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Research Article

Solvability Criteria for Some Set-Valued Inequality Systems

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Arising from studying some multivalued von Neumann model, three set-valued inequality systems are introduced, and two solvability questions are considered. By constructing some auxiliary functions and studying their minimax and saddle-point properties, solvability criteria composed of necessary and sufficient conditions regarding these inequality systems are obtained.

1. Introduction

Arising from considering some multivalued von Neumann model, this paper aims to study three set-valued inequality systems and try to find their solvability criteria. Before starting with this subject, we need to review some necessary backgrounds as follows.

We denote by $R^k = (R^k, \|\cdot\|)$ the k-dimensional Euclidean space, $R^{k*} = R^k$ its dual, and $\langle \cdot, \cdot \rangle$ the duality pairing on $\langle R^{k*}, R^k \rangle$; moreover, we denote that $R^k_+ = \{x \in R^k : x_i \geq 0 \,\forall i\}$ and $\operatorname{int} R^k_+$ is its interior. We also define $\geq$ (or $>$) in R^k by $x \geq y \Leftrightarrow x - y \in R^k_+$ (or by $x > y \Leftrightarrow x - y \in \operatorname{int} R^k_+$).

It is known that the generalized (linear or nonlinear) von Neumann model, which is composed of an inequality system and a growth factor problem described by

$$\begin{aligned} &\text{(a)}\ \exists x \in X \Longrightarrow Bx - Ax \geq c, \\ &\text{(b)}\ \lambda > 1 \quad \text{s.t. } \exists x \in X \Longrightarrow Bx \geq \lambda Ax + c, \end{aligned} \quad \text{respectively,} \tag{1.1}$$

is one of the most important issues in the input-output analysis [1–3], where $c \in R^m_+$, $X \subseteq R^n_+$ (m may not be equal to n), and B, A are two nonnegative or positive maps from X to R^m_+.

A series of researches on (1.1) have been made by the authors of [1–5] for the linear case (i.e., B, A are $m \times n$ matrices) and by the authors of [6, 7] for the nonlinear case (i.e., B, A are some types of nonlinear maps). Since (a) or (b) of (1.1) is precisely a special example of the inequality $\lambda \in [1, +\infty)$ s.t. $\exists x \in X \Rightarrow S_\lambda x \widehat{=} (B - \lambda A)x = Bx - \lambda Ax \geq c$ if we restrict $\lambda = 1$ or $\lambda > 1$, it is enough for (1.1) to consider the inequality system. This idea can be extended to the set-valued version. Indeed, if B and A are replaced by set-valued maps G and F, respectively, then (1.1) yields a class of multivalued von Neumann model, and it solves a proper set-valued inequality system to study. With this idea, by [8] (as a set-valued extension to [6, 7]) we have considered the following multivalued inequality system:

$$\begin{gathered} c \in R_+^m \quad \text{s.t.} \\ \exists x \in X, \exists y \in Tx \Longrightarrow y \geq c \end{gathered} \tag{1.2}$$

and obtained several necessary and sufficient conditions for its solvability, where $X \subset R_+^n$ and $T : X \to 2^{R^m}$ is a class of set-valued maps from X to R^m. Along the way, three further set-valued inequality systems that we will study in the sequel can be stated as follows.

Let X, T be as above, and let $G, F : X \to 2^{R_+^m}$ be set-valued maps from X to R_+^m, then we try to find the solvability criteria (i.e., the necessary and sufficient conditions) that $c \in R_+^m$ solves

$$\begin{gathered} \exists x \in X, \quad \exists y \in Tx, \\ \exists i_0 \in \{1, 2, \ldots, m\} \Longrightarrow y \geq c, \quad y_{i_0} = c_{i_0}, \end{gathered} \tag{1.3}$$

$$\begin{gathered} \exists x \in X, \\ \exists y \in (G - F)x \Longrightarrow y \geq c, \end{gathered} \tag{1.4}$$

or

$$\begin{gathered} \exists x \in X, \quad \exists y \in (G - F)x, \\ \exists i_0 \in \{1, 2, \ldots, m\} \Longrightarrow y \geq c, \quad y_{i_0} = c_{i_0}. \end{gathered} \tag{1.5}$$

When T and G, F are single-valued maps, then (1.3)–(1.5) return to the models of [6, 7]. When T and G, F are set-valued maps, there are three troubles if we try to obtain some meaningful solvability criteria regarding (1.3)–(1.5) just like what we did in [8].

(1) For (1.2) and (1.3), it is possible that only (1.2) has solution for some $c \in R_+^m$. Indeed, if X is compact and T is continuous, compact valued with $TX \subset \operatorname{int} R_+^m$, then TX is compact and there is $c \in R_+^m$ with $c < y$ for $y \in TX$. Hence c solves (1.2) but does not solve (1.3).

(2) It seems that the solvability criteria (namely, necessary and sufficient results concerning existence) to (1.4) can be obtained immediately by [8] with the replacement $T = G - F$. However, this type of result is trivial because it depends only on the property of $G - F$ but not on the respective information of G and F. This opinion is also applicable to (1.3) and (1.5).

(3) Clearly, (1.3) (or (1.5)) is more fine and more useful than (1.2) (or (1.4)). However, the method used for (1.2) in [8] (or the possible idea for (1.4)) to obtain solvability criteria fails to be applied to find the similar characteristic results for (1.3) (or (1.5)) because there are some examples (see Examples 3.5 and 4.4) to show that, without any additional restrictions,

no necessary and sufficient conditions concerning existence for them can be obtained. This is also a main cause that the author did not consider (1.3) and (1.5) in [8].

So some new methods should be introduced if we want to search out the solvability criteria to (1.3)–(1.5). In the sections below, we are devoted to study (1.3)–(1.5) by considering two questions under two assumptions as follows:

Question 1. Whether there exist any criteria that c solves (1.3) in some proper way?

Question 2. Like Question 1, whether there exist any solvability criteria to (1.4) or (1.5) that depend on the respective information of G and F?

Assumption 1. $c \in R_+^m$ is a fixed point and $X \subset R_+^n$ is a convex compact subset.

Assumption 2. Consider the following: $T : X \to 2^{R^m}, G : X \to 2^{R_+^m}$, and $F : X \to 2^{\operatorname{int} R_+^m}$ are upper semicontinuous and convex set-valued maps with nonempty convex compact values.

By constructing some functions and studying their minimax properties, some progress concerning both questions has been made. The paper is arranged as follows. We review some concepts and known results in Section 2 and prove three Theorems composed of necessary and sufficient conditions regarding the solvability of (1.3)–(1.5) in Sections 3 and 4. Then we present the conclusion in Section 5.

2. Terminology

Let $P \subseteq R^m$, $X \subseteq R^n$, and $Y_i \subseteq R^{m_i}$ $(i = 1, 2)$. Let $f, f_\alpha : X \to R$ $(\alpha \in \Lambda)$, $\varphi = \varphi(p, x) : P \times X \to R$, and $\psi = \psi(p, (u, v)) : P \times (Y_1 \times Y_2) \to R$ be functions and $T : X \to 2^{R^m}$ a set-valued map. We need some concepts concerning f, f_α $(\alpha \in \Lambda)$ and φ and ψ such as convex or concave and upper or lower semicontinuous (in short, u.s.c. or l.s.c.) and continuous (i.e., both u.s.c. and l.s.c.), whose definitions can be found in [9–11], therefore, the details are omitted here. We also need some further concepts to T, φ, and ψ as follows.

Definition 2.1. (1) T is said to be closed if its graph defined by $\operatorname{graph} T = \{(u, v) \in X \times R^m : u \in X, v \in Tx\}$ is closed in $R^n \times R^m$. Moreover, T is said to be upper semicontinuous (in short, u.s.c.) if, for each $x \in X$ and each neighborhood $V(Tx)$ of Tx, there exists a neighborhood $U(x)$ of x such that $T(U(x) \cap X) \subseteq V(Tx)$.

(2) Assume that $Y \subseteq R^m$ $(Y \neq \emptyset)$, and define $\sigma^\#(Y, p) = \sup_{y \in Y} \langle p, y \rangle$, $\sigma^\flat(Y, p) = \inf_{y \in Y} \langle p, y \rangle$ $(p \in R^m)$. Then T is said to be upper hemicontinuous (in short, u.h.c.) if $x \mapsto \sigma^\#(Tx, p)$ is u.s.c. on X for any $p \in R^n$.

(3) T is said to be convex if $X \subseteq R^n$ is convex and $\alpha T x^{(1)} + (1 - \alpha) T x^{(2)} \subseteq T[\alpha x^{(1)} + (1 - \alpha) x^{(2)}]$ for any $\alpha \in [0, 1]$ and $x^{(i)} \in X$ $(i = 1, 2)$.

