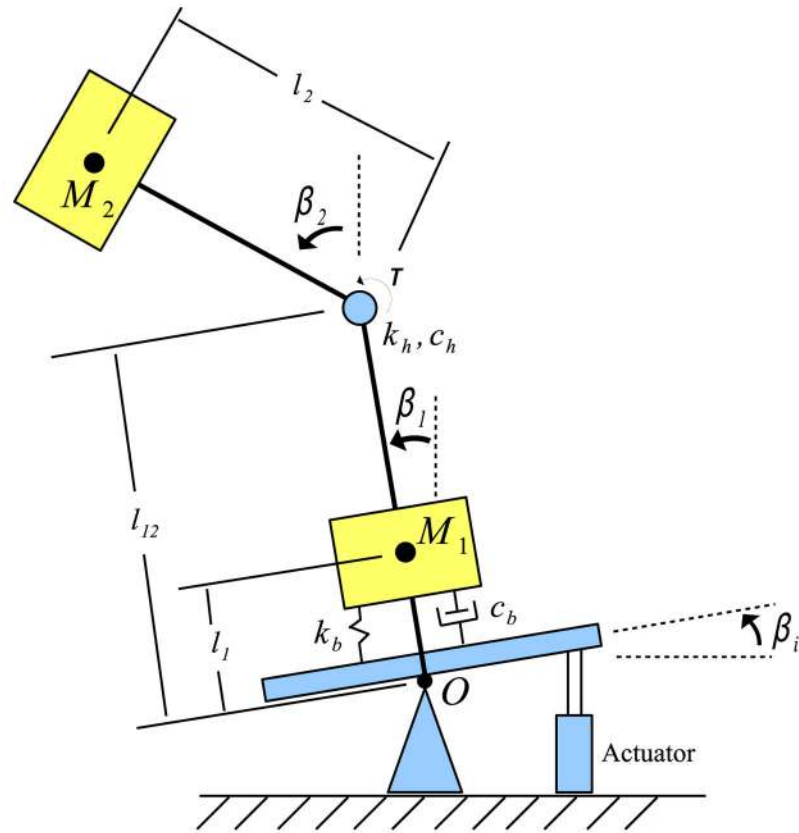
22. Dattorro, J. Convex optimization and Euclidean distance geometry. Meboo Publishing; USA: 2005.
23. Delchamps, DF. Analytic stabilization and the algebraic riccati equation. Proceedings of the 22nd IEEE Conference on Decision and Control; 1983; IEEE; 1983. p. 1396-1401.
24. Ljung, L. System identification: theory for the user. PTR Prentice Hall; Upper Saddle River, NJ: 1999.
25. Hjalmarsson H. From experiment design to closed-loop control. Automatica. 2005; 41(3):393–438.
26. Polik, I. Sedumi home page. [Online]. 2010 Jun. Available: <http://sedumi.ie.lehigh.edu/>
27. Löfberg, J. Yalmip wiki main/home page. [Online]. 2012 Nov. Available: <http://users.isy.liu.se/johan/yalmip/>
28. Zatsiorsky, VM. Kinetics of human motion. Human Kinetics; 2002.



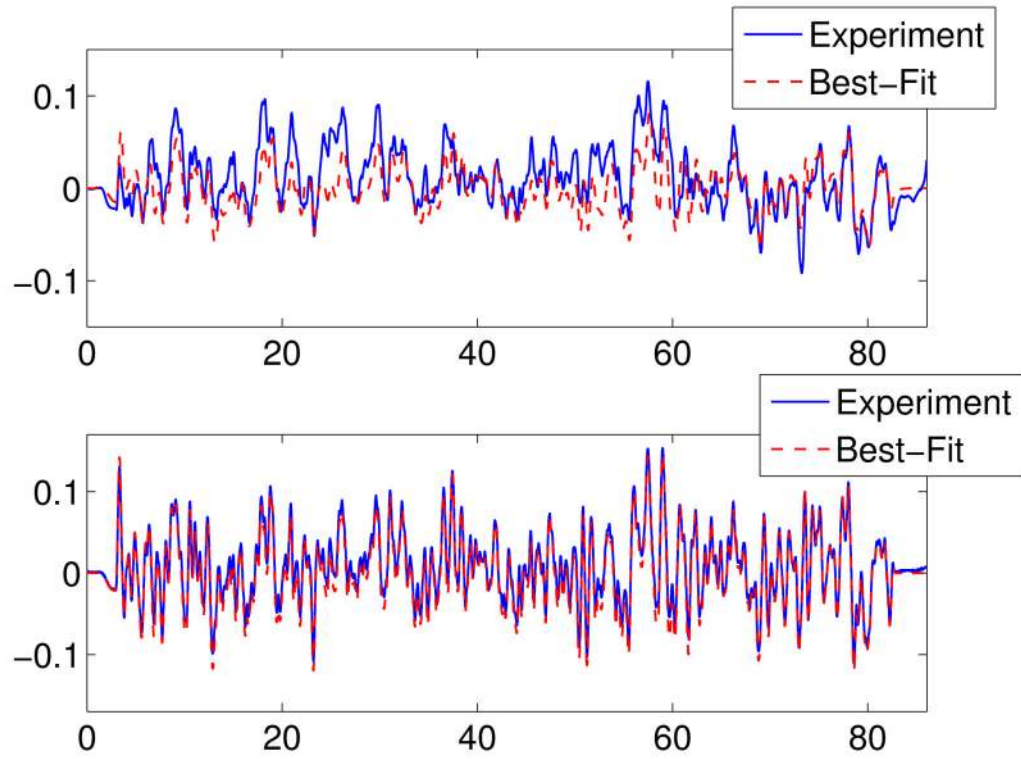
**Fig. 1.**  
CONVERGENCE OF THE ERROR NORM DURING THE GRADIENT DESCENT  
PROCESS.



