Solvability of elliptic systems with square integrable boundary data

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Abstract. We consider second order elliptic divergence form systems with complex measurable coefficients A that are independent of the transversal coordinate, and prove that the set of A for which the boundary value problem with L_2 Dirichlet or Neumann data is well posed, is an open set. Furthermore we prove that these boundary value problems are well posed when A is either Hermitean, block or constant. Our methods apply to more general systems of partial differential equations and as an example we prove perturbation results for boundary value problems for differential forms.

1. Introduction

We first review the situation for scalar equations. Consider the divergence form second order elliptic equation

$$(1) \qquad \operatorname{div}_{t,x} A(x) \nabla_{t,x} U(t,x) = \sum_{j,k=0}^{n} \partial_j A_{j,k}(x) \partial_k U(t,x) = 0,$$

on the upper half space $\mathbf{R}_+^{1+n} := \{(t,x) \in \mathbf{R} \times \mathbf{R}^n ; t > 0\}$, $n \geq 1$, where the matrix $A = (A_{j,k}(x))_{j,k=0}^{n} \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{1+n}))$ is assumed to be t-independent and strictly accretive with complex coefficients. In this generality, when no regularity is assumed of the coefficients, the natural conditions to impose on U at the boundary $\mathbf{R}^n$ are one of the following:

- Dirichlet problem (Dir-A): $U(0,x) = u(x)$ for a given function $u(x)$.
- Neumann problem (Neu-A): $-\sum_{k=0}^{n} A_{0,k}(x) \partial_k U(0,x) = \phi(x)$, where $\phi(x)$ is given.
- Dirichlet regularity problem (Reg-A): $\partial_j U(0,x) = \partial_j u(x)$, $1 \leq j \leq n$, where $u(x)$ is given.

In this paper, we consider these boundary value problems (BVPs) in $L_2(\mathbf{R}^n)$, i.e. the boundary data are $u \in L_2(\mathbf{R}^n)$, $\phi \in L_2(\mathbf{R}^n)$ and $u \in \dot{H}^1(\mathbf{R}^n)$ respectively, and

for well-posedness a unique function $U(t,x)$ with certain L_2 estimates is required. Detailed definitions are given in Section 2.

These BVPs arise naturally when considering BVPs for the Laplace equation on a Lipschitz domain Ω in $\mathbf{R}^n$. As the main problem here is a local one, the result for such domains can be derived from the scale invariant case of a Lipschitz graph domain, i.e. we assume that $\Omega=\{(t,x);t>g(x)\}$ is the domain above the graph of some Lipschitz function g. Through the change of variables $U(t,x):=V(t+g(x),x)$, a harmonic function V in Ω corresponds to U in $\mathbf{R}^{n+1}_+$ satisfying (1) with coefficients $A=(1+|\nabla_x g|^2, -\nabla_x g^t; -\nabla_x g, I)$, and the respective boundary conditions carry over from $\partial\Omega$ to $\mathbf{R}^n$. The coefficients appearing from this pullback technique are referred to as being of *Jacobian type*, and are in particular real and symmetric, as well as independent of the transversal coordinate t. In this case, solvability of the Dirichlet problem was first proved by Dahlberg [15], and solvability of the Neumann and regularity problems was first proved by Jerison and Kenig [23]. Later Verchota [34] showed that these BVPs are solvable with the layer potential integral equation method. For general real symmetric matrices A, not being of the Jacobian type, well-posedness of the Dirichlet problem was first proved by Jerison and Kenig [24], and of the Neumann and regularity problems by Kenig and Pipher [27].

It is natural to ask whether the BVPs for the fundamental elliptic equation (1) are well posed for more general coefficients. Obvious generalizations are coefficients $A(t,x)$ with t-dependence, as well as more general non-symmetric or complex-coefficient matrices. In both cases, it is known that well-posedness does not hold in general. Caffarelli, Fabes and Kenig [11] observed that some regularity in the t-coordinate is necessary for well-posedness, and Kenig, Koch, Pipher and Toro [26] and Kenig and Rule [28] gave examples where well-posedness fail in any L_p for sufficiently non-symmetric, but t-independent, real coefficients in the plane which are discontinuous at $x=0$. However, on the positive side they show that, for given real non-symmetric coefficients in the plane, the Dirichlet problem is well posed in L_p for sufficiently large p, whereas the Neumann and regularity problems are well posed for p sufficiently close to 1.