(4) (a) If $\inf_{p \in P} \sup_{x \in X} \varphi(p, x) = \sup_{x \in X} \inf_{p \in P} \varphi(p, x)$, then one claims that the minimax equality of φ holds. Denoting by $v(\varphi)$ the value of the preceding equality, one also says that the minimax value $v(\varphi)$ of φ exists. If $(\overline{p}, \overline{x}) \in P \times X$ such that $\sup_{x \in X} \varphi(\overline{p}, x) = \inf_{p \in P} \varphi(p, \overline{x})$, then one calls $(\overline{p}, \overline{x})$ a saddle point of φ. Denote by $S(\varphi)$ the set of all saddle points of φ (i.e., $S(\varphi) = \{(\overline{p}, \overline{x}) \in P \times X : \sup_{x \in X} \varphi(\overline{p}, x) = \inf_{p \in P} \varphi(p, \overline{x})\}$), and define $S(\varphi)|_X \widehat{=} \{\overline{x} \in X : \exists \overline{p} \in P \text{ s.t. } (\overline{p}, \overline{x}) \in S(\varphi)\}$, the restriction of $S(\varphi)$ to X if $S(\varphi)$ is nonempty.

(b) Replacing X by $Y_1 \times Y_2$ and $\varphi(p,x)$ by $\psi(p,(u,v))$, with the similar method one can also define $v(\psi)$ (the minimax value of ψ), $S(\psi)$ (the saddle-point set of ψ), and $S(\psi)|_{Y_1\times Y_2}$ (the restriction of $S(\psi)$ to $Y_1 \times Y_2$).

(5) If Y is a convex set and A a subset of Y, one claims that A is an extremal subset of Y if $x, y \in Y$ and $tx + (1-t)y \in A$ for some $t \in (0,1)$ entails $x, y \in A$. $x_0 \in Y$ is an extremal point of Y if $A = \{x_0\}$ is an extremal subset of Y, and the set of all extremal points of Y is denoted by ext Y.

Remark 2.2. (1) Since $p \in R^m \Leftrightarrow -p \in R^m$ and $\sigma^{\#}(Tx,-p) = -\sigma^{\flat}(Tx,p)$, we can see that $T : X \subset R^n \to 2^{R^m}$ is u.h.c. if and only if $x \mapsto \sigma^{\flat}(Tx,p)$ is l.s.c. on X for any $p \in R^m$.

(2) For the function $\varphi = \varphi(p,x)$ on $P\times X$, $v(\varphi)$ exists if and only if $\inf_{p\in P}\sup_{x\in X}\varphi(p,x) \le \sup_{x\in X}\inf_{p\in P}\varphi(p,x)$, and $(\overline{p},\overline{x}) \in S(\varphi)$ if and only if $\sup_{x\in X}\varphi(\overline{p},x) \le \inf_{p\in P}\varphi(p,\overline{x})$ if and only if $\varphi(\overline{p},x) \le \varphi(\overline{p},\overline{x}) \le \varphi(p,\overline{x})$ for any $(p,x) \in P \times X$. If $S(\varphi) \neq \emptyset$, then $v(\varphi)$ exists, and $v(\varphi) = \varphi(\overline{p},\overline{x}) = \sup_{x\in X}\varphi(\overline{p},x) = \inf_{p\in P}\varphi(p,\overline{x})$ for any $(\overline{p},\overline{x}) \in S(\varphi)$. The same properties are also true for $\psi = \psi(p,(u,v))$ on $P \times (Y_1 \times Y_2)$. Moreover, we have

$$\begin{aligned} \forall\, \overline{x} \in S(\varphi)|_X, \quad & \inf_{p\in P}\varphi(p,\overline{x}) = v(\varphi), \\ \forall (\overline{u},\overline{v}) \in S(\psi)|_{Y_1\times Y_2}, \quad & \inf_{p\in P}\psi(p,(\overline{u},\overline{v})) = v(\psi). \end{aligned} \tag{2.1}$$

We also need three known results as follows.

Lemma 2.3. (1) *(see [9]) If T is u.s.c., then T is u.h.c.*

(2) *(see [9]) If T is u.s.c. with closed values, then T is closed.*

(3) *(see [9]) If $\overline{TX}$ (the closure of TX) is compact and T is closed, then T is u.s.c.*

(4) *If $X \subset R^n$ is compact and $T : X \to 2^{R^m}$ is u.s.c. with compact values, then TX is compact in R^m.*

(5) *If X is convex (or compact) and $T_1, T_2 : X \subset R^n \to 2^{R^m}$ are convex (or u.s.c. with compact values), then $\alpha T_1 + \beta T_2$ are also convex (or u.s.c.) for all $\alpha, \beta \in R$.*

Proof. We only need to prove (5).

(a) If T_i $(i = 1,2)$ are convex, $\alpha, \beta \in R$, x_i $(i = 1,2) \in X$, and $t \in [0,1]$, then

$$\begin{aligned} (\alpha T_1 + \beta T_2)[tx_1 + (1-t)x_2] &= \alpha T_1[tx_1 + (1-t)x_2] + \beta T_2[tx_1 + (1-t)x_2] \\ &\supseteq \alpha[tT_1x_1 + (1-t)T_1x_2] + \beta[tT_2x_1 + (1-t)T_2x_2] \\ &= t(\alpha T_1 + \beta T_2)x_1 + (1-t)(\alpha T_1 + \beta T_2)x_2. \end{aligned} \tag{2.2}$$

Hence $\alpha T_1 + \beta T_2$ is convex.

(b) Now we assume that X is compact.

In case $T : X \to 2^{R^m}$ is u.s.c. with compact values and $\alpha \in R$, then by (2), (4), T is closed and the range $(\alpha T)X$ of αT is compact. If $\alpha = 0$, then $(\alpha T)x = 0$ for any $x \in X$; hence, αT is u.s.c. If $\alpha \neq 0$, supposing that $(x^j, y^j) \in \text{graph}(\alpha T)$ with $(x^j, y^j) \to (x^0, y^0)$ $(j \to \infty)$, then $(x^j, y^j/\alpha) \in \text{graph}\, T$ such that $(x^j, y^j/\alpha) \to (x^0, y_0/\alpha)$ as $j \to \infty$, which implies that $y_0 \in \alpha T x^0$. Hence, αT is closed and also u.s.c. because of (3).

In case $T_i \ \ (i = 1,2) : X \to 2^{R^m}$ are u.s.c. with compact values, if $(x^k, y^k) \in \text{graph}(T_1 + T_2)$ with $(x^k, y^k) \to (x^0, y^0)$ $(k \to \infty)$, then $x^0 \in X$ and there exist $u^k \in T_1x^k$, $v^k \in T_2x^k$ such that $y^k = u^k + v^k$ for all $k = 1,2,\ldots$. By (4), T_1X and T_2X are compact, so we can suppose $u^k \to u^0$ and $v^k \to v^0$ as $k \to \infty$. By (2), both T_i $(i = 1,2)$ are closed, this implies that $y^0 = u^0 + v^0 \in (T_1 + T_2)x^0$, and thus $T_1 + T_2$ is closed. Hence by (3), $T_1 + T_2$ is u.s.c. because $(T_1 + T_2)X = \bigcup_{x\in X}(T_1 + T_2)x \subseteq T_1X + T_2X$ and $T_1X + T_2X$ is compact. □

Lemma 2.4 (see [8, Theorems 4.1 and 4.2]). *Let $X \subset R^n_+$, $P \subset R^m_+$ be convex compact with $R_+P = R^m_+$, $\Sigma^{m-1} = \{p \in R^m_+ : \Sigma^m_{i=1}p_i = 1\}$, and $c \in R^m_+$. Assume that $T : X \to 2^{R^m}$ is convex and u.s.c. with nonempty convex compact values, and define $\varphi(p,x) = \phi_c(p,x)$ on $P \times X$ by $\phi_c(p,x) = \sup_{y\in Tx}\langle p, y - c\rangle$ for $(p,x) \in P \times X$. Then*

(1) *$v(\phi_c)$ exists and $S(\phi_c)$ is a convex compact subset of $P \times X$,*

(2) *c solves (1.2) $\Leftrightarrow v(\phi_c) \geq 0 \Leftrightarrow \phi_c(\overline{p},\overline{x}) \geq 0$ for $(\overline{p},\overline{x}) \in S(\phi_c)$.*

In particular, both (1) and (2) are also true if $P = \Sigma^{m-1}$.