**Fig. 2.**  
EXAMPLE SETUP FOR SEATED BALANCE TESTING USING THE MIKROLAR R-3000 ROTOPOD.



**Fig. 3.** SEATED-BALANCE MODEL OF THE HUMAN SUBJECT. THE VALUES OF THE GIVEN PARAMETERS ARE ADAPTED FROM [28, 3].



**Fig. 4.** Plot of the experimental and best-fit angular responses. The upper figure shows the upper body angle vs. time, while the lower figure is the lower body angle vs. time.

TABLE I

ALGORITHM FOR THE UPDATE OF  $\theta$ .

<b>Input:</b>	<p>1 The known system matrices <math>A \in \mathbb{R}^{n \times n}</math> and <math>B \in \mathbb{R}^{n \times m}</math>.</p> <p>2 The estimated feedback gain vector <math>K_e^v = \text{vec}(K_e) \in \mathbb{R}^{mn}</math>.</p>
<b>Output:</b>	<p>1 The optimal output parameter vector <math>\hat{\theta}</math>.</p>
1:	solve the set of LMIs in (20) for $Q_s$ and $R_s$ .
2:	let $Q_1 = Q_s$ and $R_1 = R_s$
3:	form $\theta^1$ from $Q_1$ and $R_1$
4:	<b>for</b> $k = 1, \dots, k_{\text{end}}$ <b>do</b>
5:	compute $K^v = \text{vec}(LQR(A, B, Q_k, R_k))$
6:	compute $P$ solving $A^T P + PA - PB R_k^{-1} B^T P + Q_k = 0$
7:	<b>for</b> $i = 1, \dots, \frac{n(n+1)}{2} + \frac{m(m+1)}{2}$ <b>do</b>
8:	compute $\bar{A} = A - B R_k^{-1} B^T P_n$
9:	compute $P'$ solving $\bar{A}^T P' + P' \bar{A} + \begin{bmatrix} -PB(R_k^{-1})' B^T P + Q_k' \end{bmatrix} = 0$
10:	compute $(K^v)' = \text{vec} \left( (R_k^{-1})' B^T P + R_k^{-1} B^T P' \right)$
11:	compute $e = K^v - K_e^v$
12:	compute $\frac{\partial (e^T e)}{\partial \theta_i^k} = [(K^v)']^T e + e^T (K^v)'$ .
13:	let the element $\theta_i^{k+1} = \theta_i^k - \lambda \frac{\partial (e^T e)}{\partial \theta_i^k}$
14:	<b>end for</b>
15:	form $Q_{k+1}$ and $R_{k+1}$ from $\theta^{k+1}$
16:	<b>if</b> $Q_{k+1} \prec 0$ <b>then</b>
17:	let $Q_{k+1} = Q_k$
18:	let $\theta_i^{k+1} = \theta_i^k, \forall i = 1, \dots, \frac{n(n+1)}{2}$
19:	<b>end if</b>
20:	<b>if</b> $R_{k+1} \preceq 0$ <b>then</b>
21:	let $R_{k+1} = R_k$
22:	let $\theta_i^{k+1} = \theta_i^k, \forall i = \frac{n(n+1)}{2} + 1, \dots, \frac{n(n+1)}{2} + \frac{m(m+1)}{2}$
23:	<b>end if</b>

```
24:   end for
25:   let  $K \hat{=} \text{LQR}(A, B, Q_{k_{\text{end}+1}}, R_{k_{\text{end}+1}})$ 
```

Author Manuscript

Author Manuscript

Author Manuscript

Author Manuscript

**TABLE II**OPTIMAL ESTIMATED PARAMETERS  $\zeta_e$  OF THE HUMAN SUBJECT.

Parameter	Value
$K_{1,e}$	$-1.8 \times 10^5$
$K_{2,e}$	$-5.573 \times 10^3$
$K_{3,e}$	$2.295 \times 10^5$
$K_{4,e}$	$7.37 \times 10^4$
$k_{b,e}$	$2.958 \times 10^4 \text{ Nm/rad}$
$c_{b,e}$	$8.09 \times 10^3 \text{ Nms/rad}$

Author Manuscript

Author Manuscript

Author Manuscript

Author Manuscript