In this paper, we consider only t-independent coefficients, but allow on the other hand for arbitrary complex, strictly accretive coefficients $A\in L_\infty(\mathbf{R}^n;\mathcal{L}(\mathbf{C}^{1+n}))$. As remarked, well-posedness does not hold in general for the BVPs in this case. But our main result Theorem 2.2 shows that the sets of well-posedness

$$(2) \qquad \mathrm{WP}(X):=\{A\,;X-A \text{ is well posed in } L_2(\mathbf{R}^n)\}\subset L_\infty(\mathbf{R}^n;\mathcal{L}(\mathbf{C}^{1+n})),$$

where X denotes one of the three BVPs Dir, Neu or Reg, are all open sets. As discussed above, the sets of well-posedness contain all real symmetric coefficients. Our theorem thus in particular shows well-posedness for small complex perturba-

tions of real symmetric coefficients. This has also been proved in [2] by Alfonseca, Auscher, Axelsson, Hofmann and Kim, with other methods using layer potential operators, and in [3] by Auscher, Axelsson and Hofmann. In fact, our methods here give the new result that well-posedness holds for complex Hermitean matrices and their perturbations.

One may ask what is the motivation for considering complex coefficients. However interesting it may be to know well-posedness for complex matrices, a main motivation is that this feeds back to give perturbation estimates for real matrices. In fact, to show that the solution U varies continuously, for fixed boundary data, as $A(x)$ varies continuously in L_∞ within the subspace of real symmetric matrices, there is no known method which does not use bounds for complex BVPs. The observation being used is that bounds for complex BVPs imply analytic dependence on A and in particular Lipschitz regularity with respect to $A \in L_\infty$.

Turning to other consequences of Theorem 2.2, well-posedness is well known to hold for all constant coefficients $A(x) = A$, and our theorem thus yields well-posedness for perturbations here as well. The Dirichlet problem was first shown to be well posed for small perturbations of constant matrices, by Fabes, Jerison and Kenig [20], using the method of multilinear expansions. The Neumann and regularity problems were tackled in [2] and [3].

It is also known that (Neu-A) and (Reg-A) are well posed in L_2 for complex matrices of block form, i.e. such that $A_{0,j} = 0 = A_{j,0}$ for all $1 \le j \le n$. This is a nontrivial result and is in fact equivalent to the Kato square root problem, proved by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [5]. Our theorem thus yields well-posedness for small perturbations of complex block form matrices, and is in this sense a generalization of the Kato square root estimate. With the further assumption of pointwise resolvent kernel bounds, this result is also implicit in [2]. However, our methods in this paper require no such pointwise estimates.

Let us now discuss the methods underlying Theorem 2.2. For the proof we use, following [3], boundary equation methods involving a Cauchy operator E_A. The name Cauchy operator is used since E_A coincides with the Cauchy singular integral operator when $A = I$ and $n = 1$. The first step of the proof is to rewrite the second order equation (1) as the equivalent first order system

$$(3) \qquad \begin{cases} \operatorname{div}_{t,x} A(x) F(t,x) = 0, \\ \operatorname{curl}_{t,x} F(t,x) = 0, \end{cases}$$

taking the gradient vector field $F(t,x) := \nabla_{t,x} U(t,x)$ as the unknown function instead of the potential U. The Cauchy operator E_A is related to (3) in the same way that the classical Cauchy integral operator is related to the Cauchy–Riemann equations. Just as the Cauchy singular integral operator is a Fourier multiplier

with symbol $\mathrm{sgn}(\xi)$, that is, belongs to the functional calculus of d/dx, the operator E_A belongs to the functional calculus a first order differential operator T_A. The bisectorial operator $-T_A$ in $L_2(\mathbf{R}^n; \mathbf{C}^{1+n})$ is the infinitesimal generator for the system (3) in the sense that these equations are equivalent to $\partial_t F + T_A F = 0$. The fundamental problem is to prove that this operator T_A has a bounded holomorphic functional calculus, and as a consequence that the Cauchy operator E_A is bounded. Given this, the perturbation results for BVPs follow as a consequence.

In [3], it was proved that $\|E_A\| < \infty$ when $\|A - A_0\|_\infty < \varepsilon$ and A_0 is either real symmetric, block or constant. This paper made use of a rather lengthy perturbation argument, and also used square function estimates of Dahlberg, Jerison and Kenig [16] and estimates of harmonic measure of Jerison and Kenig [24], for solutions to (1) in the real symmetric case.

In this paper we prove the boundedness of the holomorphic functional calculus of T_A, for all complex A, directly from the quadratic estimates proved by Axelsson, Keith and McIntosh [10]. In this way, our results build on the proof of the Kato square root problem [5]. That $\|E_A\| < \infty$ for all complex A may come as a surprise, in view of the above mentioned counterexamples to well-posedness of the BVPs for non-symmetric coefficients. However, it is important to note that E_A itself has nothing to do with BVPs, it is an infinitesimal generator associated with the differential equation, and is not related to the boundary conditions (except in the case of block form matrices). As a consequence of the boundedness of $E_A = \mathrm{sgn}(T_A)$, we prove in Theorem 2.3 that there is a Hardy type splitting

$$L_2(\mathbf{R}^n; \mathbf{C}^{1+n}) \ni f = F^+|_{\mathbf{R}^n} + F^-|_{\mathbf{R}^n}$$