Lemma 2.5. *(1) (see [10, 11]) If $x \mapsto f_\alpha(x)$ is convex or l.s.c. (resp., concave or u.s.c.) on X for $\alpha \in \Lambda$ and $\sup_{\alpha\in\Lambda} f_\alpha(x)$ (resp., $\inf_{\alpha\in\Lambda} f_\alpha(x)$) is finite for $x \in X$, then $x \mapsto \sup_{\alpha\in\Lambda} f_\alpha(x)$ (resp., $x \mapsto \inf_{\alpha\in\Lambda} f_\alpha(x)$) is also convex or l.s.c. (resp., concave or u.s.c.) on X.*

(2) *(see [11]) If $g : X\times Y \subset R^n\times R^m \to R$ is l.s.c. (or u.s.c.) and Y is compact, then $h : U \to R$ defined by $h(x) = \inf_{y\in Y} g(x,y)$ (or $k : U \to R$ defined by $k(x) = \sup_{y\in Y} g(x,y)$) is also l.s.c. (or u.s.c.).*

(3) *(see [9–11], Minimax Theorem) Let $P \subset R^m$, $X \subset R^n$ be convex compact, and let $\varphi(p,x)$ be defined on $P \times X$. If, for each $x \in X$, $p \mapsto \varphi(p,x)$ is convex and l.s.c. and, for each $p \in P$, $x \mapsto \varphi(p,x)$ is concave and u.s.c., then $\inf_{p\in P}\sup_{x\in X}\varphi(p,x) = \sup_{x\in X}\inf_{p\in P}\varphi(p,x)$ and there exists $(\overline{p},\overline{x}) \in P \times X$ such that $\sup_{x\in X}\varphi(\overline{p},x) = \inf_{p\in P}\varphi(p,\overline{x})$.*

3. Solvability Theorem to (1.3)

Let Σ^{m-1} be introduced as in Lemma 2.4, and define the functions $\phi_c(p,x)$ on $\Sigma^{m-1} \times X$ and $\phi_{c,x}(p,y)$ on $\Sigma^{m-1} \times Tx$ $(x \in X)$ by

$$\begin{aligned} &\text{(a) } \phi_c(p,x) = \sigma^{\#}(Tx - c, p) = \sup_{y\in Tx}\langle p, y - c\rangle \quad \text{for } (p,x) \in \Sigma^{m-1} \times X, \\ &\text{(b) } \phi_{c,x}(p,y) = \langle p, y - c\rangle \quad \text{for } (p,y) \in \Sigma^{m-1} \times Tx,\ x \in X. \end{aligned} \tag{3.1}$$

Remark 3.1. By both Assumptions in Section 1, Definition 2.1, and Lemmas 2.4 and 2.5, we can see that

(1) $\phi_c(p,x) = \sup_{y\in Tx}\phi_{c,x}(p,y)$ for all $(p,x) \in P \times X$,

(2) $v(\phi_c)$ and $v(\phi_{c,x})$ exist, and $S(\phi_c)$ and $S(\phi_{c,x})$ are nonempty,

(3) c solves (1.2) if and only if $v(\phi_c) \geq 0$ if and only if $(\overline{p},\overline{x}) \in S(\phi_c)$ with $\phi_c(\overline{p},\overline{x}) \geq 0$.

Hence, $S(\phi_c)|_X$ and $S(\phi_{c,x})|_{Tx}$ $(x \in X)$ are nonempty. Moreover, we have the following.

Theorem 3.2. *For* (1.3), *the following three statements are equivalent to each other:*

(1) $v(\phi_c) = 0$,

(2) *for all* $\overline{x} \in S(\phi_c)|_X$, *for all* $\overline{y} \in S(\phi_{c,\overline{x}})|_{T\overline{x}}$, $\exists i_0 \in \{1, 2, \dots, m\} \Rightarrow \overline{y} \geq c$, $\overline{y}_{i_0} = c_{i_0}$,

(3) $\exists \overline{x} \in S(\phi_c)|_X$, $\exists \overline{y} \in S(\phi_{c,\overline{x}})|_{T\overline{x}}$, $\exists i_0 \in \{1, 2, \dots, m\} \Rightarrow \overline{y} \geq c$, $\overline{y}_{i_0} = c_{i_0}$.

Remark 3.3. Clearly, each of (2) and (3) implies that c solves (1.3) because $\overline{x} \in X$ and $\overline{y} \in T\overline{x}$. So we conclude from Theorem 3.2 that c solves (1.3) in the way of (2) or in the way of (3) if and only if $v(\phi_c) = 0$.

Proof of Theorem 3.2. We only need to prove (1)⇒(2) and (3)⇒(1).

(1)⇒(2). Assume that (1) holds. By (3.1) and Remarks 2.2 and 3.1, it is easy to see that

$$\forall \overline{x} \in S(\phi_c)|_X, \quad \inf_{p \in \Sigma^{m-1}} \sup_{y \in T\overline{x}} \langle p, y - c \rangle = \inf_{p \in \Sigma^{m-1}} \phi_c(p, \overline{x}) = v(\phi_c) = 0,$$
$$\forall \overline{y} \in S(\phi_{c,\overline{x}})|_{T\overline{x}}, \quad \inf_{p \in \Sigma^{m-1}} \langle p, \overline{y} - c \rangle = v(\phi_{c,\overline{x}}) = \inf_{p \in \Sigma^{m-1}} \sup_{y \in T\overline{x}} \langle p, y - c \rangle. \tag{3.2}$$

Then for each $\overline{x} \in S(\phi_c)|_X$ and each $\overline{y} \in S(\phi_{c,\overline{x}})|_{T\overline{x}}$, we have

$$\inf_{p \in \Sigma^{m-1}} \langle p, \overline{y} - c \rangle = v(\phi_{c,\overline{x}}) = v(\phi_c) = 0. \tag{3.3}$$

By taking $p^i = e^i = (\overbrace{0, \dots, 0, 1}^{i}, 0, \dots, 0) \in \Sigma^{m-1}$ $(i = 1, 2, \dots, m)$, it follows that $\overline{y}_i \geq c_i$ $(i = 1, 2, \dots, m)$. Hence, $\overline{y} \geq c$. On the other hand, it is easy to verify that $\Sigma^* = \{p \in \Sigma^{m-1} : \langle p, \overline{y} - c \rangle = 0\}$ is a nonempty extremal subset of Σ^{m-1}. The Crain-Milmann Theorem (see [12]) shows that $\operatorname{ext} \Sigma^*$ is nonempty with $\operatorname{ext} \Sigma^* \subset \operatorname{ext} \Sigma^{m-1} = \{e^1, e^2, \dots, e^m\}$. So there exists $i_0 \in \{1, 2, \dots, m\}$ such that $p = e^{i_0} \in \operatorname{ext} \Sigma^*$. This implies by (3.3) that $\overline{y}_{i_0} = c_{i_0}$, and therefore (2) follows.

(3)⇒(1). Let $\overline{x} \in S(\phi_c)|_X$, $\overline{y} \in S(\phi_{c,\overline{x}})|_{T\overline{x}}$, and let i_0 be presented in (3). Since c solves (1.3) and also solves (1.2), by (3.1) and Remarks 2.2 and 3.1, we obtain

$$\begin{aligned} 0 = \langle e^{i_0}, \overline{y} - c \rangle &\geq \inf_{p \in \Sigma^{m-1}} \langle p, \overline{y} - c \rangle \\ &= v(\phi_{c,\overline{x}}) = \inf_{p \in \Sigma^{m-1}} \sup_{y \in T\overline{x}} \phi_{c,\overline{x}}(p, y) \\ &= \inf_{p \in \Sigma^{m-1}} \phi_c(p, \overline{x}) = v(\phi_c) \geq 0, \end{aligned} \tag{3.4}$$

where $e^{i_0} = (\overbrace{0, \dots, 0, 1}^{i_0}, 0, \dots, 0) \in \Sigma^{m-1}$. Hence, $v(\phi_c) = 0$ and the theorem follows. □

Remark 3.4. From the Theorem, we know that $v(\phi_c) = 0$ implies that c solves (1.3). However, without any additional restricting conditions, the inverse may not be true.

Example 3.5. Let $X = [0,1]^2$, $c = (s,s)$ $(s \in [0,1))$, and let $T : X \to 2^{R^2}$ be defined by $Tx = [1/2,1]^2$ for $x = (x_1, x_2) \in X$. Then T is an u.s.c. and convex set-valued map with convex compact values, and for each $p = (p^1, p^2) \in \Sigma^1$, $x = (x_1, x_2) \in X$ and $c = (s,s)$ $(s \in [0,1))$, $\phi_c(p,x) = \sigma^{\#}(Tx - c, p) = (p_1 + p_2) - s(p_1 + p_2) = (1 - s)$. Hence, $v(\phi_c) = 1 - s$ for all $s \in [0,1)$ and therefore,

$$\begin{gathered} v(\phi_c) > 0 \text{ and } c \text{ solves (1.3) for } s \in \left[\frac{1}{2}, 1\right), \\ v(\phi_c) > 0 \text{ and } c \text{ does not solve (1.3) for } s \in \left[0, \frac{1}{2}\right). \end{gathered} \tag{3.5}$$

This implies that $v(\phi_c) = 0$ (or $v(\phi_c) > 0$) may not be the necessary (or the sufficient) condition that c solves (1.3).

4. Solvability Theorems to (1.4) and (1.5)

Taking $T = G - F$, then from both Assumptions, Lemma 2.4, and Theorem 3.2, we immediately obtain the necessary and sufficient conditions to the solvability of (1.4) and (1.5). However, just as indicated in Section 1, this type of result is only concerned with $G - F$. To get some further solvability criteria to (1.4) and (1.5) depending on the respective information of G and F, we define the functions $H_c(p,x)$ on $\Sigma^{m-1} \times X$ and $H_{c,x}(p,(u,v))$ on $\Sigma^{m-1} \times (Gx \times Fx)$ $(x \in X)$ by

$$\begin{aligned} &\text{(a)}\ H_c(p,x) = \frac{\sigma^{\#}(Gx,p) - \langle p, c\rangle}{\sigma^{\flat}(Fx,p)} \quad \text{for } (p,x) \in \Sigma^{m-1} \times X, \\ &\text{(b)}\ H_{c,x}(p,(u,v)) = \frac{\langle p, u - c\rangle}{\langle p, v\rangle} \quad \text{for } (p,(u,v)) \in \Sigma^{m-1} \times (Gx \times Fx),\ x \in X. \end{aligned} \tag{4.1}$$

By both Assumptions, we know that $\sigma^{\#}(Gx,p) = \sup_{u \in Gx}\langle p, u\rangle$ and $\sigma^{\flat}(Fx,p) = \inf_{v \in Fx}\langle p, v\rangle$ are finite with $\sigma^{\#}(Gx,p) \geq 0$ and $\langle p, v\rangle \geq \sigma^{\flat}(Fx,p) > 0$ $(v \in Fx)$ for $x \in X$ and $p \in \Sigma^{m-1}$, so the functions $H_c(p,x)$ and $H_{c,x}(p,(u,v))$ $(x \in X)$ defined by (4.1) are well defined.

