Pacific Journal of Mathematics

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Volume 280 No. 1 January 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 280, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.280.115

SOLUTIONS WITH LARGE NUMBER OF PEAKS FOR THE SUPERCRITICAL HÉNON EQUATION

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This paper is concerned with the Hénon equation

$$\begin{cases} -\Delta u = |y|^\alpha u^{p+\varepsilon}, \ u > 0, & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0), \end{cases}$$

where $B_1(0)$ is the unit ball in $\mathbb{R}^N$ ($N \geq 4$), $p = (N+2)/(N-2)$ is the critical Sobolev exponent, $\alpha > 0$ and $\varepsilon > 0$. We show that if ε is small enough, this problem has a positive peak solution which presents a new phenomenon: the number of its peaks varies with the parameter ε at the order $\varepsilon^{-1/(N-1)}$ when $\varepsilon \to 0^+$. Moreover, all peaks of the solutions approach the boundary of $B_1(0)$ as ε goes to 0^+.

1. Introduction and main results

We study the existence of positive solutions to a type of nonlinear elliptic problem whose typical form is the supercritical problem

$$\begin{cases} -\Delta u = |y|^\alpha u^{p+\varepsilon}, \ u > 0, & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{1-1}$$

where $p = (N+2)/(N-2)$, $\alpha > 0$, $\varepsilon > 0$ and $B_1(0)$ is the unit ball in $\mathbb{R}^N$ ($N \geq 4$).

It is well known that the problem

$$\begin{cases} -\Delta u = |y|^\alpha u^{q}, \ u > 0, & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{1-2}$$

was proposed by M. Hénon [1973] when he studied rotating stellar structures and is hence called the Hénon equation, and it has attracted a lot of interest in recent years. Ni [1982] first considered (1-2) and proved that it possesses a positive radial solution when $q \in (1, (N+2+2\alpha)/(N-2))$. Due to the appearance of the weighted term $|y|^\alpha$, the classical moving plane method in [Gidas et al. 1979] cannot be applied to problem (1-2). It is natural to ask whether problem (1-2) has nonradial solutions. The existence of a nonradial solution for $1 < q < p$ was obtained by

MSC2010: primary 35J60; secondary 35J65, 58E05.
Keywords: peak solutions, supercritical Hénon equation, reduction method.

Smets, Willem and Su [2002] provided α is large enough. When $q = p - \varepsilon$, Cao and Peng [2003] showed that the ground state solution is nonradial and blows up near the boundary of $B_1(0)$ as $\varepsilon \to 0$. Later on, Peng [2006] constructed multiple boundary peak solutions for problem (1-2). When $q = p$, Serra [2005] proved that problem (1-2) has a nonradial solution provided α is large enough. More recently, Wei and Yan [2013] showed that there are infinitely many nonradial solutions for problem (1-2) with $\alpha > 0$. For other results related to Hénon type problems, see [Byeon and Wang 2006; 2005; Cao et al. 2009; Hirano 2009; Pistoia and Serra 2007] and the references therein.

On the other hand, using the Pohozaev identity [1965], we know that for $q \geq \frac{N+2+2\alpha}{N-2}$ there are no solutions to problem (1-2) in star-shaped domains with respect to the origin. So it seems more interesting to consider whether there are solutions for q in the range $\big(\frac{N+2}{N-2}, \frac{N+2+2\alpha}{N-2}\big)$. However, much less is known about that case. When $q = \frac{N+2+2\alpha}{N-2} - \varepsilon$, Gladiali and Grossi [2012] showed that there exists one solution concentrating at $y = 0$ provided $0 < \alpha \leq 1$. By the results in [Gladiali et al. 2013], the same results still hold when α is not an even integer. In [Li and Peng 2009], the asymptotic behavior of the radial solutions obtained by Ni [1982] was analyzed as $\varepsilon \to 0^+$.

The purpose of this paper is to study the supercritical problem (1-1) and try to construct solutions whose number of peaks varies with ε as $\varepsilon \to 0^+$. In fact, we will consider the more general problem

$$
\begin{cases} -\Delta u = K(|y|)u^{p+\varepsilon}, \ u > 0, & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{1-3}
$$

where $K(r) \in C^1[0, 1]$ and $K(1) > 0$.

Without loss of generality, we can assume that

$$K(1) = 1.$$

The main result of this paper is as follows.

Theorem 1.1. *Assume that $N \geq 4$. If $K(r)$ satisfies $K(1) > 0$ and $K'(1) > 0$, then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, problem* (1-3) *has a solution u_ε whose number of local maximal points is of the order $\varepsilon^{-1/(N-1)}$ as $\varepsilon \to 0^+$. In particular, problem* (1-1) *has solutions with a large number of peaks for small $\varepsilon > 0$.*

Remark 1.2. For the case $\alpha = 0$, the well-known Pohozaev identity [1965] implies that (1-1) has no solutions for $\varepsilon > 0$. It was also shown in [Ben Ayed et al. 2003] that problem (1-1) has no single-peak solutions for ε small enough. Our results mean that the weight $|y|^\alpha$ has a great influence on the existence of peak solutions for problem (1-1).

Let us outline the main idea in the proof of Theorem 1.1. We introduce some notation first. For $x \in \mathbb{R}^N$ and $\Lambda > 0$, set

$$U_{x,\Lambda}(y) = C_N \left(\frac{\Lambda}{1 + \Lambda^2 |y - x|^2} \right)^{(N-2)/2}, \quad C_N = (N(N-2))^{(N-2)/4}.$$

It's well known that $U_{x,\Lambda}(y)$ are the only solutions of

$$-\Delta u = u^{(N+2)/(N-2)}, \; u > 0, \quad \text{in } \mathbb{R}^N.$$

Let

$$k = [\varepsilon^{-1/(N-1)}],$$

where $[a]$ denotes the integer part of a real number a. By the transformation $u(y) \mapsto \varepsilon^{2/(4+(N-2)\varepsilon)} u(\varepsilon^{1/(N-2)} y)$ and setting $B_* = B_{\varepsilon^{-1/(N-2)}}$, we see that (1-3) becomes

$$\begin{cases} -\Delta u = K(\varepsilon^{1/(N-2)}|y|) u^{p+\varepsilon}, \; u > 0, & \text{in } B_*(0), \\ u = 0 & \text{on } \partial B_*(0). \end{cases} \tag{1-4}$$

We denote by $PU_{x,\Lambda}$, the projection of $U_{x,\Lambda}$, the solution of the problem

$$\begin{cases} \Delta PU_{x,\Lambda} = \Delta U_{x,\Lambda} & \text{in } B_*(0), \\ PU_{x,\Lambda} = 0 & \text{on } \partial B_*(0). \end{cases} \tag{1-5}$$

Set $y = (y', y'')$, $y'' \in \mathbb{R}^{N-2}$. Define

$$\mathcal{H}_s = \Big\{ u : u \in H_0^1(B_*(0)), \; u \text{ is even in } y_h, \; h = 2, 3, \ldots, N, \\ u(r\cos\theta, r\sin\theta, y'') = u\Big(r\cos\Big(\theta + \frac{2\pi j}{k}\Big), r\sin\Big(\theta + \frac{2\pi j}{k}\Big), y'' \Big) \Big\}.$$

Let

$$x_j = \left(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, \ldots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, and let

$$W_{r,\Lambda}(y) = \sum_{j=1}^{k} PU_{x_j,\Lambda}.$$

In what follows, we always assume that

$$r \in [\varepsilon^{-1/(N-2)}(1 - r_0 \varepsilon^{1/(N-1)}), \varepsilon^{-1/(N-2)}(1 - r_1 \varepsilon^{1/(N-1)})]$$

for some constants $r_1 > r_0 > 0$, and that

$$L_0 \le \Lambda \le L_1$$

for some constants $L_1 > L_0 > 0$.

We will prove Theorem 1.1 by verifying the following result.

Theorem 1.3. *Under the same assumptions as Theorem 1.1, there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0)$, *problem* (1-4) *has a solution* u_ε *of the form*

$$u_\varepsilon = W_{r_\varepsilon,\Lambda_\varepsilon} + \phi_\varepsilon,$$

where $\phi_\varepsilon \in \mathcal{H}_s$, $\|\phi_\varepsilon\|_{L^\infty} \to 0$ *as* $\varepsilon \to 0^+$, $L_0 \le \Lambda_\varepsilon \le L_1$ *and*

$$r_\varepsilon \in [\varepsilon^{-1/(N-2)}(1 - r_0\varepsilon^{1/(N-1)}), \varepsilon^{-1/(N-2)}(1 - r_1\varepsilon^{1/(N-1)})].$$

Remark 1.4. In our result, the number of peaks k of the solution u_ε varies with the parameter ε at the order $\varepsilon^{-1/(N-1)}$ when $\varepsilon \to 0^+$. This is a new phenomenon for the Hénon equation and is in contrast to the subcritical or critical case. For example, in [Peng 2006], where $\varepsilon < 0$, it was proved that for any prescribed integer $k > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (-\varepsilon_0, 0)$, problem (1-4) has a solution which has exactly k peaks.