of boundary functions f into traces of $F^\pm$ satisfying (3) in $\mathbf{R}_\pm^{1+n}$, with estimates $\|f\|_2 \approx \|F^+|_{\mathbf{R}^n}\|_2 + \|F^-|_{\mathbf{R}^n}\|_2$. That a BVP is well posed is the question whether the full traces $F^+|_{\mathbf{R}^n}$ of solutions to the equations in $\mathbf{R}_+^{n+1}$ are in one-to-one correspondence with the normal components (Neumann problem) or tangential parts (Dirichlet regularity problem), respectively. It is this one-to-one correspondence which may fail for some complex A. What we prove here and use for the proof of Theorem 2.2, is that the Hardy subspaces $\{F^+|_{\mathbf{R}^n}\}$, and the projections $f \mapsto F^+|_{\mathbf{R}^n}$ onto them, depend analytically on A.

Finally we note that the methods developed here go beyond scalar elliptic equations like (1). The natural framework here is rather BVPs for elliptic systems of partial differential equations, as should be clear from (3). Thus we shall formulate our results for divergence form elliptic systems of m second order equations (but the reader interested in scalar equations only can set $m = 1$ throughout). In this setting, our well-posedness results are mostly new. Previously known results are

limited to systems with coefficients of Jacobian type, or more generally constant coefficient systems on Lipschitz domains.

Well-posedness of (Dir-A), (Reg-A) and (Neu-A) with L_2 boundary values have been obtained for the Stokes' system by Fabes, Kenig and Verchota [21], and of (Dir-A) and (Reg-A) for the Lamé system by Dahlberg, Kenig and Verchota [18]. For general constant coefficient symmetric second order systems, solvability results for (Neu-A) and (Reg-A) are found in Fabes [19]. Under the weaker Legendre–Hadamard ellipticity condition, (Dir-A) and (Reg-A) where solved by Gao [22]. As for non-symmetric systems, Verchota and Vogel [35] obtained L_p solvability results for (Dir-A), (Reg-A) and (Neu-A) in the spirit of [26] and [28] for certain non-symmetric constant coefficient Legendre–Hadamard systems of two equations on C^1 polygons in the plane. For general elliptic systems, the Kato problem was solved by Auscher, Hofmann, McIntosh and Tchamitchian [6], a consequence being the well-posedness of (Neu-A) and (Reg-A) for elliptic systems with block form coefficients.

Note that the pullback technique, from the Lipschitz domain Ω to $\mathbf{R}^{n+1}_+$, described above, works for more general divergence form equations or systems. In this case, coefficients $\tilde{A}(x)$ in Ω are transformed into coefficients

$$A(x) := \begin{pmatrix} 1 & -(\nabla_x g(x))^t \\ 0 & I \end{pmatrix} \tilde{A}(x) \begin{pmatrix} 1 & 0 \\ -\nabla_x g(x) & I \end{pmatrix}$$

in $\mathbf{R}^{n+1}_+$. We also remark that our methods are by no means limited to divergence form equations. In Section 6, we give solvability results for exterior differential systems (22) for differential forms, as an example of this. This generalizes the first order system (3), which is the special case of (22) for 1-forms. Furthermore, we note that time-harmonic Maxwell's equations on a Lipschitz domain can be written as a system of equations (22) for 1- and 2-forms with lower order terms added, through the above pullback technique. Thus, although not directly applicable, (22) is closely related to Maxwell's equations. Solvability of Maxwell's equations on Lipschitz domains is due to Mitrea [32], and more general BVPs for Dirac equations were solved by McIntosh and Mitrea [30]. In fact, the Cauchy integral boundary equation method used in this paper, as well as in [3], was developed for solving BVPs for Maxwell's and Dirac's equations in the Ph.D. thesis [7] of the second author. Further elaborations of the ideas presented in this paper, along the lines of thought in [3] and [9], working with general inhomogeneous differential forms taking values in the full exterior algebra and allowing lower order terms, one should be able to extend the theory to cover both Dirac's and Maxwell's equations.

2. Notation and results

We begin by giving the precise definition of well-posedness of the BVPs discussed in the introduction, or rather the corresponding BVPs for systems. Throughout this paper, we use the notation $X \approx Y$ and $X \lesssim Y$ for estimates to mean that there exists a constant $C > 0$, independent of the variables in the estimate, such that $X/C \leq Y \leq CX$ and $X \leq CY$, respectively.

We write $\{e_0, e_1, ..., e_n\}$ for the standard basis for $\mathbf{R}^{1+n}$ with e_0 "upward" pointing into $\mathbf{R}_+^{1+n}$, and write $t = x_0$ for the vertical coordinate. For the vertical derivative, we write $\partial_0 = \partial_t$. For vectors $v = (v_j^\alpha)_{0 \leq j \leq n}^{1 \leq \alpha \leq m}$, we write v_0 and $v_\parallel$ for the normal and tangential parts of v, i.e. $(v_0)_0^\alpha = v_0^\alpha$ and $(v_0)_j^\alpha = 0$ when $1 \leq j \leq n$, whereas $(v_\parallel)_j^\alpha = v_j^\alpha$ when $1 \leq j \leq n$ and $(v_\parallel)_0^\alpha = 0$. Frequently, we shall identify normal vector fields $v = v_0$ with the corresponding scalar functions $v_0 = (v_0^\alpha)_{\alpha=1}^m$.