In view of Definition 2.1, we denote by $v(H_c)$ (or $v(H_{c,x})$) the minimax value of $\varphi(p,x) = H_c(p,x)$ (or $\psi(p,(u,v)) = H_{c,x}(p,(u,v))$) if it exists, $S(H_c)$ (or $S(H_{c,x})$) the saddle point set if it is nonempty, and $S(H_c)|_X$ (or $S(H_{c,x})|_{Gx \times Fx}$ $(x \in X)$) the restriction of $S(H_c)$ to X (or $S(H_{c,x})$ to $Gx \times Fx$). Then we have the solvability result to (1.4) and (1.5) as follows.

Theorem 4.1. (i) *$v(H_c)$ exists if and only if $S(H_c) \neq \emptyset$.*

(ii) (1) *c solves (1.4) if and only if $v(H_c)$ exists with $v(H_c) \geq 1$ if and only if $S(H_c) \neq \emptyset$ with $H_c(\overline{p}, \overline{x}) \geq 1$ for $(\overline{p}, \overline{x}) \in S(H_c)$.*

(2) *In particular, if $v(H_c)$ exists with $v(H_c) \geq 1$, then for each $\overline{x} \in S(H_c)|_X$, there exists $\overline{y} \in (G - F)\overline{x}$ such that $\overline{y} \geq c$.*

Theorem 4.2. *For* (1.5)*, the following three statements are equivalent to each other:*

(1) $v(H_c) = 1$,

(2) $S(H_c) \neq \emptyset$, $S(H_{c,\overline{x}}) \neq \emptyset$ $(\overline{x} \in S(H_c)|_X)$, *and for all* $\overline{x} \in S(H_c)|_X$, *for all* $(\overline{u}, \overline{v}) \in S(H_{c,\overline{x}})|_{G\overline{x}\times F\overline{x}}$, $\exists i_0 \in \{1,\ldots,m\} \Rightarrow \overline{u} - \overline{v} \geq c$, $\overline{u}_{i_0} - \overline{v}_{i_0} = c_{i_0}$,

(3) $S(H_c) \neq \emptyset$, $S(H_{c,\overline{x}}) \neq \emptyset$ $(\overline{x} \in S(H_c)|_X)$, *and* $\exists \widehat{x} \in S(H_c)|_X$, $\exists (\widehat{u}, \widehat{v}) \in S(H_{c,\widehat{x}})|_{G\widehat{x}\times F\widehat{x}}$, $\exists i_0 \in \{1,\ldots,m\} \Rightarrow \widehat{u} - \widehat{v} \geq c$, $\widehat{u}_{i_0} - \widehat{v}_{i_0} = c_{i_0}$.

That is, c solves (1.5) *in the way of (2) or in the way of (3) if and only if* $v(H_c) = 1$.

Remark 4.3. It is also needed to point out that $v(H_c) = 1$ is not the necessary condition of c making (1.5) solvable without any other restricting conditions.

Example 4.4. Let $X = [0,1]^2$, $c = (s,s) \in R_+^2$ $(s \in (5/12, 3/4))$, and $G, F : X \to 2^{R_+^m}$ be defined by $Gx \equiv [3/4,1]^2$, $Fx \equiv [1/4,1/3]^2$ for $x = (x^1, x^2) \in X$. Then both G and F are u.s.c. convex set-valued maps with convex compact values, and for any $p = (p^1, p^2) \in \Sigma^1$, $x = (x^1, x^2) \in X$, and $c = (s,s) \in R_+^2$, we have $(G-F)x = Gx - Fx = [3/4,1]^2 - [1/4,1/3]^2 = [5/12,3/4]^2$, $\sigma^\#(Gx,p) = 1$, $\sigma^\flat(Fx,p) = 1/4$, and $\langle p,c\rangle = s$. Therefore,

$$\begin{gathered} H_c(p,x) = \frac{\sigma^\#(Gx,p) - \langle p,c\rangle}{\sigma^\flat(Fx,p)} = 4(1-s) \quad \text{for } p \in \Sigma^1,\ x \in X, \\ v(H_c) \text{ exists with } v(H_c) = 4(1-s) > 1 \quad \text{for } \frac{5}{12} < s < \frac{3}{4}. \end{gathered} \tag{4.2}$$

This implies that, for each $c = (s,s)$ $(s \in (5/12,3/4))$, c solves (1.5) but $v(H_c) > 1$.

The proof of both Theorems 4.1 and 4.2 can be divided into eight lemmas.

Let $t \in R_+$, $T_t = G - tF$, and $c \in R_+^m$. Consider the auxiliary inequality system

$$\begin{gathered} \exists x \in X, \\ \exists y \in T_t x = Gx - tFx \Longrightarrow y \geq c. \end{gathered} \tag{4.3}$$

Then $t \in R_+$ solves (4.3) if and only if c solves (1.2) for $T = T_t$, and in particular, $t = 1$ solves (4.3) if and only if c solves (1.4). Define $\varphi(p,x) = K_t(p,x)$ on $\Sigma^{m-1} \times X$ by

$$K_t(p,x) = \sigma^\#(T_t x - c, p) = \sigma^\#(Gx,p) - t\sigma^\flat(Fx,p) - \langle p,c\rangle, \quad (p,x) \in \Sigma^{m-1} \times X, \tag{4.4}$$

denote by $v(K_t)$ the minimax value of $\varphi = K_t$ if it exists, and denote by $S(K_t)$ the saddle point set if it is nonempty. Then we have the following.

Lemma 4.5. (1) *For each* $t \in R_+ = [0,+\infty)$, $v(K_t)$ *exists and* $S(K_t)$ *is nonempty. Moreover,* $t \in R_+$ *solves* (4.3) *if and only if* $v(K_t) \geq 0$ *if and only if* $K_t(\overline{p},\overline{x}) \geq 0$ *for* $(\overline{p},\overline{x}) \in S(K_t)$.

(2) *The function* $t \mapsto v(K_t)$ *is continuous and strictly decreasing on* R_+ *with* $v(K_{+\infty}) = \lim_{t\to+\infty} v(K_t) = -\infty$.

Proof. (1) By both Assumptions and Lemmas 2.3(4) and 2.3(5), $T_t = G - tF$ is convex and u.s.c. with nonempty convex compact values for each $t \in R_+$. Since $K_t(p,x) = \sup_{y\in Gx-tFx}\langle p, y-c\rangle$

for $(p, x) \in \Sigma^{m-1} \times X$, applying Lemma 2.4 to $T = T_t$ and substituting $K_t(p, x)$ for $\phi_c(p, x)$, we know that (1) is true.

(2) We prove (2) in three steps as follows.

(a) By Lemma 2.3(1), G and F are u.h.c., which implies by Definition 2.1(2) and Remark 2.2(1) that for each $p \in \Sigma^{m-1}$, $(t, x) \mapsto K_t(p, x) = \sigma^{\#}(Gx, p) - t\sigma^{\flat}(Fx, p) - \langle p, c\rangle$ is u.s.c. on $R_+ \times X$. Then from Lemma 2.5(1)(2), we know that both functions

$$(t, x) \longmapsto \inf_{p\in\Sigma^{m-1}} K_t(p, x) \quad \text{on } R_+ \times X,$$
$$\text{and } t \longmapsto \sup_{x\in X} \inf_{p\in\Sigma^{m-1}} K_t(p, x) \quad \text{on } R_+ \text{ are u.s.c.} \tag{4.5}$$

Since GX and FX are compact by both Assumptions and Lemma 2.3(4), $C_{GX} = \sup_{u\in GX}\|u\|$ and $C_{FX} = \sup_{v\in FX}\|v\|$ are finite. Then for any $x \in X$, $p, p_0 \in \Sigma^{m-1}$, and $t, t_0 \in R_+$, we have

$$\sigma^{\#}(Gx, p) = \sup_{u\in Gx}(\langle p_0, u\rangle + \langle p - p_0, u\rangle) \leq \sigma^{\#}(Gx, p_0) + \|p - p_0\|C_{GX},$$
$$\sigma^{\flat}(Fx, p) = \inf_{v\in Fx}(\langle p_0, v\rangle + \langle p - p_0, v\rangle) \geq \sigma^{\flat}(Fx, p_0) - \|p - p_0\|C_{FX}. \tag{4.6}$$

This implies that, for each $x \in X$,

$$\left|\sigma^{\#}(Gx, p) - \sigma^{\#}(Gx, p_0)\right| \leq \|p - p_0\|C_{GX},$$
$$\left|\sigma^{\flat}(Fx, p) - \sigma^{\flat}(Fx, p_0)\right| \leq \|p - p_0\|C_{FX}. \tag{4.7}$$

Hence for each $x \in X$, $(t, p) \mapsto K_t(p, x) = \sigma^{\#}(Gx, p) - t\sigma^{\flat}(Fx, p) - \langle p, c\rangle$ is continuous on $R_+ \times \Sigma^{m-1}$. Also from Lemmas 2.5(1) and 2.5(2), it follows that both functions

$$(t, p) \longmapsto \sup_{x\in X} K_t(p, x) \quad \text{on } R_+ \times \Sigma^{m-1},$$
$$\text{and } t \longmapsto \inf_{p\in\Sigma^{m-1}} \sup_{x\in X} K_t(p, x) \quad \text{on } R_+ \text{ are l.s.c.} \tag{4.8}$$

So we conclude from (4.5), (4.8), and Statement (1) that $t \mapsto v(K_t)$ is continuous on R_+.