Remark 1.5. The results of this paper can be considered as a perturbation of those in [Wei and Yan 2013]. In fact the number of bubbles k can be taken to be

$$k = [\delta^{-1/(N-2)}]$$

for any $|\varepsilon| < \delta \ll 1$. When $\varepsilon = 0$, we recover Wei and Yan's result.

We use a reduction argument to prove Theorem 1.3. More precisely, we follow the method in [Wei and Yan 2010b; 2013] to construct peak solutions for problem (1-4). In those papers, where no parameter appears in the considered problem, Wei and Yan used k, the number of peaks of the solutions, as the parameter to construct infinitely many positive peak solutions. This idea is very novel and effective for obtaining infinitely many solutions to several types of problems; see [Peng and Wang 2013; Wei and Yan 2010a; 2011]. Unlike the situation in [Wei and Yan 2010b; 2013], here we deal with the supercritical case; we cannot use the variational argument. Instead, we will use the Fredholm theory of compact operators in a suitable Banach space and will employ a direct technique to eliminate the Lagrange multipliers caused from the reduction procedure. Another aspect that differs from [Wei and Yan 2010b; 2013] is that, as we mentioned before, in our proof, we use ε as the parameter in the construction of peak solutions, but in this paper the number of peaks depends on the parameter ε. As a final remark, we point out that for $\alpha = 0$, del Pino, Felmer and Musso [2003] have constructed two-peaked solutions for problem (1-1) in a special domain. Hence, we believe that the effect of the weight $|y|^\alpha$ on the existence of solutions is something like that of the domain.

This paper has the following structure. In Section 2, we carry out the finite-dimensional reduction procedure. The main results will be proved in Section 3. We put the energy expansion and some basic estimates used in Sections 2 and 3 in Appendices A and B.

2. Finite-dimensional reduction

In this section, we perform a finite-dimensional reduction. Let

$$\|u\|_* = \sup_{y\in B_*(0)} \left(\sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{1}{2}(N-2)+\tau}} \right)^{-1} |u(y)| \tag{2-1}$$

and

$$\|v\|_{**} = \sup_{y\in B_*(0)} \left(\sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{1}{2}(N+2)+\tau}} \right)^{-1} |v(y)|, \tag{2-2}$$

where $\tau = (N-2)/(N-1)$. We denote by L^∞_* and L^∞_{**} the function spaces defined on $B_*(0)$ with finite $\|\cdot\|_*$ and $\|\cdot\|_{**}$ norm, respectively.

Let

$$Z_{i,1} = \frac{\partial PU_{x_i,\Lambda}}{\partial r}, \qquad Z_{i,2} = \frac{\partial PU_{x_i,\Lambda}}{\partial \Lambda}.$$

First, we consider the linear problem

$$\begin{cases} -\Delta\phi - (p+\varepsilon)K(\varepsilon^{\frac{1}{N-2}}|y|)W^{p-1+\varepsilon}_{r,\Lambda}\phi = h + \sum_{j=1}^{2} c_j \sum_{i=1}^{k} U^{p-1}_{x_i,\Lambda} Z_{i,j} \text{ in } B_*(0), \\ \phi \in \mathcal{H}_s, \qquad \left\langle \sum_{i=1}^{k} U^{p-1}_{x_i,\Lambda} Z_{i,l}, \phi \right\rangle = 0, \quad l = 1, 2, \end{cases} \tag{2-3}$$

for some numbers c_i, where

$$\langle u, v\rangle = \int_{B_*(0)} uv.$$

Lemma 2.1. *Assume there is a sequence $\varepsilon = \varepsilon_n \to 0$ such that ϕ_ε solves* (2-3) *for $h = h_\varepsilon$. If $\|h_\varepsilon\|_{**}$ goes to zero as ε goes to zero, so does $\|\phi_\varepsilon\|_*$.*

Proof. The proof of this lemma is very similar to the proof of Lemma 2.1 in [Wei and Yan 2013].

We argue by contradiction. Suppose that there are $\varepsilon \to 0$, $h = h_\varepsilon$, $\Lambda_\varepsilon \in [L_0, L_1]$ and $r_\varepsilon \in [\varepsilon^{-1/(N-2)}(1-r_0\varepsilon^{1/(N-1)}), \varepsilon^{-1/(N-2)}(1-r_1\varepsilon^{1/(N-1)})]$ such that ϕ_ε solves (2-3) for $h = h_\varepsilon$, $\Lambda = \Lambda_\varepsilon$, $r = r_\varepsilon$ with $\|h_\varepsilon\|_{**} \to 0$ and $\|\phi_\varepsilon\|_* \geq c > 0$. Without loss of generality, we may assume that $\|\phi_\varepsilon\|_* = 1$.

Now rewrite (2-3) in the following integral form:

$$\phi_\varepsilon(y) = (p+\varepsilon)\int_{B_*(0)} G_\varepsilon(y,z)K(\varepsilon^{-1/(N-2)}|z|)W^{p-1+\varepsilon}_{r,\Lambda}(z)\phi_\varepsilon(z)\,dz \\ + \int_{B_*(0)} G_\varepsilon(y,z)\Big(h_\varepsilon(z) + \sum_{j=1}^{2} c_j \sum_{i=1}^{k} Z_{i,j}(z)U^{p-1}_{x_i,\Lambda}(z)\Big)\,dz.$$

By Lemma B.3, we find

$$\begin{aligned}
&\left|(p+\varepsilon)\int_{B_*(0)} G_\varepsilon(y,z)K(\varepsilon^{-1/(N-2)}|z|)W_{r,\Lambda}^{p-1+\varepsilon}(z)\phi_\varepsilon(z)\,dz\right| \\
&\qquad\le (p+\varepsilon)\int_{B_*(0)} \frac{1}{|y-z|^{N-2}}K(\varepsilon^{-1/(N-2)}|z|)W_{r,\Lambda}^{p-1+\varepsilon}(z)\,|\phi_\varepsilon(z)|\,dz \\
&\qquad\le C\|\phi_\varepsilon\|_*\int_{B_*(0)} \frac{1}{|y-z|^{N-2}}W_{r,\Lambda}^{p-1+\varepsilon}\sum_{j=1}^k \frac{1}{(1+|z-x_j|)^{\frac12(N-2)+\tau}}\,dz \\
&\qquad\le C\|\phi_\varepsilon\|_*\sum_{j=1}^k \frac{1}{(1+|z-x_j|)^{\frac12(N-2)+\tau+\vartheta}}.
\end{aligned}$$

It follows from Lemma B.2 that

$$\begin{aligned}
\left|\int_{B_*(0)} G_\varepsilon(y,z)h_\varepsilon(z)\,dz\right| &\le \int_{B_*(0)} \frac{1}{|y-z|^{N-2}}|h_\varepsilon(z)|\,dz \\
&\le C\|h_\varepsilon\|_{**}\sum_{j=1}^k \frac{1}{(1+|z-x_j|)^{\frac12(N-2)+\tau}}
\end{aligned}$$

and

$$\left|\int_{B_*(0)} G_\varepsilon(y,z)\sum_{j=1}^k Z_{i,l}(z)U_{x_i,\Lambda}^{p-1}(z)\,dz\right| \le C\sum_{j=1}^k \frac{1}{(1+|z-x_j|)^{\frac12(N-2)+\tau}}.$$

Next, we estimate c_ℓ, $\ell=1,2$. Multiplying (2-3) by $Z_{1,t}$ and integrating, we obtain that c_ℓ satisfies

$$\begin{aligned}
\text{(2-4)}\quad &\sum_{\ell=1}^2\sum_{i=1}^k \langle U_{x_i,\Lambda}^{p-1}Z_{i,\ell}, Z_{1,t}\rangle c_\ell \\
&\qquad= \big\langle -\Delta\phi_\varepsilon-(p+\varepsilon)K(\varepsilon^{-1/(N-2)}|y|)W_{r,\Lambda}^{p-1+\varepsilon}\phi_\varepsilon, Z_{1,t}\big\rangle - \langle h_\varepsilon, Z_{1,t}\rangle.
\end{aligned}$$

It follows from Lemma B.1 that

$$\begin{aligned}
|\langle h_\varepsilon, Z_{1,\ell}\rangle| &\le C\|h_\varepsilon\|_{**}\int_{\mathbb{R}^N} \frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^k \frac{1}{(1+|z-x_j|)^{\frac12(N+2)+\tau}}\,dz \\
&\le C\|h_\varepsilon\|_{**}.
\end{aligned}$$