For systems, gradient and divergence act as $(\nabla_{t,x} U)_j^\alpha = \partial_j U^\alpha$ and $(\operatorname{div}_{t,x} F)^\alpha = \sum_{j=0}^n \partial_j F_j^\alpha$, with corresponding tangential versions $\nabla_x U = (\nabla_{t,x} U)_\parallel$ and $(\operatorname{div}_x F)^\alpha = \sum_{j=1}^n \partial_j F_j^\alpha$. With $\operatorname{curl}_{t,x} F = 0$ we understand that $\partial_k F_j^\alpha = \partial_j F_k^\alpha$, for all $j, k = 0, ..., n$, $\alpha = 1, ..., m$. Similarly, write $\operatorname{curl}_x F_\parallel = 0$ if $\partial_k F_j^\alpha = \partial_j F_k^\alpha$, for all $j, k = 1, ..., n$, $\alpha = 1, ..., m$.

We consider divergence form second order elliptic systems

$$(4) \qquad \sum_{j,k=0}^n \sum_{\beta=1}^m \partial_j A_{j,k}^{\alpha,\beta}(x) \partial_k U^\beta(t, x) = 0, \quad \alpha = 1, ..., m,$$

on the half spaces $\mathbf{R}_\pm^{1+n} := \{(t, x) \in \mathbf{R} \times \mathbf{R}^n; \pm t > 0\}$, $n \geq 1$, where the matrix $A = (A_{j,k}^{\alpha,\beta}(x))_{j,k=0,...,n}^{\alpha,\beta=1,...,m} \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ is assumed to be t-independent with complex coefficients and *strictly accretive* on $\mathsf{N}(\operatorname{curl}_\parallel)$, in the sense that there exists

$\varkappa > 0$ such that

$$(5) \qquad \sum_{j,k=0}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathbf{R}^n} \mathrm{Re}\big(A_{j,k}^{\alpha,\beta}(x) f_k^{\beta}(x) \overline{f_j^{\alpha}(x)}\big)\, dx \geq \varkappa \sum_{j=0}^{n} \sum_{\alpha=1}^{m} \int_{\mathbf{R}^n} \big|f_j^{\alpha}(x)\big|^2\, dx$$

for all $f \in \mathsf{N}(\mathrm{curl}_{\|}) := \{g \in L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}); \mathrm{curl}_x(g_{\|}) = 0\}$.

Equivalently, this means that the Gårding inequality

$$\int_{\mathbf{R}^n} \mathrm{Re}(A(\mathrm{div}_x\, u_{\|} + \nabla_x u_0), (\mathrm{div}_x\, u_{\|} + \nabla_x u_0))\, dx \geq \varkappa \int_{\mathbf{R}^n} (|\mathrm{div}_x\, u_{\|}|^2 + |\nabla_x u_0|^2)\, dx$$

holds for $u = (u_0, u_{\|}) \in \dot{H}^1(\mathbf{R}^n; \mathbf{C}^{(1+n)m})$, since div_x has dense range in $L_2(\mathbf{R}^n; \mathbf{C}^m)$ and ∇_x has dense range in $\{g \in L_2(\mathbf{R}^n; \mathbf{C}^{nm}); \mathrm{curl}_x\, g = 0\}$. Splitting $\mathbf{C}^{(1+n)m}$ into normal parts $\mathbf{C}^m$ and tangential parts $\mathbf{C}^{nm}$, we write

$$A(x)v = \begin{pmatrix} A_{00}(x) & A_{0\|}(x) \\ A_{\|0}(x) & A_{\|\|}(x) \end{pmatrix} \begin{pmatrix} v_0 \\ v_{\|} \end{pmatrix}.$$

It is then clear that (5) implies that A_{00} is pointwise strictly accretive and that $A_{\|\|}$ satisfies the strict Gårding inequality

$$(6) \qquad \begin{cases} \mathrm{Re}(A_{00}(x)v, v) \geq \varkappa |v|^2, & v \in \mathbf{C}^m,\ \text{a.e. } x \in \mathbf{R}^n, \\ \int_{\mathbf{R}^n} \mathrm{Re}(A_{\|\|} \nabla_x u_0, \nabla_x u_0)\, dx \geq \varkappa \int_{\mathbf{R}^n} |\nabla_x u_0|^2\, dx, & u_0 \in \dot{H}^1(\mathbf{R}^n; \mathbf{C}^m). \end{cases}$$

The condition (5) lies between *pointwise strict accretivity*, i.e.