(b) Assume that $t_2 > t_1 \geq 0$. Since $FX \subset \operatorname{int} R_+^m$ is compact, it is easy to see that $\varepsilon_0 = \inf\{\langle p, v\rangle : (p, v) \in \Sigma^{m-1} \times FX\} > 0$. Thus for any $(p, x) \in \Sigma^{m-1} \times X$, we have

$$K_{t_1}(p, x) = \sigma^{\#}(Gx, p) - t_2\sigma^{\flat}(Fx, p) - \langle p, c\rangle + (t_2 - t_1)\sigma^{\flat}(Fx, p)$$
$$\geq K_{t_2}(p, x) + (t_2 - t_1)\varepsilon_0, \tag{4.9}$$

which implies that $v(K_{t_1}) > v(K_{t_2})$, and hence $t \mapsto v(K_t)$ is strict decreasing on R_+.

(c) Let $\varepsilon_1 = \sup\{\langle p, u\rangle : p \in \Sigma^{m-1}, u \in GX\}$ and $\varepsilon_2 = \inf\{\langle p, c\rangle : p \in \Sigma^{m-1}\}$. By both Assumptions, ε_1 and ε_2 are finite. Thus for any $t > 0$ and $(p, x) \in \Sigma^{m-1} \times X$, we have

$$\begin{aligned} K_t(p,x) &= \sigma^{\#}(Gx,p) - t\sigma^{\flat}(Fx,p) - \langle p,c\rangle \\ &\le \varepsilon_1 - t\varepsilon_0 - \varepsilon_2. \end{aligned} \tag{4.10}$$

This implies that $v(K_t) \le \varepsilon_1 - t\varepsilon_0 - \varepsilon_2$. Therefore, $v(K_{+\infty}) = \lim_{t\to+\infty} v(K_t) = -\infty$. This completes the proof. □

Lemma 4.6. (1) $p \mapsto H_c(p,x)$ $(x \in X)$ *and* $p \mapsto \sup_{x\in X} H_c(p,x)$ *are l.s.c. on* Σ^{m-1}.

(2) $x \mapsto H_c(p,x)$ $(p \in \Sigma^{m-1})$ *and* $x \mapsto \inf_{p\in P} H_c(p,x)$ *are u.s.c. on* X.

(3) $v(H_c)$ *exists if and only if* $S(H_c)$ *is nonempty.*

Proof. (1) Since, for each $x \in X$ and $(u,v) \in Gx \times Fx$, the function $p \mapsto \langle p, u-c\rangle/\langle p, v\rangle$ is continuous on Σ^{m-1}, from Lemma 2.5(1), we can see that $p \mapsto H_c(p,x) = \sup_{(u,v)\in Gx\times Fx}(\langle p, u-c\rangle/\langle p,v\rangle)$ $(x \in X)$ and $p \mapsto \sup_{x\in X} H_c(p,x)$ are l.s.c., hence (1) is true.

(2) Assume that $\{(p^k, x^k)\} \subset \Sigma^{m-1} \times X$ is a sequence with $(p^k, x^k) \to (p^0, x^0)$ $(k \to \infty)$, then for each k, there exist $u^k \in Gx^k$ and $v^k \in Fx^k$ such that $\sigma^{\#}(Gx^k, p^k) = \langle p^k, u^k\rangle$, $\sigma^{\flat}(Fx^k, p^k) = \langle p^k, v^k\rangle$. Since GX, FX are compact and $Gx^k \subset GX$, $Fx^k \subset FX$ $(k \ge 1)$, we may choose $\{u^{k_j}\} \subset \{u^k\}$ and $\{v^{k_j}\} \subset \{v^k\}$ such that

$$\begin{gathered} u^{k_j} \longrightarrow u^0, \quad v^{k_j} \longrightarrow v^0 \quad (k \to \infty), \\ \limsup_{k\to\infty}\langle p^k, u^k\rangle = \lim_{j\to\infty}\langle p^{k_j}, u^{k_j}\rangle, \qquad \liminf_{k\to\infty}\langle p^k, v^k\rangle = \lim_{j\to\infty}\langle p^{k_j}, v^{k_j}\rangle. \end{gathered} \tag{4.11}$$

By Lemma 2.3(2), both G and F are closed. Hence, $(x^{k_j}, u^{k_j}) \to (x^0, u^0) \in \operatorname{graph} G$ and $(x^{k_j}, v^{k_j}) \to (x^0, v^0) \in \operatorname{graph} F$, which in turn imply that $u^0 \in Gx^0$, $v^0 \in Fx^0$ and

$$\begin{aligned} \limsup_{k\to\infty} \sigma^{\#}\left(Gx^k, p^k\right) &= \lim_{j\to\infty}\langle p^{k_j}, u^{k_j}\rangle = \langle p^0, u^0\rangle \le \sigma^{\#}\left(Gx^0, p^0\right), \\ \liminf_{k\to\infty} \sigma^{\flat}\left(Fx^k, p^k\right) &= \lim_{j\to\infty}\langle p^{k_j}, v^{k_j}\rangle = \langle p^0, v^0\rangle \ge \sigma^{\flat}\left(Fx^0, p^0\right). \end{aligned} \tag{4.12}$$

Combining this with $\sigma^{\flat}(Fx, p) > 0$ for $(p, x) \in \Sigma^{m-1} \times X$, it follows that

$$\limsup_{k\to\infty}\frac{\sigma^{\#}(Gx^k, p^k) - \langle p^k, c\rangle}{\sigma^{\flat}(Fx^k, p^k)} \le \frac{\limsup_{k\to\infty}\left[\sigma^{\#}(Gx^k, p^k) - \langle p^k, c\rangle\right]}{\liminf_{k\to\infty}\sigma^{\flat}(Fx^k, p^k)} \le \frac{\sigma^{\#}(Gx^0, p^0) - \langle p^0, c\rangle}{\sigma^{\flat}(Fx^0, p^0)}. \tag{4.13}$$

Hence by (4.1), $(p,x) \mapsto H_c(p,x)$ is u.s.c. on $\Sigma^{m-1} \times X$, so is $x \mapsto \inf_{p\in\Sigma^{m-1}} H_c(p,x)$ on X thanks to Lemma 2.5(1).

(3) Assume that $v(H_c)$ exists. By (1) and (2), there exist $\overline{p} \in \Sigma^{m-1}$ and $\overline{x} \in X$ such that

$$\sup_{x\in X} H_c(\overline{p},x) = \inf_{p\in\Sigma^{m-1}} \sup_{x\in X} H_c(p,x) = v(H_c) = \sup_{x\in X} \inf_{p\in\Sigma^{m-1}} H_c(p,x) = \inf_{p\in\Sigma^{m-1}} H_c(p,\overline{x}). \tag{4.14}$$

By Remark 2.2(2), $(\overline{p},\overline{x}) \in S(H_c)$. Hence $S(H_c)$ is nonempty. The inverse is obvious. This completes the proof. □

Lemma 4.7. (1) *If c solves*(1.4), *then $v(H_c)$ exists with $v(H_c) \geq 1$.*

(2) *If $v(H_c)$ exists with $v(H_c) \geq 1$, then $S(H_c) \neq \emptyset$ and $H_c(\overline{p},\overline{x}) \geq 1$ for $(\overline{p},\overline{x}) \in S(H_c)$.*

Proof. (1) If c solves (1.4), then $t = 1$ solves (4.3). From Lemma 4.5, we know that $v(K_1) \geq 0$, and there is a unique $t_0 \geq 1$ such that $v(K_{t_0}) = 0$. Moreover, also from Lemma 4.5, t_0 is the biggest number that makes (4.3) solvable, and thus $t \in R_+$ solves (4.3) if and only if $t \in [0,t_0]$. We will prove that $v(H_c)$ exists with $v(H_c) = t_0$. Let

$$v_* = \sup_{x\in X} \inf_{p\in\Sigma^{m-1}} H_c(p,x), \qquad v^* = \inf_{p\in\Sigma^{m-1}} \sup_{x\in X} H_c(p,x), \tag{4.15}$$

then $v_* \leq v^*$. It is needed to show that $v^* \leq t_0 \leq v_*$.