On the other hand, using Lemma B.3, we obtain

$$\begin{aligned}
&\big\langle -\Delta\phi_\varepsilon-(p+\varepsilon)K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p-1+\varepsilon}\phi_\varepsilon, Z_{1,\ell}\big\rangle \\
&\qquad= \big\langle -\Delta Z_{1,\ell}-(p+\varepsilon)K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p-1+\varepsilon}Z_{1,\ell}, \phi_\varepsilon\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \bigl\langle -\Delta Z_{1,\ell} - pK(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p-1} Z_{1,\ell}, \phi_\varepsilon \bigr\rangle \\
&\qquad\qquad + p\bigl\langle K(\varepsilon^{1/(N-2)}|y|)(W_{r,\Lambda}^{p-1} - W_{r,\Lambda}^{p-1+\varepsilon}) Z_{1,\ell}, \phi_\varepsilon \bigr\rangle \\
&\qquad\qquad\qquad\qquad - \varepsilon\bigl\langle K(\varepsilon^{1/(N-2)}|y|) W_{r,\Lambda}^{p-1+\varepsilon}) Z_{1,\ell}, \phi_\varepsilon \bigr\rangle \\
&= o(\|\phi_\varepsilon\|_*).
\end{aligned}
$$

However, there is a constant $c' > 0$ such that

$$
\sum_{i=1}^{k} \langle U_{x_i,\Lambda}^{p-1} Z_{i,t}, Z_{1,\ell} \rangle = (c' + o(1))\delta_{t\ell}.
$$

Hence we find from (2-4) that

$$
c_\ell = o(\|\phi_\varepsilon\|_*) + O(\|h_\varepsilon\|_{**}).
$$

Therefore,

$$
\|\phi_\varepsilon\|_* \le o(1) + \|h_\varepsilon\|_{**} + \frac{\displaystyle\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N-2)+\tau+\vartheta}}}{\displaystyle\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N-2)+\tau}}}. \tag{2-5}
$$

Noting that $\|\phi_\varepsilon\|_* = 1$, we obtain from (2-5) that there is $R > 0$ such that

$$
\|\phi_\varepsilon(y)\|_{L^\infty(B_R(x_i))} \ge a > 0 \quad \text{for some } i. \tag{2-6}
$$

Furthermore, for this particular i, the translated version $\bar{\phi}_\varepsilon(y) = \phi_\varepsilon(y - x_i)$ converges uniformly on any compact set to a solution u of

$$
-\Delta u - pU_{0,\Lambda}^{p-1} u = 0 \quad \text{in } \mathbb{R}^N \tag{2-7}
$$

for some $\Lambda \in [L_0, L_1]$. Since u is perpendicular to the kernel of (2-7), we have $u \equiv 0$, which contradicts $\|u(y)\|_{L^\infty(B_R(x_i))} \ge a > 0$. □

The following proposition is a direct consequence of combining Proposition 4.1 in [del Pino et al. 2003] with Lemma 2.1.

Proposition 2.2. *There exists $\varepsilon_0 > 0$ and a constant $C > 0$ such that for all $\varepsilon \le \varepsilon_0$ and all $h_\varepsilon \in L_{**}^\infty$, problem* (2-3) *has a unique solution $\phi_\varepsilon \equiv \mathcal{L}_\varepsilon(h_\varepsilon) \in L_*^\infty$. Moreover,*

$$
\|\mathcal{L}_\varepsilon(h_\varepsilon)\|_* \le C\|h_\varepsilon\|_{**}, \qquad |c_\ell| \le C\|h_\varepsilon\|_{**}. \tag{2-8}
$$

In order to prove the main theorem, we will prove that problem (1-4) admits a solution of the form $u = W_{r,\Lambda} + \phi$, where $W_{r,\Lambda} = \sum_{j=1}^{k} PU_{x_j,\Lambda}$ and $\phi \in \mathcal{H}_s$ is small and satisfies $\langle U_{x_i,\Lambda}^{p-1} Z_{i,l}, \phi\rangle = 0$, $i = 1, 2, \dots, k$, $l = 1, 2$.

We consider the perturbation problem

$$\begin{cases} -\Delta(W_{r,\Lambda}+\phi) = K(\varepsilon^{\frac{1}{N-2}}|y|)(W_{r,\Lambda}+\phi)^{p+\varepsilon} + \sum\limits_{\ell=1}^{2} c_\ell \sum\limits_{i=1}^{k} U_{x_i,\Lambda}^{p-1} Z_{i,\ell} \text{ in } B_*(0), \\ \phi \in \mathcal{H}_s, \qquad \Big\langle \sum\limits_{i=1}^{k} U_{x_i,\Lambda}^{p-1} Z_{i,\ell}, \phi \Big\rangle = 0, \quad \ell = 1, 2. \end{cases} \tag{2-9}$$

Proposition 2.3. *There is $\varepsilon_0 > 0$ such that for any $\varepsilon \le \varepsilon_0$, any $\Lambda \in [L_0, L_1]$, and*

$$r \in \big[\varepsilon^{-\frac{1}{N-2}}(1 - r_0 \varepsilon^{\frac{1}{N-1}}), \varepsilon^{-\frac{1}{N-2}}(1 - r_1 \varepsilon^{\frac{1}{N-1}})\big],$$

problem (2-9) has a unique solution $\phi = \phi_{r,\Lambda}$ satisfying

$$\|\phi\|_* \le C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}, \quad |c_\ell| \le C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)},$$

where $\sigma > 0$ is a small constant.

Let

$$N_\varepsilon(\phi) = K\big(\varepsilon^{\frac{1}{N-2}}|y|\big)\big((W_{r,\Lambda} + \phi)^{p+\varepsilon} - W_{r,\Lambda}^{p+\varepsilon} - (p+\varepsilon) W_{r,\Lambda}^{p-1+\varepsilon}\phi\big),$$

$$l_\varepsilon = K\big(\varepsilon^{\frac{1}{N-2}}|y|\big) W_{r,\Lambda}^{p+\varepsilon} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{p}.$$

Then problem (2-9) can be written as

$$\begin{cases} -\Delta\phi - (p+\varepsilon)K(\varepsilon^{\frac{1}{N-2}}|y|) W_{r,\Lambda}^{p-1+\varepsilon}\phi \\ \qquad\qquad = N_\varepsilon(\phi) + l_\varepsilon + \sum\limits_{\ell=1}^{2} c_\ell \sum\limits_{i=1}^{k} U_{x_i,\Lambda}^{p-1} Z_{i,\ell} \quad \text{in } B_*(0), \\ \phi \in \mathcal{H}_s, \qquad \Big\langle \sum\limits_{i=1}^{k} U_{x_i,\Lambda}^{p-1} Z_{i,\ell}, \phi \Big\rangle = 0, \quad \ell = 1, 2. \end{cases} \tag{2-10}$$

We will use the contraction mapping theorem to prove that problem (2-9) is uniquely solvable under the condition that $\|\phi\|_*$ is small enough. So we need to estimate $N_\varepsilon(\phi)$ and l_ε.

Lemma 2.4. *If $N \ge 4$, then*

$$\|N_\varepsilon(\phi)\|_{**} \le C\|\phi\|_*^{\min\{p+\varepsilon, 2\}}.$$

Proof. We have

$$|N_\varepsilon(\phi)| \le \begin{cases} C|\phi|^{p+\varepsilon}, & N \ge 7, \\ C(W_{r,\Lambda}^{p-2+\varepsilon}\phi^2 + |\phi|^{p+\varepsilon}), & N = 4, 5, 6. \end{cases}$$

Firstly, we consider $N \geq 7$. By the Hölder inequality, we have

$$
\begin{aligned}
|N_\varepsilon(\phi)| &\leq C\|\phi\|_*^{p+\varepsilon}\bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N-2)+\tau}}\bigg)^{p+\varepsilon} \\
&\leq C\|\phi\|_*^{p+\varepsilon}\bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N+2)+\tau}}\bigg) \\
&\qquad \times \bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{p+\varepsilon}{p+\varepsilon-1}(\frac{1}{2}(N-2)+\tau)-\frac{1}{p+\varepsilon-1}(\frac{1}{2}(N+2)+\tau)}}\bigg)^{p+\varepsilon-1} \\
&\leq C\|\phi\|_*^{p+\varepsilon}\bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N+2)+\tau}}\bigg).
\end{aligned}
$$

Thus, the result follows.