$$(7) \qquad \mathrm{Re}(A(x)v, v) \geq \varkappa |v|^2 \quad \text{for all } v \in \mathbf{C}^{(1+n)m} \text{ and a.e. } x \in \mathbf{R}^n,$$

and the $\mathbf{R}_+^{n+1}$ Gårding inequality

$$(8) \qquad \iint_{\mathbf{R}_+^{n+1}} \mathrm{Re}(A(x)\nabla_{t,x} g(t,x), \nabla_{t,x} g(t,x))\, dt\, dx \geq \varkappa \iint_{\mathbf{R}_+^{n+1}} |\nabla_{t,x} g(t,x)|^2\, dt\, dx$$

for $g \in \dot{H}^1(\mathbf{R}_+^{n+1}; \mathbf{C}^m)$. Clearly, (7) implies (5), which in turn implies (8), as is seen by taking $f(x) = \nabla_{t,x} g(t,x)$ for fixed t and then integrating over t. Furthermore (8), implies (6), which is seen by taking $g(t,x) := \psi(\varepsilon t) u_0(x)$ and integrating away t, for some $\psi \in C_0^{\infty}(\mathbf{R}_+)$. Letting $\varepsilon \to \infty$ and $\varepsilon \to 0$, respectively, proves (6). In fact, only the Gårding inequality (8) for $g \in H_0^1(\mathbf{R}_+^{n+1}; \mathbf{C}^m)$ is needed for this argument.

When $n = 1$, (5) is equivalent to strong accretivity (7) since in this case ∇_x has dense range in $L_2(\mathbf{R}; \mathbf{C}^m)$ and $\mathsf{N}(\mathrm{curl}_{\|}) = L_2(\mathbf{R}; \mathbf{C}^{2m})$. On the other hand, if A is of block form, i.e. $A_{0\|} = A_{\|0} = 0$, then (5) is equivalent to (6) and to the $\mathbf{R}_+^{n+1}$ Gårding inequality (8), for $H^1(\mathbf{R}_+^{n+1}; \mathbf{C}^m)$ as well as for $H_0^1(\mathbf{R}_+^{n+1}; \mathbf{C}^m)$, since (6) implies (5). It is also known that the $\mathbf{R}_+^{n+1}$ Gårding inequality (8) implies strong accretivity (7) when $m = 1$, so for scalar equations (5), (8) and (7) are all equivalent.

On the functions $U = (U^{\alpha})_{\alpha=1}^{m}$ satisfying (4), we impose one of the following boundary conditions.

- (Dir-A): $U^\alpha(0,x)=u^\alpha(x)$ for a given function $u\in L_2(\mathbf{R}^n;\mathbf{C}^m)$.
- (Neu-A): $-\sum_{k=0}^n\sum_{\beta=1}^m A_{0,k}^{\alpha,\beta}(x)\partial_k U^\beta(0,x)=\phi^\alpha(x)$, where $\phi\in L_2(\mathbf{R}^n;\mathbf{C}^m)$
is given.
- (Reg-A): $\partial_j U^\alpha(0,x)=\partial_j u^\alpha(x)$, $1\le j\le n$, where $u\in\dot{H}^1(\mathbf{R}^n;\mathbf{C}^m)$ is given.

The boundary value problems (Neu-A) and (Reg-A) can be viewed as problems concerning a first order partial differential system, and this is the point of view we take here. Indeed, consider the gradient vector fields

$$F(t,x)=\nabla_{t,x}U(t,x)\colon \mathbf{R}_+^{1+n}\longrightarrow \mathbf{C}^{(1+n)m}.$$

Since the scalar potentials U are in one-to-one correspondence with the curl-free vector fields $F(t,x)$, modulo constants, we can take F rather than U as the unknown, and equation (4) for U is rewritten as the equivalent first order system (3) for F. Since the coefficients $A(x)$ are independent of t, it is natural to view F from the semigroup point of view $F(t,x)=F_t(x)=\nabla_{t,x}U(t,x)\in C^1(\mathbf{R}_+;L_2(\mathbf{R}^n;\mathbf{C}^{(1+n)m}))$.

Definition 2.1. (i) We say that the boundary value problem (Neu-A) is *well posed* if for each boundary data $\phi\in L_2(\mathbf{R}^n;\mathbf{C}^m)$, there exists a unique function

$$F(t,x)=F_t(x)=\nabla_{t,x}U(t,x)\in C^1(\mathbf{R}_+;L_2(\mathbf{R}^n;\mathbf{C}^{(1+n)m}))$$

which satisfies (3) for $t>0$, and has limits $\lim_{t\to\infty}F_t=0$ and $\lim_{t\to0}F_t=f$ in L_2 norm, where the full boundary trace f satisfies the boundary condition $-(Af)_0=\phi$. More precisely, by F satisfying (3), we mean that $\partial_t(AF)_0=-\operatorname{div}_x(AF)_\parallel$, $\partial_t F_\parallel=\nabla_x F_0$ and $\operatorname{curl}_x F_\parallel=0$, where ∂_t is taken in the strong sense, and x-derivatives in the distributional sense.