Since t_0 solves (4.3), there exist $x^0 \in X$, $u^0 \in Gx^0$, and $v^0 \in Fx^0$ such that $u^0 - t_0 v^0 \geq c$. Hence for each $p \in \Sigma^{m-1}$, $\sigma^{\#}(Gx^0,p) - t_0\sigma^{\flat}(Fx^0,p) - \langle p,c\rangle \geq \langle p, u^0 - t_0 v^0 - c\rangle \geq 0$. As $\sigma^{\flat}(Fx^0,p) > 0$ for $p \in \Sigma^{m-1}$, it follows from (4.1) that $H_c(p,x^0) = (\sigma^{\#}(Gx^0,p) - \langle p,c\rangle)/\sigma^{\flat}(Fx^0,p) \geq t_0$ ($p \in \Sigma^{m-1}$) and thus

$$v_* \geq \inf_{p\in\Sigma^{m-1}} H_c\left(p,x^0\right) \geq t_0. \tag{4.16}$$

On the other hand, by (4.15), for each $p \in \Sigma^{m-1}$ we have $\sup_{x\in X} H_c(p,x) \geq v^*$. By (4.1) and Lemma 4.6(2), there exists $x_p \in X$ such that

$$\frac{\sigma^{\#}(Gx_p,p) - \langle p,c\rangle}{\sigma^{\flat}(Fx_p,p)} = H_c(p,x_p) = \sup_{x\in X} H_c(p,x) \geq v^*. \tag{4.17}$$

It deduces from (4.4) that for each $p \in \Sigma^{m-1}$

$$\sup_{x\in X} K_{v^*}(p,x) \geq K_{v^*}(p,x_p) = \sigma^{\#}(Gx_p,p) - v^*\sigma^{\flat}(Fx_p,p) - \langle p,c\rangle \geq 0. \tag{4.18}$$

Hence by Lemma 4.5(1), $v(K_{v^*}) = \inf_{p\in\Sigma^{m-1}} \sup_{x\in X} K_{v^*}(p,x) \geq 0$, and $t = v^*$ solves (4.3). Since t_0 is the biggest number that makes (4.3) solvable, we have $v^* \leq t_0$. Combining this with (4.16), we obtain $v^* = v_* = t_0$. Therefore, $v(H_c)$ exists and $v(H_c) = t_0 \geq 1$.

(2) follows immediately from Lemma 4.6(3) and Remark 2.2(2). The third lemma follows. □

Lemma 4.8. *If $S(H_c) \neq \emptyset$ with $H_c(\overline{p},\overline{x}) \geq 1$ for $(\overline{p},\overline{x}) \in S(H_c)$, then c solves* (1.4)*. Moreover, for each $\overline{x} \in S(H_c)|_X$, there exists $\overline{y} \in (G-F)\overline{x}$ such that $\overline{y} \geq c$.*

Proof. By (4.1) and Remark 2.2(2), we know that, for each $(\overline{p},\overline{x}) \in S(H_c)$,

$$\begin{aligned} \frac{\sigma^{\#}(Gx,\overline{p}) - \langle \overline{p}, c\rangle}{\sigma^{\flat}(Fx,\overline{p})} &= H_c(\overline{p},x) \leq v(H_c) \leq H_c(p,\overline{x}) \\ &= \frac{\sigma^{\#}(G\overline{x},p) - \langle p, c\rangle}{\sigma^{\flat}(F\overline{x},p)}, \quad (p,x) \in \Sigma^{m-1} \times X. \end{aligned} \tag{4.19}$$

Combining this with the definition of $K_{v(H_c)}(p,x)$ (i.e., (4.4) for $t = v(H_c)$), it follows that, for each $(\overline{p},\overline{x}) \in S(H_c)$ and each $(p,x) \in \Sigma^{m-1} \times X$,

$$\begin{aligned} K_{v(H_c)}(\overline{p},x) &= \sigma^{\#}(Gx,\overline{p}) - v(H_c)\sigma^{\flat}(Fx,\overline{p}) - \langle \overline{p}, c\rangle \\ &\leq 0 \leq \sigma^{\#}(G\overline{x},p) - v(H_c)\sigma^{\flat}(F\overline{x},p) - \langle p, c\rangle \\ &= K_{v(H_c)}(p,\overline{x}). \end{aligned} \tag{4.20}$$

Hence by Definition 2.1(4) and Remark 2.2(2),

$$\forall (\overline{p},\overline{x}) \in S(H_c), \quad \sup_{x\in X} K_{v(H_c)}(\overline{p},x) = 0 = \inf_{p\in\Sigma^{m-1}} K_{v(H_c)}(p,\overline{x}). \tag{4.21}$$

It follows that $(\overline{p},\overline{x}) \in S(H_c)$ implies that $(\overline{p},\overline{x}) \in S(K_{v(H_c)})$ with $v(K_{v(H_c)}) = K_{v(H_c)}(\overline{p},\overline{x}) = 0$, and so

$$\begin{gathered} S(H_c)|_X \subset S(K_{v(H_c)})|_X, \\ \forall \overline{x} \in S(H_c)|_X, \quad \inf_{p\in\Sigma^{m-1}} K_{v(H_c)}(p,\overline{x}) = v(K_{v(H_c)}) = 0. \end{gathered} \tag{4.22}$$

Applying Lemma 4.5(1) to $T_{v(H_c)} = G - v(H_c)F$, we then conclude that $t = v(H_c)$ solves (4.3). So there exist $\overline{x} \in X$, $\overline{u} \in G\overline{x}$, and $\overline{v} \in F\overline{x}$ such that $\overline{u} - v(H_c)\overline{v} \geq c$. Hence c solves (1.4) because $Gx \subset R^m_+$, $Fx \subset \operatorname{int} R^m_+$ $(x \in X)$, and $v(H_c) \geq 1$.

For each $\overline{x} \in S(H_c)|_X$. Since $K_{v(H_c)}(p,\overline{x}) = \sup_{y\in(G-v(H_c)F)\overline{x}} \langle p, y-c\rangle$ by (4.4), applying Lemma 2.5(3) to the function $(p,y) \mapsto \langle p, y-c\rangle$ on $\Sigma^{m-1} \times (G-v(H_c)F)\overline{x}$ and associating with (4.22), we obtain

$$\begin{aligned} \sup_{y\in(G-v(H_c)F)\overline{x}} \inf_{p\in\Sigma^{m-1}} \langle p, y-c\rangle &= \inf_{p\in\Sigma^{m-1}} \sup_{y\in(G-v(H_c)F)\overline{x}} \langle p, y-c\rangle \\ &= \inf_{p\in\Sigma^{m-1}} K_{v(H_c)}(p,\overline{x}) \\ &= v(K_{v(H_c)}) \\ &= 0. \end{aligned} \tag{4.23}$$

Since $y \mapsto \inf_{p\in\Sigma^{m-1}}\langle p, y-c\rangle$ is u.s.c. on $(G-v(H_c)F)\overline{x}$, from (4.23) there exist $\overline{u}\in G\overline{x}$, $\overline{v}\in F\overline{x}$ such that $\hat{y}=\overline{u}-v(H_c)\overline{v}\in(G-v(H_c)F)\overline{x}$ satisfies

$$\inf_{p\in\Sigma^{m-1}}\langle p,\hat{y}-c\rangle=\sup_{y\in(G-v(H_c)F)\overline{x}}\ \inf_{p\in\Sigma^{m-1}}\langle p,y-c\rangle=0. \tag{4.24}$$

By taking $p^i=e^i\in\Sigma^{m-1}$ $(i=1,2,\ldots,m)$, we get $\hat{y}=\overline{u}-v(H_c)\overline{v}\geq c$, and therefore $\overline{y}=\overline{u}-\overline{v}\in(G-F)\overline{x}$ satisfies $\overline{y}\geq c$ because $v(H_c)\geq 1$. This completes the proof. □

Proof of Theorem 4.1. By Lemmas 4.6(3), 4.7, and 4.8, we know that Theorem 4.1 is true. □

To prove Theorem 4.2, besides using Lemmas 4.5–4.8, for $c\in R_+^m$ and $x\in X$, we also need to study the condition that $t\in R_+$ solves

$$\begin{gathered}\exists u\in Gx,\quad \exists v\in Fx\\ \Longrightarrow u-tv\geq c.\end{gathered} \tag{4.25}$$

Define $\psi(p,(u,v))=L_{t,c,x}(p,(u,v))$ on $P\times(Y_1\times Y_2)=\Sigma^{m-1}\times(Gx\times Fx)$ by

$$L_{t,c,x}(p,(u,v))=\langle p,u-tv-c\rangle\quad\text{for }(p,(u,v))\in\Sigma^{m-1}\times(Gx\times Fx). \tag{4.26}$$

We denote by $v(L_{t,c,x})$ the minimax values of $L_{t,c,x}$ if it exists, $S(L_{t,c,x})$ the saddle point set if it is nonempty, and $S(L_{t,c,x})|_{Gx\times Fx}$ its restriction to $Gx\times Fx$.