Suppose that $N = 4, 5, 6$. Using the fact that $N-2 > \frac{1}{2}(N-2)+\tau$, we get

$$
\begin{aligned}
|N_\varepsilon(\phi)| &\leq C\|\phi\|_*^{2}\bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N-2}}\bigg)^{p-2+\varepsilon}\bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N-2)+\tau}}\bigg)^{2} \\
&\qquad + C\|\phi\|_*^{p+\varepsilon}\bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N+2)+\tau}}\bigg) \\
&\leq C\|\phi\|_*^{2}\bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N-2)+\tau}}\bigg)^{p+\varepsilon} \\
&\qquad + C\|\phi\|_*^{p+\varepsilon}\bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N+2)+\tau}}\bigg) \\
&\leq C\|\phi\|_*^{2}\bigg(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N+2)+\tau}}\bigg).
\end{aligned}
$$

So we have proved that for $N \geq 4$,

$$\|N_\varepsilon(\phi)\|_{**} \leq C\|\phi\|_*^{\min\{p+\varepsilon,2\}}. \qquad \square$$

Lemma 2.5. *Assume that* $r \in \big[\varepsilon^{-\frac{1}{N-2}}(1-r_0\varepsilon^{\frac{1}{N-1}}), \varepsilon^{-\frac{1}{N-2}}(1-r_1\varepsilon^{\frac{1}{N-1}})\big]$. *If* $N \geq 4$, *then*

$$\|l_\varepsilon\|_{**} \leq C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}.$$

Proof. Define

$$\Omega_j = \bigg\{y : y = (y', y'') \in B_{\varepsilon^{-1/(N-2)}}(0), \bigg\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|}\bigg\rangle \geq \cos\frac{\pi}{k}\bigg\}.$$

We have

$$
\begin{aligned}
l_\varepsilon &= K(\varepsilon^{1/(N-2)}|y|)(W_{r,\Lambda}^{p+\varepsilon}-W_{r,\Lambda}^{p}) + K(\varepsilon^{1/(N-2)}|y|)\Big(W_{r,\Lambda}^{p}-\sum_{j=1}^{k} PU_{x_j,\Lambda}^{p}\Big)\\
&\quad + K(\varepsilon^{1/(N-2)}|y|)\Big(\sum_{j=1}^{k} PU_{x_j,\Lambda}^{p}-\sum_{j=1}^{k} U_{x_j,\Lambda}^{p}\Big)+\sum_{j=1}^{k} U_{x_j,\Lambda}^{p}\big(K(\varepsilon^{1/(N-2)}|y|)-1\big)\\
&=: J_0+J_1+J_2+J_3.
\end{aligned}
$$

Estimate of J_0.

$$
\begin{aligned}
|J_0| &\le C\varepsilon W_{r,\Lambda}^{p}|\ln W_{r,\Lambda}|\\
&\le C\varepsilon\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{p}\ln\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\\
&\le C\varepsilon\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac12(N+2)+\tau}}\Big)\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{4}\left(\frac{N-2}{2}-\frac{N-2}{N+2}\tau\right)}}\Big)^{\frac{4}{N-2}}\\
&\le C\varepsilon\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac12(N+2)+\tau}}.
\end{aligned}
$$

Estimate of J_1. From the symmetry, we can assume that $y\in\Omega_1$. Then,

$$
|y-x_j|\ge|y-x_1|,\qquad y\in\Omega_1,\ j\ne1.
$$

Firstly, we claim

$$
\frac{1}{1+|y-x_j|}\le\frac{C}{|x_j-x_1|},\qquad y\in\Omega_1,\ j\ne1. \tag{2-11}
$$

In fact, if $|y-x_1|\le\frac12|x_1-x_j|$, then $|y-x_j|\ge\frac12|x_j-x_1|$. If $|y-x_1|\ge\frac12|x_j-x_1|$, then $|y-x_j|\ge|y-x_1|\ge\frac12|x_j-x_1|$.

It's easy to verify that

$$
|J_1|\le C\frac{1}{(1+|y-x_1|)^4}\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}+C\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{p}.
$$

Using (2-11), taking $1<\varrho\le N-2$, we obtain for $y\in\Omega_1$,

$$
\frac{1}{(1+|y-x_1|)^4}\frac{1}{(1+|y-x_j|)^{N-2}}\le C\frac{1}{(1+|y-x_1|)^{N+2-\varrho}}\frac{1}{|x_j-x_1|^{\varrho}},\qquad j\ne1.
$$

Take $\varrho > \max\left\{\frac{1}{2}(N-1), 1\right\}$ satisfying $N+2-\varrho \geq \frac{1}{2}(N+2)+\tau$. Then

$$\begin{aligned}\frac{1}{(1+|y-x_1|)^4}\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}} &\leq \frac{C}{(1+|y-x_1|)^{N+2-\varrho}}(k\varepsilon^{1/(N-2)})^{\varrho}\\ &= \frac{C}{(1+|y-x_1|)^{N+2-\varrho}}\varepsilon^{\varrho/((N-1)(N-2))} \leq \frac{C}{(1+|y-x_1|)^{\frac{1}{2}(N+2)+\tau}}\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}.\end{aligned}$$

By the Hölder inequality, we find

$$\begin{aligned}&\left(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\right)^{p}\\ &\qquad\leq \sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N+2)+\tau}}\left(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{4}\left(\frac{N-2}{2}-\tau\frac{N-2}{N+2}\right)}}\right)^{\frac{4}{N-2}}.\end{aligned}$$

Noticing that $\dfrac{N+2}{N-2}\left(\dfrac{N-2}{2}-\tau\dfrac{N-2}{N+2}\right) > \dfrac{N-1}{2}$ if $N \geq 4$, we deduce that

$$\begin{aligned}&\left(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\right)^{p}\\ &\qquad\leq C\left(\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{N+2}{4}\left(\frac{N-2}{2}-\tau\frac{N-2}{N+2}\right)}}\right)^{\frac{4}{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N+2)+\tau}}\\ &\qquad\leq C(k\varepsilon^{1/(N-2)})^{\frac{N+2}{N-2}\left(\frac{N-2}{2}-\tau\frac{N-2}{N+2}\right)}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N+2)+\tau}}\\ &\qquad\leq C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{1}{2}(N+2)+\tau}}.\end{aligned}$$

Hence, we conclude that if $N \geq 4$,

$$\|J_1\|_{**} \leq C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}.$$

Estimate of J_2. Let $H(y,x)$ be the regular part of the Green function for $-\Delta$ in $B_1(0)$ with the zero boundary condition. Let $\bar{x}_j^*$ be the reflection point of $\bar{x}_j$ with respect to $\partial B_1(0)$. Then

$$\varepsilon H(\bar{y},\bar{x}_j) = \frac{C\varepsilon}{|\bar{y}-\bar{x}_j^*|^{N-2}} \leq \frac{C}{(1+|y-x_j|)^{N-2}}.$$

By direct calculation, we have

$$\begin{aligned}|J_2| &\le C\sum_{j=1}^{k}\frac{C\varepsilon}{(1+|y-x_j|)^4}H(\bar{y},\bar{x}_j)\\ &\le \sum_{j=1}^{k}\frac{C}{(1+|y-x_j|)^{4+(1-\gamma)(N-2)}}(\varepsilon H(\bar{y},\bar{x}_j))^{\gamma}\\ &\le C\varepsilon^{\gamma/(N-1)}\sum_{j=1}^{k}\frac{C}{(1+|y-x_j|)^{4+(1-\gamma)(N-2)}}\\ &\le C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}\sum_{j=1}^{k}\frac{C}{(1+|y-x_j|)^{\frac{1}{2}(N+2)+\tau}},\end{aligned}$$

where $\gamma > 0$ satisfies $\gamma(N-2)/(N-1) > \frac{1}{2}$ and $4+(1-\gamma)(N-2) \ge \frac{1}{2}(N+2)+\tau$.

Estimate of J_3. For $y \in \Omega_1$ and $j > 1$, using (2-11), we find

$$U^p_{x_j,\Lambda} \le \frac{C}{(1+|y-x_1|)^{\frac{1}{2}(N+2)+\tau}}\frac{1}{|x_j-x_1|^{\frac{1}{2}(N+2)-\tau}}.$$

Thus, we have

$$\begin{aligned}\Big|\sum_{j=2}^{k}\big(K(\varepsilon^{1/(N-2)}|y|)-1\big)U^p_{x_j,\Lambda}\Big| &\le \frac{C}{(1+|y-x_1|)^{\frac{1}{2}(N+2)+\tau}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{1}{2}(N+2)-\tau}}\\ &\le \frac{C}{(1+|y-x_1|)^{\frac{1}{2}(N+2)+\tau}}(k\varepsilon^{1/(N-2)})^{\frac{1}{2}(N+2)-\tau}\\ &\le \frac{C}{(1+|y-x_1|)^{\frac{1}{2}(N+2)+\tau}}\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}.\end{aligned}$$

If $y \in \Omega_1$ and $\big||y|-\varepsilon^{-1/(N-2)}\big| \ge \delta\varepsilon^{-1/(N-2)}$, where $\delta > 0$ is a fixed constant, then

$$\big||y|-|x_1|\big| \ge \big||y|-\varepsilon^{-1/(N-2)}\big| - \big||x_1|-\varepsilon^{-1/(N-2)}\big| \ge \tfrac{1}{2}\delta\varepsilon^{-1/(N-2)}.$$