(ii) We say that the boundary value problem (Reg-A) is *well posed* if for each boundary data $\nabla_x u\in L_2(\mathbf{R}^n;\mathbf{C}^{nm})$, there exists a unique function

$$F\in C^1(\mathbf{R}_+;L_2(\mathbf{R}^n;\mathbf{C}^{(1+n)m}))$$

which satisfies (3) for $t>0$, and has limits $\lim_{t\to\infty}F_t=0$ and $\lim_{t\to0}F_t=f$ in L_2 norm, where the full boundary trace f satisfies the boundary condition $f_\parallel=\nabla_x u$.

(iii) The Dirichlet problem (Dir-A) is said to be *well posed* if for each $u\in L_2(\mathbf{R}^n;\mathbf{C}^m)$, there is a unique function

$$U_t(x)=U(t,x)\in C^1(\mathbf{R}_+;L_2(\mathbf{R}^n;\mathbf{C}^m))$$

such that $\nabla_x U\in C^0(\mathbf{R}_+;L_2(\mathbf{R}^n;\mathbf{C}^{nm}))$, where U satisfies (4) for $t>0$, $\lim_{t\to0}U_t=u$, $\lim_{t\to\infty}U_t=0$, $\lim_{t\to\infty}\nabla_{t,x}U_t=0$ in L_2 norm, and $\int_{t_0}^{t_1}\nabla_x U_s\,ds$ converges in L_2 when $t_0\to0$ and $t_1\to\infty$. More precisely, by U satisfying (4), we mean that

$$\int_t^\infty ((A\nabla_{s,x}U_s)_\parallel,\nabla_x v)\,ds=-((A\nabla_{t,x}U_t)_0,v)$$

for all $v\in C_0^\infty(\mathbf{R}^n;\mathbf{C}^m)$.

The Dirichlet problem (Dir-A) will also be rewritten as a BVP for the first order system (3). However, here it is not appropriate to consider the gradient vector field $\nabla_{t,x}U$, since the boundary condition is a condition on the potential U itself. Instead we use the point of view of harmonic conjugate functions, and write $F=Ue_0+F_\parallel$, where the tangential vector fields $F_\parallel$ are conjugate functions in a generalized sense and F satisfies (3), which is viewed as a generalized Cauchy–Riemann system. Details of this are given in Lemma 4.2, where it is shown that the Dirichlet problem (Dir-A) for U is equivalent to an auxiliary Neumann problem (Neu$^\perp$-A) for F.

Our main result, which we prove in Section 4, is the following.

Theorem 2.2. *The sets* WP(Reg), WP(Neu) *and* WP(Dir), *as define d in* (2), *are all open subsets of* $L_\infty(\mathbf{R}^n;\mathcal{L}(\mathbf{C}^{(1+n)m}))$. *Each of the sets of well-posedness contains*

(i) *all Hermitean matrices* $A(x)=A(x)^*$ (*and in particular all real symmetric matrices*),

(ii) *all block matrices where* $A_{0,j}^{\alpha,\beta}(x)=0=A_{j,0}^{\alpha,\beta}(x)$, $1\le j\le n$, $1\le\alpha,\beta\le m$, *and*

(iii) *all constant matrices* $A(x)=A$.

The notion of well-posedness used here departs from the standard variational one. However, we show in Section 5 that the solutions obtained here coincide with the solutions obtained through the Lax–Milgram theorem when A belongs to the connected component of WP which contains I. This connected component includes the three classes (i), (ii) and (iii) specified in Theorem 2.2. The notion of well-posedness used here coincides with that in [3] for the BVPs (Neu-A) and (Reg-A). However, for (Dir-A) the meaning of well-posedness differs slightly from that in [3], as we impose an extra integrability condition on $\nabla_x U$ here.

A natural function space for solutions $F_t(x)\in C^1(\mathbf{R}_+;L_2(\mathbf{R}^n;\mathbf{C}^{(1+n)m}))$ to the BVPs is $L_\infty(\mathbf{R}_+;L_2)$, with norm $\sup_{t>0}\|F_t\|_2$. Two other norms which are important are the square function norm $\|F_t\|^2:=\int_0^\infty\|F_t\|_2^2\,t^{-1}\,dt$ and the norm $\|\widetilde{N}_*(F)\|_2$, using the modified *non-tangential maximal function*

$$\widetilde{N}_*(F)(x):=\sup_{t>0}t^{-(1+n)/2}\|F\|_{L_2(Q(t,x))},$$

where $Q(t,x):=[(1-c_0)t,(1+c_0)t]\times B(x;c_1t)$ for some fixed constants $c_0\in(0,1)$ and $c_1>0$.

The key result underlying Theorem 2.2, which we prove in Section 3, is the following result on Hardy type splittings of $L_2(\mathbf{R}^n)$.