Lemma 4.9. *Let $c\in R_+^m$ and $x\in X$ be fixed. Then one has the following.*

(1) *For each $t\in R_+$, $v(L_{t,c,x})$ exists and $S(L_{t,c,x})$ is nonempty.*

(2) *$t\in R_+$ solves* (4.25) *if and only if $v(L_{t,c,x})\geq 0$ if and only if $(\overline{p},(\overline{u},\overline{v}))\in S(L_{t,c,x})$ implies that $L_{t,c,x}(\overline{p},(\overline{u},\overline{v}))\geq 0$.*

(3) *$t\mapsto v(L_{t,c,x})$ is continuous and strict decreasing on R_+ with $v(L_{+\infty,c,x})=-\infty$.*

Proof. Define $T_{t,x}$ from $Gx\times Fx\subset R_+^{2m}$ to R^m by

$$T_{t,x}(u,v)=u-tv,\quad (u,v)\in Gx\times Fx. \tag{4.27}$$

Then $T_{t,x}$ is a single-valued continuous map with the convex condition defined by Definition 2.1(3) because $T_{t,x}[\alpha(u^1,v^1)+(1-\alpha)(u^2,v^2)]=\alpha T_{t,x}(u^1,v^1)+(1-\alpha)T_{t,x}(u^2,v^2)$ for all $\alpha\in[0,1]$ and $(u^i,v^i)\in Gx\times Fx$ $(i=1,2)$.

Since $Gx \times Fx$ is convex and compact in R_+^{2m}, replacing $x \in X(\subset R_+^n)$ by $(u,v) \in Gx \times Fx(\subset R_+^{2m})$, Tx by $T_{t,x}(u,v)$, and $\phi_c(p,x) = \sup_{y\in Tx}\langle p, y-c\rangle$ by $L_{t,c,x}(p,(u,v)) = \langle p, T_{t,x}(u,v) - c\rangle$, from Lemma 2.4, we know that both (1) and (2) are true. Moreover, with the same method as in proving Lemma 4.5(2), we can show that (3) is also true. (In fact, since $(t,p,(u,v)) \mapsto \langle p, u - tv - c\rangle$ is continuous on $R_+ \times \Sigma^{m-1} \times (Gx \times Fx)$ and $Gx \times Fx$ and Σ^{m-1} are compact, by (4.26) and Lemmas 2.5(1) and 2.5(2), we can see that

$$\begin{aligned} &t \longmapsto \sup_{(u,v)\in Gx\times Fx}\inf_{p\in\Sigma^{m-1}} L_{t,c,x}(p,(u,v)) \text{ is u.s.c.,} \\ &t \longmapsto \inf_{p\in\Sigma^{m-1}}\sup_{(u,v)\in Gx\times Fx} L_{t,c,x}(p,(u,v)) \text{ is l.s.c.} \end{aligned} \tag{4.28}$$

Hence by (1), $t \mapsto v(L_{t,c,x})$ is continuous on R_+.

Let ε_0, ε_1, and ε_2 be defined as in the proof of Lemma 4.5(2).

If $t_2 > t_1 \geq 0$, also by (4.26), we can see that $L_{t_1,c,x}(p,(u,v)) \geq L_{t_2,c,x}(p,(u,v)) + (t_2 - t_1)\varepsilon_0$ for $(p,(u,v)) \in \Sigma^{m-1} \times (Gx \times Fx)$. It follows that $v(L_{t_1,c,x}) \geq v(L_{t_2,c,x}) + (t_2 - t_1)\varepsilon_0$ and thus $t \mapsto v(L_{t,c,x})$ is strict decreasing on R_+.

If $t > 0$, then $L_{t,c,x}(p,(u,v)) = \langle p, u - tv - c\rangle \leq \varepsilon_1 - t\varepsilon_0 - \varepsilon_2$ for $(p,(u,v)) \in \Sigma^{m-1} \times (Gx \times Fx)$. This implies that $v(L_{t,c,x}) \leq \varepsilon_1 - t\varepsilon_0 - \varepsilon_2$ with $v(L_{+\infty,c,x}) = -\infty$.) Hence the fifth lemma follows. □

Lemma 4.10. (1) *$v(H_{c,x})$ exists if and only if $S(H_{c,x})$ is nonempty, where $H_{c,x}$ is defined by* (4.1)*(b).*

(2) *If $t = 1$ solves* (4.25) *for $c \in R_+^m$ and $x \in X$, then $v(H_{c,x})$ exists with $v(H_{c,x}) \geq 1$, $S(H_{c,x})$ is nonempty with $H_{c,x}(\overline{p},(\overline{u},\overline{v})) \geq 1$ for $(\overline{p},(\overline{u},\overline{v})) \in S(H_{c,x})$, and $\inf_{p\in\Sigma^{m-1}} H_{c,x}(p,(\overline{u},\overline{v})) = v(H_{c,x})$ for $(\overline{u},\overline{v}) \in S(H_{c,x})|_{Gx\times Fx}$.*

Proof. Since $(p,(u,v)) \mapsto H_{c,x}(p,(u,v)) = \langle p, u-c\rangle / \langle p, v\rangle$ is continuous on $\Sigma^{m-1} \times (Gx \times Fx)$, by Lemma 2.5(1), it is easy to see that

$$\begin{aligned} &\text{(a)} \quad \begin{aligned} &(u,v) \longmapsto H_{c,x}(p,(u,v)) \quad (p \in \Sigma^{m-1}), \\ &(u,v) \longmapsto \inf_{p\in\Sigma^{m-1}} H_{c,x}(p,(u,v)) \text{ are u.s.c. on } Gx \times Fx, \end{aligned} \\ &\text{(b)} \quad \begin{aligned} &p \longmapsto H_{c,x}(p,(u,v)) \quad ((u,v) \in Gx \times Fx), \\ &p \longmapsto \sup_{(u,v)\in Gx\times Fx} H_{c,x}(p,(u,v)) \text{ are l.s.c. on } \Sigma^{m-1}. \end{aligned} \end{aligned} \tag{4.29}$$

(1) By (4.29) and with the same method as in proving Lemma 4.6(3), we can show that (1) is true. (Indeed, we only need to prove the necessary part. If $v(H_{c,x})$ exists, then by (4.29), there exists $(\overline{p},(\overline{u},\overline{v})) \in \Sigma^{m-1} \times (Gx \times Fx)$ such that $\sup_{(u,v)\in Gx\times Fx} H_{c,x}(\overline{p},(u,v)) = v(H_{c,x}) = \inf_{p\in\Sigma^{m-1}} H_{c,x}(p,(\overline{u},\overline{v}))$. Hence $S(H_{c,x})$ is nonempty.)

(2) If $t = 1$ solves (4.25) for $c \in R_+^m$ and $x \in X$, then from Lemma 4.9 we know that $v(L_{1,c,x})$ exists with $v(L_{1,c,x}) \geq 0$, and there is a unique $\widetilde{t}_0 \geq 1$ such that $v(L_{\widetilde{t}_0,c,x}) = 0$. In particular, $\widetilde{t}_0$ is the biggest number that makes (4.25) solvable for c and x.

Applying the same method as in proving Lemma 4.7(1), we can show that $v(H_{c,x})$ exists with $v(H_{c,x}) = \widetilde{t}_0 \geq 1$. (In fact, let

$$\widetilde{v}_* = \sup_{(u,v)\in Gx\times Fx} \inf_{p\in\Sigma^{m-1}} H_{c,x}(p,(u,v)), \qquad \widetilde{v}^* = \inf_{p\in\Sigma^{m-1}} \sup_{(u,v)\in Gx\times Fx} H_{c,x}(p,(u,v)). \tag{4.30}$$

Then $\widetilde{v}_* \leq \widetilde{v}^*$. We need to show that $\widetilde{v}_* \geq \widetilde{t}_0 \geq \widetilde{v}^*$.