So, we obtain

$$\big|U^p_{x_1,\Lambda}\big(K(\varepsilon^{1/(N-2)}|y|)-1\big)\big| \le \frac{C}{(1+|y-x_1|)^{\frac{1}{2}(N+2)+\tau}}\varepsilon^{(\frac{1}{2}(N+2)-\tau)/(N-2)}.$$

If $y \in \Omega_1$ and $\big||y| - \varepsilon^{-1/(N-2)}\big| \le \delta\varepsilon^{-1/(N-2)}$, then

$$\begin{aligned}\big|K(\varepsilon^{1/(N-2)}|y|) - 1\big| &\le C\big|\varepsilon^{1/(N-2)}|y| - 1\big| \\ &\le C\varepsilon^{1/(N-2)}\big(\big||y| - |x_1|\big| + \big||x_1| - \varepsilon^{-1/(N-2)}\big|\big) \\ &\le C\varepsilon^{1/(N-2)}\big||y| - |x_1|\big| + C\varepsilon^{1/(N-1)} \\ &\le C\varepsilon^{1/(N-2)}\big||y| - |x_1|\big| + C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)},\end{aligned}$$

and

$$\big||y| - |x_1|\big| \le |y|\big|-\varepsilon^{-1/(N-2)}\big| + \big||x_1| - \varepsilon^{-1/(N-2)}\big| \le 2\delta\varepsilon^{-1/(N-2)}.$$

Since

$$\begin{aligned}\frac{\varepsilon^{1/(N-2)}\big||y| - |x_1|\big|}{(1+|y-x_1|)^{N+2}} &\le C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}\frac{\big||y| - |x_1|\big|^{\frac{1}{2}+\sigma}}{(1+|y-x_1|)^{N+2}} \\ &\le \frac{C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}}{(1+|y-x_1|)^{N+\frac{3}{2}-\sigma}} \le \frac{C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}}{(1+|y-x_1|)^{\frac{1}{2}(N+2)+\tau}},\end{aligned}$$

we get

$$\big|U^p_{x_1,\Lambda}\big(K(\varepsilon^{1/(N-2)}|y|) - 1\big)\big| \le \frac{C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}}{(1+|y-x_1|)^{\frac{1}{2}(N+2)+\tau}}.$$

As a result, we deduce

$$\|J_3\|_{**} \le C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}. \qquad \square$$

Proof of Proposition 2.3. Recall that

$$k = [\varepsilon^{-1/(N-1)}], \quad N \ge 4.$$

Let

$$E = \Big\{u \in \mathcal{H}_s \cap L^\infty_* : \|u\|_* \le \varepsilon^{1/(2(N-2))} \text{ and } \int_{B_*(0)} U^{p-1}_{x_i,\Lambda} Z_{i,\ell} u = 0,\ i = 1, \dots, k,\ \ell = 1, 2\Big\}.$$

Then, (2-10) is equivalent to

$$\phi = A_\varepsilon(\phi) =: \mathcal{L}_\varepsilon(N_\varepsilon(\phi)) + \mathcal{L}_\varepsilon(l_\varepsilon),$$

where $\mathcal{L}_\varepsilon$ is defined in Proposition 2.2. We will prove that A_ε is a contraction map from E to E. First, $A_\varepsilon(E) \subset E$ because

$$\begin{aligned}\|A_\varepsilon(\phi)\|_* &\le C\|N_\varepsilon(\phi)\|_{**} + C\|l_\varepsilon\|_{**} \\ &\le C\|\phi\|_*^{\min\{p+\varepsilon,2\}} + C\|l_\varepsilon\|_{**} \le C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)} \le \varepsilon^{1/(2(N-2))}.\end{aligned}$$

Next we write

$$\|A_\varepsilon(\phi_1)-A_\varepsilon(\phi_2)\|_* = \|\mathcal{L}_\varepsilon(N_\varepsilon(\phi_1))-\mathcal{L}_\varepsilon(N_\varepsilon(\phi_2))\|_* \le C\|N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)\|_{**}.$$

If $N\ge 7$, then

$$|N_\varepsilon'(t)|\le C|t|^{p-1+\varepsilon}.$$

Thus, we have

$$\begin{aligned}
&|N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)|\\
&\quad\le C(|\phi_1|^{p-1+\varepsilon}+|\phi_2|^{p-1+\varepsilon})|\phi_1-\phi_2|\\
&\quad\le C(\|\phi_1\|_*^{p-1+\varepsilon}+\|\phi_2\|_*^{p-1+\varepsilon})\|\phi_1-\phi_2\|_*\biggl(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac12(N-2)+\tau}}\biggr)^{p+\varepsilon}\\
&\quad\le C(\|\phi_1\|_*^{p-1+\varepsilon}+\|\phi_2\|_*^{p-1+\varepsilon})\|\phi_1-\phi_2\|_*\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac12(N+2)+\tau}}.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\|A_\varepsilon(\phi_1)-A_\varepsilon(\phi_2)\|_* &\le C\|N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)\|_{**}\\
&\le C(\|\phi_1\|_*^{p-1+\varepsilon}+\|\phi_2\|_*^{p-1+\varepsilon})\|\phi_1-\phi_2\|_*\le \tfrac12\|\phi_1-\phi_2\|_*.
\end{aligned}$$

For $N=4,5,6$,

$$|N_\varepsilon'(t)|\le CW_{r,\Lambda}^{p-2+\varepsilon}|t|+C|t|^{p-1+\varepsilon}.$$

So we have

$$\begin{aligned}
&|N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)|\\
&\quad\le C(|\phi_1|^{p-1+\varepsilon}+|\phi_2|^{p-1+\varepsilon})|\phi_1-\phi_2|+C(|\phi_1|+|\phi_2|)W_{r,\Lambda}^{p-2+\varepsilon}|\phi_1-\phi_2|\\
&\quad\le C(\|\phi_1\|_*^{p-1+\varepsilon}+\|\phi_2\|_*^{p-1+\varepsilon})\|\phi_1-\phi_2\|_*\biggl(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac12(N-2)+\tau}}\biggr)^{p+\varepsilon}\\
&\qquad+C(\|\phi_1\|_*+\|\phi_2\|_*)\|\phi_1-\phi_2\|_*W_{r,\Lambda}^{p-2+\varepsilon}\biggl(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac12(N-2)+\tau}}\biggr)^{2}\\
&\quad\le C(\|\phi_1\|_*+\|\phi_2\|_*)\|\phi_1-\phi_2\|_*\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac12(N+2)+\tau}}.
\end{aligned}$$

In either case, we see that A_ε is a contraction map. By the contraction mapping theorem, there is a unique $\phi\in E$ such that

$$\phi=A_\varepsilon(\phi).$$

Moreover, it follows from Proposition 2.2 that

$$\|\phi\|_* \le C\|l_\varepsilon\|_{**} + C\|N_\varepsilon(\phi)\|_{**} \le C\|l_\varepsilon\|_{**} + C\|\phi\|_*^{\min\{p+\varepsilon,2\}},$$

which implies

$$\|\phi\|_* \le C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}, \quad |c_\ell| \le C\varepsilon^{(\frac{1}{2}+\sigma)/(N-2)}. \qquad \square$$

3. Proof of the main results

In this section, we will choose (r, Λ) such that $W_{r,\Lambda} + \phi_{r,\Lambda}$ is a solution of (1-4), where $\phi_{r,\Lambda}$ is the map obtained in Proposition 2.3.

Lemma 3.1. *If* (r, Λ) *satisfies*

(3-1)
$$\int_{B_*(0)} \left(\nabla(W_{r,\Lambda} + \phi_{r,\Lambda}) \nabla \frac{\partial W_{r,\Lambda}}{\partial r} - K(\varepsilon^{1/(N-2)}|y|)(W_{r,\Lambda} + \phi_{r,\Lambda})^{p+\varepsilon} \frac{\partial W_{r,\Lambda}}{\partial r} \right) = 0,$$

(3-2)
$$\int_{B_*(0)} \left(\nabla(W_{r,\Lambda} + \phi_{r,\Lambda}) \nabla \frac{\partial W_{r,\Lambda}}{\partial \Lambda} - K(\varepsilon^{1/(N-2)}|y|)(W_{r,\Lambda} + \phi_{r,\Lambda})^{p+\varepsilon} \frac{\partial W_{r,\Lambda}}{\partial \Lambda} \right) = 0,$$

then $W_{r,\Lambda} + \phi_{r,\Lambda}$ *is a solution of* (1-4).

Proof. It follows from Proposition 2.3 that if (3-1) and (3-2) hold, then by symmetry,

$$c_1 \left\langle U_{x_1,\Lambda}^{p-1} \frac{\partial PU_{x_1,\Lambda}}{\partial r}, \frac{\partial W_{r,\Lambda}}{\partial r} \right\rangle = 0 = c_2 \left\langle U_{x_1,\Lambda}^{p-1} \frac{\partial PU_{x_1,\Lambda}}{\partial \Lambda}, \frac{\partial W_{r,\Lambda}}{\partial \Lambda} \right\rangle, \tag{3-3}$$

which implies that $c_1 = c_2 = 0$. Hence $W_{r,\Lambda} + \phi_{r,\Lambda}$ is a solution of (1-4). $\square$

In the rest of this section, we need to solve (3-1) and (3-2).