Theorem 2.3. *Let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ be a t-independent, complex matrix function which is strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$.*

Then each $f \in \mathsf{N}(\mathrm{curl}_\parallel)$ is in one-to-one correspondence with a pair of vector fields $F_t^\pm(x) = F^\pm(t,x) \in C^1(\mathbf{R}_\pm; L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}))$ in $\mathbf{R}_\pm^{1+n}$ satisfying (3) and having L_2 limits $\lim_{t\to 0\pm} F_t^\pm = f^\pm$ and $\lim_{t\to\pm\infty} F_t^\pm = 0$, such that

$$f = f^+ + f^-.$$

This splitting is topological, i.e. $\|f\|_2 \approx \|f^+\|_2 + \|f^-\|_2$, and the vector fields $F^\pm$ satisfy the norm equivalences

$$\|f\|_2 \approx \sup_{t>0} \|F_t\|_2 \approx \|t\partial_t F_t\| \approx \|\widetilde{N}_*(F)\|_2.$$

Moreover, the Hardy projections $E_A^\pm\colon f \mapsto F^\pm = F_A^\pm$ depend locally Lipschitz continuously on A in the sense that

$$\|F_{A_2}^\pm - F_{A_1}^\pm\|_{\mathcal{X}} \le C\|A_2 - A_1\|_{L_\infty(\mathbf{R}^n)}\|f\|_2,$$

where $C = C(\varkappa_{A_1}, \varkappa_{A_2}, \|A_1\|_\infty, \|A_2\|_\infty)$ and where $\|F\|_{\mathcal{X}}$ denotes any of the four norms above.

In particular, for the Neumann problem we have the following estimates of solutions. If $A \in \mathrm{WP(Neu)}$, then the solution $F = \nabla_{t,x}U$ with boundary data $\phi = -(Af)_0$, where $f := F|_{\mathbf{R}^n}$, has estimates

$$\int_{\mathbf{R}^n} |\phi|^2\, dx \approx \int_{\mathbf{R}^n} |f|^2\, dx \approx \sup_{t>0} \int_{\mathbf{R}^n} |F_t|^2\, dx$$

$$\approx \int_{\mathbf{R}^n} |\widetilde{N}_*(F)|^2\, dx \approx \iint_{\mathbf{R}_+^{1+n}} |\partial_t F|^2 t\, dt\, dx.$$

Similar estimates hold for the solution to the Dirichlet regularity problem with boundary data $\nabla_x u$. For the Dirichlet problem, Theorem 2.3 shows the following estimates.

Corollary 2.4. *Let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ be a t-independent, complex matrix function which is strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$ and assume that $A \in \mathrm{WP(Dir)}$. Then any function $U_t(x) = U(t,x) \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m))$ solving (4), with properties as in Definition 2.1, has estimates*

$$\int_{\mathbf{R}^n} |u|^2\, dx \approx \sup_{t>0} \int_{\mathbf{R}^n} |U_t|^2\, dx \approx \int_{\mathbf{R}^n} |\widetilde{N}_*(U)|^2\, dx \approx \iint_{\mathbf{R}_+^{1+n}} |\nabla_{t,x}U|^2 t\, dt\, dx,$$

where $u=U|_{\mathbf{R}^n}$. If furthermore A is real (not necessarily symmetric) and $m=1$, then Moser's local boundedness estimate [33] gives the pointwise estimate $\widetilde{N}_*(U)(x)\approx N_*(U)(x)$, where the standard non-tangential maximal function is $N_*(U)(x):=\sup_{|y-x|<ct}|U(t,y)|$, for fixed $0<c<\infty$.

Theorem 2.2 shows in particular that $A\in\mathrm{WP}(\mathrm{Dir})$ if A is real symmetric. Even for real symmetric A our methods yield a new proof of the estimate between the square function and the non-tangential maximal function above, first proved by Dahlberg, Jerison and Kenig [16], using estimates of harmonic measure by Jerison and Kenig [24], in the scalar case $m=1$. In [3], these estimates were used to prove that $\|E_A\|<\infty$ for real symmetric A. Here we reverse the argument: we prove that $\|E_A\|<\infty$ independently and deduce from this the equivalences of norms.

In the case $m>1$ of systems, Dahlberg, Kenig, Pipher and Verchota [17] have shown equivalence in L_p norm of the non-tangential maximal function and the square function, for general constant coefficient real symmetric systems on Lipschitz domains. Also, such equivalence has been shown for certain non-symmetric systems of two equations in the plane by Verchota and Vogel [36].

3. Cauchy operators and Hardy spaces

The boundary equation methods for solving BVPs which are used here and in [3], are based on Cauchy operators E_A, with associated Hardy type subspaces. In this section we prove quadratic estimates for E_A and deduce from this Theorem 2.3. How these Cauchy operators are used to prove Theorem 2.2 is shown in Section 4 and in particular in Lemma 4.3. Note that the operators E_A themselves depend only on the differential system (3), and have nothing to do with the boundary conditions. Thus they are the same for both Neumann and Dirichlet problems.