Since $\widetilde{t}_0$ solves (4.25) for c and x, there exist $u_x \in Gx$ and $v_x \in Fx$ such that $u_x - \widetilde{t}_0 v_x \geq c$. It follows that $H_{c,x}(p,(u_x,v_x)) = \langle p, u_x - c\rangle / \langle p, v_x\rangle \geq \widetilde{t}_0$ for any $p \in \Sigma^{m-1}$, hence $\widetilde{v}_* \geq \inf_{p\in\Sigma^{m-1}} H_{c,x}(p,(u_x,v_x)) \geq \widetilde{t}_0$. On the other hand, by the definition of $\widetilde{v}^*$, we have $\sup_{(u,v)\in Gx\times Fx} H_{c,x}(p,(u,v)) \geq \widetilde{v}^*$ for any $p \in \Sigma^{m-1}$. By (4.29)(a) and (4.1)(b), there exists $(u_p, v_p) \in Gx \times Fx$ such that $\langle p, u_p - c\rangle / \langle p, v_p\rangle = H_{c,x}(p,(u_p,v_p)) = \sup_{(u,v)\in Gx\times Fx} H_{c,x}(p,(u,v)) \geq \widetilde{v}^*$, which implies by (4.26) that $\sup_{(u,v)\in Gx\times Fx} L_{\widetilde{v}^*,c,x}(p,(u,v)) \geq \langle p, u_p - \widetilde{v}^* v_p - c\rangle \geq 0$ for any $p \in \Sigma^{m-1}$. Hence from Lemma 4.9, $v(L_{\widetilde{v}^*,c,x}) \geq 0$, $t = \widetilde{v}^*$ solves (4.25), and $\widetilde{t}_0 \geq \widetilde{v}^*$. Therefore, $v(H_{c,x})$ exist with $v(H_{c,x}) = \widetilde{t}_0 \geq 1$.) So we conclude from (1) and Remark 2.2 that (2) is true. This completes the proof. □

Lemma 4.11. *If $v(H_c) = 1$, then Theorem 4.2(2) is true.*

Proof. (i) If $v(H_c) = 1$, then by Lemma 4.6(3) and Remark 2.2, $S(H_c) \neq \emptyset$ and

$$\forall(\overline{p},\overline{x}) \in S(H_c), \quad \frac{\sigma^\#(Gx,\overline{p}) - \langle \overline{p}, c\rangle}{\sigma^\flat(Fx,\overline{p})} \leq 1 \leq \frac{\sigma^\#(G\overline{x},p) - \langle p, c\rangle}{\sigma^\flat(F\overline{x},p)} \quad \text{for } (p,x) \in \Sigma^{m-1} \times X. \tag{4.31}$$

By the same proof of (4.21) we can show that

$$\forall(\overline{p},\overline{x}) \in S(H_c), \quad \sup_{x\in X} K_1(\overline{p},x) = 0 = \inf_{p\in\Sigma^{m-1}} K_1(p,\overline{x}). \tag{4.32}$$

Combining this with Lemma 4.5(1) and using Remark 2.2(2), we have

$$S(H_c)|_X \subseteq S(K_1)|_X, \quad \forall\, \overline{x} \in S(H_c)|_X, \qquad \inf_{p\in\Sigma^{m-1}} K_1(p,\overline{x}) = v(K_1) = 0. \tag{4.33}$$

As $K_1(p,\overline{x}) = \sup_{y\in(G-F)\overline{x}-c} \langle p, y\rangle$ by (4.4), applying Lemma 2.5(3) to the function $(p,y) \mapsto \langle p, y\rangle$ on $\Sigma^{m-1} \times ((G-F)\overline{x} - c)$, we obtain that, for each $\overline{x} \in S(H_c)|_X$,

$$\sup_{y\in(G-F)\overline{x}-c} \inf_{p\in\Sigma^{m-1}} \langle p, y\rangle = \inf_{p\in\Sigma^{m-1}} \sup_{y\in(G-F)\overline{x}-c} \langle p, y\rangle = \inf_{p\in\Sigma^{m-1}} K_1(p,\overline{x}) = 0. \tag{4.34}$$

Since $y \mapsto \inf_{p\in\Sigma^{m-1}} \langle p, y\rangle$ is u.s.c. on $(G-F)\overline{x}-c$, from (4.34) there exists $\overline{y} \in (G-F)\overline{x}-c$ such that

$$\inf_{p\in\Sigma^{m-1}} \langle p, \overline{y}\rangle = \sup_{y\in(G-F)\overline{x}-c} \inf_{p\in\Sigma^{m-1}} \langle p, y\rangle = 0. \tag{4.35}$$

Hence, $\overline{y} \geq 0$. This implies that $t = 1$ solves (4.25) for c and any $\overline{x} \in S(H_c)|_X$. So we conclude from Lemma 4.10 and Remark 2.2(2) that

$$\begin{gathered}\forall \overline{x} \in S(H_c)|_X, \quad v(H_{c,\overline{x}}) \text{ exists and } S(H_{c,\overline{x}}) \text{ is nonempty}, \\ \forall(\overline{u},\overline{v}) \in S(H_{c,\overline{x}})|_{G\overline{x}\times F\overline{x}}, \quad \inf_{p\in\Sigma^{m-1}} H_{c,\overline{x}}(p,(\overline{u},\overline{v})) = v(H_{c,\overline{x}}) \geq 1.\end{gathered} \tag{4.36}$$

On the other hand, by (4.1), $H_c(p,x) = \sup_{(u,v)\in Gx\times Fx} H_{c,x}(p,(u,v))$. Combining this with (4.36), it follows that, for each $\overline{x} \in S(H_c)|_X$ and each $(\overline{u},\overline{v}) \in S(H_{c,\overline{x}})|_{G\overline{x}\times F\overline{x}}$,

$$\begin{aligned} 1 &\leq \inf_{p\in\Sigma^{m-1}} H_{c,\overline{x}}(p,(\overline{u},\overline{v})) = v(H_{c,\overline{x}}) \\ &= \inf_{p\in\Sigma^{m-1}} \sup_{(u,v)\in G\overline{x}\times F\overline{x}} H_{c,\overline{x}}(p,(u,v)) \\ &= \inf_{p\in\Sigma^{m-1}} H_c(p,\overline{x}) = v(H_c) = 1. \end{aligned} \tag{4.37}$$

Hence, also by (4.1), $\inf_{p\in\Sigma^{m-1}}(\langle p,\overline{u}-c\rangle/\langle p,\overline{v}\rangle) = \inf_{p\in\Sigma^{m-1}} H_{c,\overline{x}}(p,(\overline{u},\overline{v})) = 1$. This implies that, for each $p \in \Sigma^{m-1}$, $\langle p,\overline{u}-c\rangle/\langle p,\overline{v}\rangle \geq 1$ and there exists $\widetilde{p} \in \Sigma^{m-1}$ such that $\langle \widetilde{p},\overline{u}-c\rangle/\langle \widetilde{p},\overline{v}\rangle = 1$. So we obtain

$$\forall \overline{x} \in S(H_c)|_X,\ \forall(\overline{u},\overline{v}) \in S(H_{c,\overline{x}})|_{G\overline{x}\times F\overline{x}}, \quad \inf_{p\in\Sigma^{m-1}} \langle p,\overline{u}-\overline{v}-c\rangle = 0. \tag{4.38}$$

By using the same method as in proving (1)⇒(2) of Theorem 3.2, we conclude that $\overline{u}-\overline{v} \geq c$ and there exists $i_0 \in \{1,2,\ldots,m\}$ such that $\overline{u}_{i_0} - \overline{v}_{i_0} = c_{i_0}$. Hence Theorem 4.2(2) is true. □

Lemma 4.12. *If Theorem 4.2(3) holds,then $v(H_c) = 1$.*

Proof. If Theorem 4.2(3) holds, then c solves both (1.4) and (1.5), and by Lemma 4.7(1), $v(H_c)$ exists with $v(H_c) \geq 1$.

Now we let $\widehat{x} \in S(H_c)|_X$, $(\widehat{u},\widehat{v}) \in S(H_{c,\widehat{x}})|_{G\widehat{x}\times F\widehat{x}}$, and $i_0 \in \{1,2,\ldots,m\}$ satisfy $\widehat{u}-\widehat{v} \geq c$ and $\widehat{u}_{i_0} - \widehat{v}_{i_0} = c_{i_0}$, then we have $\langle p,\widehat{u}-\widehat{v}-c\rangle \geq 0$ for $p \in \Sigma^{m-1}$ and $\langle e^{i_0},\widehat{u}-\widehat{v}-c\rangle = 0$ (where $e^{i_0} = (\overbrace{0,\ldots,0,1}^{i_0},0,\ldots,0) \in \Sigma^{m-1}$). This implies by (4.1) that

$$\inf_{p\in\Sigma^{m-1}} H_{c,\widehat{x}}(p,(\widehat{u},\widehat{v})) = \inf_{p\in\Sigma^{m-1}} \frac{\langle p,\widehat{u}-c\rangle}{\langle p,\widehat{v}\rangle} = 1. \tag{4.39}$$

Combining this with the fact that $v(H_c) \geq 1$ and using Remark 2.2 and (4.1), we obtain that

$$\begin{aligned} 1 &= \inf_{p\in\Sigma^{m-1}} H_{c,\widehat{x}}(p,(\widehat{u},\widehat{v})) = v(H_{c,\widehat{x}}) \\ &= \inf_{p\in\Sigma^{m-1}} \sup_{(u,v)\in G\widehat{x}\times F\widehat{x}} H_{c,\widehat{x}}(p,(u,v)) \\ &= \inf_{p\in\Sigma^{m-1}} H_c(p,\widehat{x}) = v(H_c) \geq 1. \end{aligned} \tag{4.40}$$

Hence, $v(H_c) = 1$. □

Proof of Theorem 4.2. Since (2)⇒(3) of is clear, Theorem 4.2 follows immediately from Lemmas 4.11 and 4.12. □

5. Conclusion

Based on the generalized and multivalued input-output inequality models, in this paper we have considered three types of set-valued inequality systems (namely, (1.3)–(1.5)) and two corresponding solvability questions. By constructing some auxiliary functions and studying their minimax and saddle point properties with the nonlinear analysis approaches, three solvability theorems (i.e., Theorems 3.2, 4.1, and 4.2) composed of necessary and sufficient conditions regarding these inequality systems have been obtained.

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