Proposition 3.2. *Equations* (3-1) *and* (3-2) *are equivalent to*

$$-\frac{\varepsilon H(\bar{x}_1, \bar{x}_1)}{\Lambda^{N-1}} + \sum_{i=2}^{k} \frac{\varepsilon G(\bar{x}_i, \bar{x}_1)}{\Lambda^{N-1}} + O(\varepsilon^{(1+\sigma)/(N-2)}) = 0 \tag{3-4}$$

and

$$\frac{B_2\varepsilon}{\Lambda^{N-2}} \frac{\partial H(\bar{x}_1, \bar{x}_1)}{\partial d} + B_3 K'(1) + \sum_{i=2}^{k} \frac{B_2\varepsilon}{\Lambda^{N-2}} \frac{\partial G(\bar{x}_i, \bar{x}_1)}{\partial d} + O(\varepsilon^{\sigma/(N-2)}) = 0, \tag{3-5}$$

respectively, where $d = 1 - \varepsilon^{1/(N-2)} r$, B_1, B_2 *and* B_3 *are the same positive constants as in Proposition A.1 and* $\sigma > 0$ *is a small constant.*

Proof. Here we prove only the first one. The second can be proved similarly by noting that $\partial/\partial d = -\varepsilon^{-1/(N-2)}\partial/\partial r$.

First, we see that

$$\int_{B_*(0)} \nabla(W_{r,\Lambda}+\phi_{r,\Lambda})\nabla\frac{\partial W_{r,\Lambda}}{\partial\Lambda} = \int_{B_*(0)} \nabla W_{r,\Lambda}\nabla\frac{\partial W_{r,\Lambda}}{\partial\Lambda},$$

and

$$\begin{aligned}\int_{B_*(0)} & K(\varepsilon^{1/(N-2)}|y|)(W_{r,\Lambda}+\phi_{r,\Lambda})^{p+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\\ &= \int_{B_*(0)} K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\\ &\qquad + (p+\varepsilon)\int_{B_*(0)} K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p-1+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\phi_{r,\Lambda}\\ &\qquad + O\Big(\int_{B_*(0)} W_{r,\Lambda}^{p-1+\varepsilon}|\phi_{r,\Lambda}|^2\Big).\end{aligned}$$

On the other hand, noticing that $\phi_{r,\Lambda}\in E$, we have

$$\begin{aligned}\int_{B_*(0)} & K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p-1+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\phi_{r,\Lambda}\\ &= \int_{B_*(0)} K(\varepsilon^{1/(N-2)}|y|)\Big(W_{r,\Lambda}^{p-1+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{p-1}\frac{\partial U_{x_j,\Lambda}}{\partial\Lambda}\Big)\phi_{r,\Lambda}\\ &\qquad + \sum_{j=1}^{k}\int_{B_*(0)}\big(K(\varepsilon^{1/(N-2)}|y|)-1\big)U_{x_j,\Lambda}^{p-1}\frac{\partial U_{x_j,\Lambda}}{\partial\Lambda}\phi_{r,\Lambda}\\ &= k\int_{\Omega_1} K(\varepsilon^{1/(N-2)}|y|)\Big(W_{r,\Lambda}^{p-1+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{p-1}\frac{\partial U_{x_j,\Lambda}}{\partial\Lambda}\Big)\phi_{r,\Lambda}\\ &\qquad + k\int_{\Omega_1}\big(K(\varepsilon^{1/(N-2)}|y|)-1\big)U_{x_1,\Lambda}^{p-1}\frac{\partial U_{x_1,\Lambda}}{\partial\Lambda}\phi_{r,\Lambda},\end{aligned}$$

$$\begin{aligned}\Big|\int_{\Omega_1} & K(\varepsilon^{1/(N-2)}|y|)\Big(W_{r,\Lambda}^{p-1+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{p-1}\frac{\partial U_{x_j,\Lambda}}{\partial\Lambda}\Big)\phi_{r,\Lambda}\Big|\\ &\le C\int_{\Omega_1}\Big(U_{x_1,\Lambda}^{p-1}(U_{x_1,\Lambda}-PU_{x_1,\Lambda}) + U_{x_1,\Lambda}^{p-1}\sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=2}^{k} U_{x_j,\Lambda}^{p}\Big)|\phi_{r,\Lambda}|\\ &\qquad + O\Big(\varepsilon\int_{\Omega_1} W_{r,\Lambda}^{p-1}\ln W_{r,\Lambda}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\phi_{r,\Lambda}\Big)\\ &\le C\varepsilon^{1/(N-2)(1+\sigma)},\end{aligned}$$

and

$$\begin{aligned}\Big|\int_{\Omega_1}\big(K(\varepsilon^{1/(N-2)}|y|)-1\big)U_{x_1,\Lambda}^{p-1}\frac{\partial U_{x_1,\Lambda}}{\partial\Lambda}\phi_{r,\Lambda}\Big| &\le \Big|\int_{\left||y|-\varepsilon^{-1/(N-2)}\right|\le\varepsilon^{-1/(2(N-2))}}\big(K(\varepsilon^{1/(N-2)}|y|)-1\big)U_{x_1,\Lambda}^{p-1}\frac{\partial U_{x_1,\Lambda}}{\partial\Lambda}\phi_{r,\Lambda}\Big|\\ &\quad+\Big|\int_{\left||y|-\varepsilon^{-1/(N-2)}\right|\ge\varepsilon^{-1/(2(N-2))}}\big(K(\varepsilon^{1/(N-2)}|y|)-1\big)U_{x_1,\Lambda}^{p-1}\frac{\partial U_{x_1,\Lambda}}{\partial\Lambda}\phi_{r,\Lambda}\Big|\\ &\le C\varepsilon^{1/(N-2)(1+\sigma)}.\end{aligned}$$

So, we have proved

$$\begin{aligned}&\int_{B_*(0)}\Big(\nabla(W_{r,\Lambda}+\phi_{r,\Lambda})\nabla\frac{\partial W_{r,\Lambda}}{\partial\Lambda}-K(\varepsilon^{1/(N-2)}|y|)(W_{r,\Lambda}+\phi_{r,\Lambda})^{p+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\Big)\\ &\quad=\int_{B_*(0)}\Big(\nabla W_{r,\Lambda}\nabla\frac{\partial W_{r,\Lambda}}{\partial\Lambda}-K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\Big)+O(k\varepsilon^{(1+\sigma)/(N-2)}),\end{aligned}$$

and the result follows from Proposition A.1. □

Proof of Theorem 1.3. Note (see [Wei and Yan 2013]) that

$$H(\bar x_1,\bar x_1)=\frac{1}{2^{N-2}d^{N-2}}(1+O(d))$$

and

$$\frac{a_0}{j^N}+O\Big(\frac{d}{j^{N-2}}\Big)\le\frac{1}{k^{N-2}}G(\bar x_j,\bar x_1)\le\frac{a_1}{j^N}+O\Big(\frac{d}{j^{N-2}}\Big),$$

where $a_1\ge a_0>0$. Hence, we find that there is a constant $B_4>0$ such that

$$\sum_{j=2}^{k}G(\bar x_j,\bar x_1)=k^{N-2}\Big(\frac{B_4}{|\bar x_1|^{N-2}}+O\Big(\frac{1}{k^{N-1}}\Big)+O(d)\Big)=B_4k^{N-2}+O(k^{N-2}d).$$

Consequently, (3-4) and (3-5) are equivalent to

$$-\frac{A_1\varepsilon}{\Lambda^{N-1}d^{N-2}}+\frac{A_2k^{N-2}\varepsilon}{\Lambda^{N-1}}+O(\varepsilon^{(1+\sigma)/(N-2)})=0 \tag{3-6}$$

and

$$-\frac{A_3\varepsilon}{\Lambda^{N-2}d^{N-1}}+A_4+O(\varepsilon^{\sigma/(N-2)})=0, \tag{3-7}$$

respectively, for positive constants A_i, $i=1,2,3,4$. Recall that $d=1-\varepsilon^{1/(N-2)}r$. Define $\eta=dk$. Thus, (3-6) and (3-7) read

$$-\frac{A_1}{\Lambda^{N-1}\eta^{N-2}}+\frac{A_2}{\Lambda^{N-1}}+O(\varepsilon^{\sigma/(N-2)})=0 \tag{3-8}$$

and

$$-\frac{A_3}{\Lambda^{N-2}\eta^{N-1}} + A_4 + O(\varepsilon^{\sigma/(N-2)}) = 0. \tag{3-9}$$

Let

$$f_1(\eta, \Lambda) = -\frac{A_1}{\Lambda^{N-1}\eta^{N-2}} + \frac{A_2}{\Lambda^{N-1}}$$

and

$$f_2(\eta, \Lambda) = -\frac{A_3}{\Lambda^{N-2}\eta^{N-1}} + A_4.$$

It is easy to check that $f_1 = 0$ and $f_2 = 0$ have a unique solution

$$\eta_0 = \left(\frac{A_1}{A_2}\right)^{\frac{1}{N-2}}, \quad \Lambda_0 = \left(\frac{A_3}{A_4\eta_0^{N-1}}\right)^{\frac{1}{N-2}}.$$

On the other hand, we have

$$\frac{\partial f_1(\eta_0, \Lambda_0)}{\partial \Lambda} = 0, \quad \frac{\partial f_2(\eta_0, \Lambda_0)}{\partial \eta} > 0,$$

and

$$\frac{\partial f_1(\eta_0, \Lambda_0)}{\partial \eta} > 0, \quad \frac{\partial f_2(\eta_0, \Lambda_0)}{\partial \Lambda} > 0.$$

Hence the linear operator of $f_1 = 0$ and $f_2 = 0$ at (η_0, Λ_0) is invertible. Therefore, (3-8) and (3-9) have a solution near (η_0, Λ_0). □

Appendix A: Energy expansion

Here and in Appendix B, we assume that

$$x_j = \left(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\right), \quad j = 1, \dots, k,$$

where $r \in \left[\varepsilon^{-\frac{1}{N-2}}(1 - r_0\varepsilon^{\frac{1}{N-1}}), \varepsilon^{-\frac{1}{N-2}}(1 - r_1\varepsilon^{\frac{1}{N-1}})\right]$ and 0 is the zero vector in $\mathbb{R}^{N-2}$.

Let

$$\bar{x}_j = \varepsilon^{\frac{1}{N-2}} x_j.$$

Let $G(x, y)$ be the Green function of $-\Delta$ in $B_1(0)$ with the Dirichlet boundary and let $H(x, y)$ be the regular part of the Green function.

Recall that

$$k = \left[\varepsilon^{-\frac{1}{N-1}}\right]$$

and

$$W_{r,\Lambda}(y) = \sum_{j=1}^{k} PU_{x_j,\Lambda}(y),$$

where $PU_{x,\Lambda}$ is the solution of (1-5). Moreover,

$$\text{(A-1)}\qquad \phi_{x_j,\Lambda}(y)=U_{x_j,\Lambda}(y)-PU_{x_j,\Lambda}(y)=\frac{\varepsilon H(\bar{x}_j,\bar{y})}{\Lambda^{\frac{1}{2}(N-2)}}+O\Big(\frac{\varepsilon^{N/(N-2)}}{d^N}\Big),$$

where $d=1-|\bar{x}_j|=1-\varepsilon^{1/(N-2)}|x_j|$.

Proposition A.1. *We have*

$$\begin{aligned}\int_{B_*(0)}&\Big(\nabla W_{r,\Lambda}\nabla\frac{\partial W_{r,\Lambda}}{\partial\Lambda}-K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\Big)\\&=kB_1\Big(-\frac{\varepsilon H(\bar{x}_1,\bar{x}_1)}{\Lambda^{N-1}}+\sum_{i=2}^{k}\frac{\varepsilon G(\bar{x}_i,\bar{x}_1)}{\Lambda^{N-1}}+O(\varepsilon^{(1+\sigma)/(N-2)})\Big),\end{aligned}$$

and

$$\begin{aligned}\int_{B_*(0)}&\Big(\nabla W_{r,\Lambda}\nabla\frac{\partial W_{r,\Lambda}}{\partial r}-K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial r}\Big)\\&=k\Big(\frac{B_2\varepsilon}{\Lambda^{N-2}}\frac{\partial H(\bar{x}_1,\bar{x}_1)}{\partial r}-B_3K'(1)\varepsilon^{1/(N-2)}\\&\qquad+\sum_{i=2}^{k}\frac{B_2\varepsilon}{\Lambda^{N-2}}\frac{\partial G(\bar{x}_i,\bar{x}_1)}{\partial r}+O(\varepsilon^{(1+\sigma)/(N-2)})\Big),\end{aligned}$$

where B_1, B_2 and B_3 are some positive constants.

Proof. The proof is quite standard now. Here we only prove the first equation. The other one can be obtained similarly.

Using symmetry, we find

$$\begin{aligned}I&:=\int_{B_*(0)}\Big(\nabla W_{r,\Lambda}\nabla\frac{\partial W_{r,\Lambda}}{\partial\Lambda}-K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\Big)\\&=k\Big(p\sum_{i=1}^{k}\int_{B_*(0)}PU_{x_1,\Lambda}^{p-1}\frac{\partial PU_{x_1,\Lambda}}{\partial\Lambda}PU_{x_i,\Lambda}-\int_{\Omega_1}K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\Big).\end{aligned}$$

It is easy to check that for $y\in\Omega_1$,

$$\begin{aligned}\frac{\partial}{\partial\Lambda}W_{r,\Lambda}^{p+1}&=\frac{\partial}{\partial\Lambda}PU_{x_1,\Lambda}^{p+1}+(p+1)\frac{\partial}{\partial\Lambda}\Big(PU_{x_1,\Lambda}^{p}\sum_{i=2}^{k}PU_{x_i,\Lambda}\Big)\\&\qquad+O\Big(U_{x_1,\Lambda}^{\frac{1}{2}(p+1)}\Big(\sum_{i=2}^{k}U_{x_i,\Lambda}\Big)^{\frac{1}{2}(p+1)}\Big).\end{aligned}$$

Thus, we have

$$
\begin{aligned}
(p+1)\int_{\Omega_1} K(\varepsilon^{1/(N-2)}|y|)W_{r,\Lambda}^{p+\varepsilon}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}
&= \int_{\Omega_1} K(\varepsilon^{1/(N-2)}|y|)\frac{\partial W_{r,\Lambda}^{p+1}}{\partial\Lambda} + O\Big(\varepsilon\int_{\Omega_1} W_{r,\Lambda}^{p+1}\ln W_{r,\Lambda}\Big)\\
&= \int_{\Omega_1} K(\varepsilon^{1/(N-2)}|y|)\frac{\partial}{\partial\Lambda}PU_{x_1,\Lambda}^{p+1}\\
&\quad + (p+1)\int_{\Omega_1} K(\varepsilon^{1/(N-2)}|y|)\frac{\partial}{\partial\Lambda}\Big(PU_{x_1,\Lambda}^{p}\sum_{i=2}^{k}PU_{x_i,\Lambda}\Big)\\
&\quad + O\Big(\int_{\Omega_1} U_{x_1,\Lambda}^{\frac12(p+1)}\Big(\sum_{i=2}^{k}U_{x_i,\Lambda}\Big)^{\frac12(p+1)}\Big) + O\Big(\varepsilon\int_{\Omega_1} W_{r,\Lambda}^{p+1}\ln W_{r,\Lambda}\Big).
\end{aligned}
$$

Note that for $y\in\Omega_1$, $|y-x_i|\ge|y-x_1|$. Using (2-11), we see that for $t\in(1,N-2)$,

$$
\sum_{i=2}^{k}U_{x_i,\Lambda}\le\frac{C}{(1+|y-x_1|)^{N-2-t}}\sum_{i=2}^{k}\frac{1}{|x_i-x_1|^t}.
$$

If we take t close to $N-2$, then

$$
\int_{\Omega_1} U_{x_1,\Lambda}^{\frac12(p+1)}\Big(\sum_{i=2}^{k}U_{x_i,\Lambda}\Big)^{\frac12(p+1)} = O((k\varepsilon^{1/(N-2)})^{Nt/(N-2)}) = O(\varepsilon^{(1+\sigma)/(N-2)}).
$$

Moreover, it is easy to show that

$$
\varepsilon\int_{\Omega_1} W_{r,\Lambda}^{p+1}\ln W_{r,\Lambda} = O(\varepsilon).
$$

As a result, we obtain

$$
\begin{aligned}
I = k\Big(&-\int_{\Omega_1} K(\varepsilon^{1/(N-2)}|y|)PU_{x_1,\Lambda}^{p}\frac{\partial PU_{x_1,\Lambda}}{\partial\Lambda}\\
&-\sum_{i=2}^{k}\int_{\Omega_1} K(\varepsilon^{1/(N-2)}|y|)PU_{x_1,\Lambda}^{p}\frac{\partial PU_{x_i,\Lambda}}{\partial\Lambda}\\
&\quad + p\sum_{i=2}^{k}\int_{\Omega_1}\big(1-K(\varepsilon^{1/(N-2)}|y|)\big)PU_{x_1,\Lambda}^{p-1}\frac{\partial PU_{x_1,\Lambda}}{\partial\Lambda}PU_{x_i,\Lambda}\\
&\qquad + p\sum_{i=2}^{k}\int_{B_*(0)\setminus\Omega_1} PU_{x_1,\Lambda}^{p-1}\frac{\partial PU_{x_1,\Lambda}}{\partial\Lambda}PU_{x_i,\Lambda} + O(\varepsilon^{(1+\sigma)/(N-2)})\Big).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_{\Omega_1} K(\varepsilon^{1/(N-2)}|y|) PU_{x_1,\Lambda}^p \frac{\partial PU_{x_1,\Lambda}}{\partial \Lambda} \\
&\quad = \int_{\Omega_1} PU_{x_1,\Lambda}^p \frac{\partial PU_{x_1,\Lambda}}{\partial \Lambda} + \int_{\Omega_1} \big(K(\varepsilon^{1/(N-2)}|y|) - 1\big) PU_{x_1,\Lambda}^p \frac{\partial PU_{x_1,\Lambda}}{\partial \Lambda} \\
&\quad = \int_{\Omega_1} PU_{x_1,\Lambda}^p \frac{\partial PU_{x_1,\Lambda}}{\partial \Lambda} \\
&\qquad\qquad + \int_{\Omega_1} \big(K(|\bar{x}_1|) - 1\big) PU_{x_1,\Lambda}^p \frac{\partial PU_{x_1,\Lambda}}{\partial \Lambda} + O(\varepsilon^{(1+\sigma)/(N-2)}) \\
&\quad = \int_{\Omega_1} PU_{x_1,\Lambda}^p \frac{\partial PU_{x_1,\Lambda}}{\partial \Lambda} - K'(1)d \int_{\Omega_1} PU_{x_1,\Lambda}^p \frac{\partial PU_{x_1,\Lambda}}{\partial \Lambda} + O(\varepsilon^{(1+\sigma)/(N-2)}) \\
&\quad = -\int_{\Omega_1} U_{x_1,\Lambda}^p \frac{\partial \phi_{x_1,\Lambda}}{\partial \Lambda} - p \int_{\Omega_1} U_{x_1,\Lambda}^p \frac{\partial U_{x_1,\Lambda}}{\partial \Lambda} \phi_{x_1,\Lambda} + O(\varepsilon^{(1+\sigma)/(N-2)}) \\
&\quad = \frac{B_1 \varepsilon H(\bar{x_1}, \bar{x}_1)}{\Lambda^{N-1}} + O(\varepsilon^{(1+\sigma)/(N-2)})
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\Omega_1} K(\varepsilon^{1/(N-2)}|y|) PU_{x_1,\Lambda}^p \frac{\partial PU_{x_i,\Lambda}}{\partial \Lambda} \\
&\qquad = \int_{\Omega_1} PU_{x_1,\Lambda}^p \frac{\partial PU_{x_i,\Lambda}}{\partial \Lambda} + \int_{\Omega_1} \big(K(\varepsilon^{1/(N-2)}|y|) - 1\big) PU_{x_1,\Lambda}^p \frac{\partial PU_{x_i,\Lambda}}{\partial \Lambda} \\
&\qquad = \int_{\Omega_1} U_{x_1,\Lambda}^p \frac{\partial U_{x_i,\Lambda}}{\partial \Lambda} - \int_{\Omega_1} U_{x_1,\Lambda}^p \frac{\partial \phi_{x_i,\Lambda}}{\partial \Lambda} + O(\varepsilon^{(1+\sigma)/(N-2)}) \\
&\qquad = -\frac{B_1 \varepsilon G(\bar{x}_i, \bar{x}_1)}{\Lambda^{N-1}} + O(\varepsilon^{(1+\sigma)/(N-2)}).
\end{aligned}
$$

Other terms can be estimated similarly. Thus, the result follows. □

Appendix B: Basic estimates

In this section, we will give some basic estimates used in the reduction procedure. We will use the same constant $C > 0$ to denote the different constants.

Lemma B.1. *Let* $g_{ij} = 1/((1+|y-x_i|)^\alpha (1+|y-x_j|)^\beta)$ *for each fixed* i *and* j, $i \neq j$, *where* $\alpha \geq 1$ *and* $\beta \geq 1$ *are two constants. Then for any* $0 < \sigma \leq \min(\alpha, \beta)$, *there is a constant* $C > 0$ *such that*

$$
g_{ij}(y) \leq \frac{C}{|x_i - x_j|^\sigma} \left(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}} \right).
$$

Lemma B.2. *For any constant* $0 < \sigma < N-2$, *there is a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}}\, dz \le \frac{C}{(1+|y|)^{\sigma}}.$$

The proofs of the above two lemmas can be found in [Wei and Yan 2010b].

Lemma B.3. *Suppose that* $\varepsilon > 0$ *and* $N \ge 4$. *Then there is a small* $\vartheta > 0$ *such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{p-1+\varepsilon}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{1}{2}(N-2)+\tau}}\, dz \le C \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{1}{2}(N-2)+\tau+\vartheta}}.$$

Proof. This is similar to the proof of Lemma B.3 in [Wei and Yan 2010b]. So we just sketch it. Note that

$$W_{r,\Lambda}(z) \le C \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{N-2}}.$$

As in [Wei and Yan 2010b], for $y \in \Omega_1$ we have $W_{r,\Lambda}(z) \le \frac{C}{(1+|z-x_1|)^{N-2-\tau_1}}$, where $0 < \tau_1 \le \frac{1}{2}(N-2)$. Thus,

$$W_{r,\Lambda}^{p-1+\varepsilon}(z) \le \frac{C}{(1+|z-x_1|)^{4-\frac{4\tau_1}{N-2}+(N-2-\tau_1)\varepsilon}}.$$

By virtue of Lemma B.1, for $y \in \Omega_1$ we get

$$\begin{aligned} W_{r,\Lambda}^{p-1+\varepsilon}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{1}{2}(N-2)+\tau}} &\le \frac{C}{(1+|z-x_1|)^{\frac{1}{2}(N+6)+\tau-\frac{4\tau_1}{N-2}+(N-2-\tau_1)\varepsilon}} \\ &\quad + \sum_{j=2}^{k} \frac{C}{(1+|z-x_1|)^{4-\frac{4\tau_1}{N-2}+(N-2-\tau_1)\varepsilon}} \frac{1}{(1+|z-x_j|)^{\frac{1}{2}(N-2)+\tau}} \\ &\le \frac{C}{(1+|z-x_1|)^{\frac{1}{2}(N+6)+\tau-\frac{4\tau_1}{N-2}+(N-2-\tau_1)\varepsilon}} \\ &\quad + \frac{C}{(1+|z-x_1|)^{\frac{1}{2}(N+6)+\tau-\frac{N+2}{N-2}\tau_1+(N-2-\tau_1)\varepsilon}} \sum_{j=2}^{k} \frac{1}{|x_j-x_1|^{\tau_1}} \\ &\le \frac{C}{(1+|z-x_1|)^{\frac{1}{2}(N+6)+\tau-\frac{N+2}{N-2}\tau_1+(N-2-\tau_1)\varepsilon}}. \end{aligned}$$

Thus, we can obtain

$$\begin{aligned}\int_{\Omega_1} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{p-1+\varepsilon}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{1}{2}(N-2)+\tau}}\, dz \\ \leq \int_{\Omega_1} \frac{1}{|y-z|^{N-2}} \frac{C}{(1+|z-x_1|)^{\frac{1}{2}(N+6)+\tau-\frac{N+2}{N-2}\tau_1+(N-2-\tau_1)\varepsilon}}\, dz \\ \leq \frac{C}{(1+|z-x_1|)^{\frac{1}{2}(N+2)+\tau-\frac{N+2}{N-2}\tau_1+(N-2-\tau_1)\varepsilon}}.\end{aligned}$$

As a result, for τ_1 satisfying $2-(N+2)/(N-2)\tau_1 > 0$, we find that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{p-1+\varepsilon}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{1}{2}(N-2)+\tau}}\, dz \\ = \sum_{i=1}^{k} \int_{\Omega_i} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{p-1+\varepsilon}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{1}{2}(N-2)+\tau}}\, dz \\ \leq \sum_{i=1}^{k} \frac{C}{(1+|z-x_1|)^{\frac{1}{2}(N+2)+\tau-\frac{N+2}{N-2}\tau_1+(N-2-\tau_1)\varepsilon}} \\ \leq C \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{1}{2}(N-2)+\tau+\vartheta}}.\end{aligned}$$

□

Acknowledgements

The authors would like to thank the referee very much for the careful reading of the paper and valuable comments, including pointing out Remark 1.5. Z. Liu was supported by funds from the NSFC (No. 11426088, No. 11501166); S. Peng was partially supported by funds from the NSFC (No. 11125101, No. 11571130).

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Received November 6, 2014. Revised May 10, 2015.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 280 No. 1 January 2016

0030-8730(2016)280:1;1-6