We start by rewriting the equations (3) in terms of an "infinitesimal generator" T_A. Write $v\in\mathbf{C}^{(1+n)m}$ as $v=(v_0,v_\parallel)^t$, where $v_0\in\mathbf{C}^m$ and $v_\parallel\in\mathbf{C}^{nm}$, and introduce the auxiliary matrices

$$\bar{A}:=\begin{pmatrix} A_{00} & A_{0\parallel} \\ 0 & I \end{pmatrix} \text{ and } \underline{A}:=\begin{pmatrix} 1 & 0 \\ A_{\parallel 0} & A_{\parallel\parallel} \end{pmatrix}, \quad \text{if } A=\begin{pmatrix} A_{00} & A_{0\parallel} \\ A_{\parallel 0} & A_{\parallel\parallel} \end{pmatrix}$$

in the normal/tangential splitting of $\mathbf{C}^{(1+n)m}$. Recall that A being strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$, as in (5), implies the accretivity estimates (6) for the diagonal blocks A_{00} and $A_{\parallel\parallel}$. Since A_{00} is pointwise strictly accretive, it is invertible, and consequently $\bar{A}$ is invertible. This is not necessarily true for $\underline{A}$.

Splitting normal and tangential derivatives in (3), we see that this system of equations is equivalent to

$$\begin{cases} \partial_t (AF)_0 + \mathrm{div}_x (AF)_\| = 0, \\ \partial_t F_\| - \nabla_x F_0 = 0, \end{cases}$$

together with the constraint $\mathrm{curl}_x F_\| = 0$. Since

(9) $$\bar{A}F = (AF)_0 + F_\| \quad \text{and} \quad \underline{A}F = F_0 + (AF)_\|,$$

we have shown that (3) is equivalent to

$$\begin{cases} \partial_t (\bar{A}F)_0 + \mathrm{div}_x (\underline{A}F)_\| = 0, \\ \partial_t (\bar{A}F)_\| - \nabla_x (\underline{A}F)_0 = 0, \end{cases}$$

together with the tangential constraint $\mathrm{curl}_x F_\| = 0$. Combining the two equations, we get

(10) $$\partial_t F + T_A F = 0,$$

where T_A is the following operator.

Definition 3.1. Let D be the self-adjoint differential operator $D := \begin{pmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{pmatrix}$ in $L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m})$ with domain

$$\mathsf{D}(D) := \{ (f_0, f_\|)^t \in L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}) \,; \nabla_x f_0, \mathrm{div}_x f_\| \in L_2 \}.$$

Define the infinitesimal generator for (3) to be the operator

$$T_A := \bar{A}^{-1} D \underline{A} = \begin{pmatrix} A_{00}^{-1}(A_{0\|}\nabla_x + \mathrm{div}_x A_{\|0}) & A_{00}^{-1} \mathrm{div}_x A_{\|\|} \\ -\nabla_x & 0 \end{pmatrix},$$

with domain $\mathsf{D}(T_A) := \underline{A}^{-1}\mathsf{D}(D)$. Let the transformed coefficient matrix be

$$\hat{A} := \underline{A}\bar{A}^{-1} = \begin{pmatrix} A_{00}^{-1} & -A_{00}^{-1} A_{0\|} \\ A_{\|0}A_{00}^{-1} & A_{\|\|} - A_{\|0}A_{00}^{-1}A_{0\|} \end{pmatrix}$$

so that

(11) $$T_A = \bar{A}^{-1}(D\hat{A})\bar{A}.$$

The following is the main algebraic result of the paper. Recall that $\mathsf{N}(\mathrm{curl}_\|) = \{ g \in L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}); \mathrm{curl}_x(g_\|) = 0 \}$ and note that $\mathsf{N}(\mathrm{curl}_\|) = \overline{\mathsf{R}(D)}$.

Proposition 3.2. *The transformed coefficient matrix $\hat{A}$ is bounded and strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$, i.e. satisfies* (5), *if and only if A is bounded and strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$. Moreover $\hat{\hat{A}}{=}A$.*

Proof. Assume that A is bounded and strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$. As noted above, $\bar{A}$ is invertible and thus $\hat{A}$ is bounded. Since $\bar{A}$ acts as identity on tangential vector fields, it is clear that $\bar{A}{:}\mathsf{N}(\mathrm{curl}_\parallel){\to}\mathsf{N}(\mathrm{curl}_\parallel)$ is an isomorphism. Strict accretivity of $\hat{A}$ on $\mathsf{N}(\mathrm{curl}_\parallel)$ now follows from the formula

$$\mathrm{Re}(\hat{A}(\bar{A}f),\bar{A}f) = \mathrm{Re}(\underline{A}f,\bar{A}f) = \mathrm{Re}((f_0, A_{00}f_0+A_{0\parallel}f_\parallel)+(A_{\parallel 0}f_0+A_{\parallel\parallel}f_\parallel, f_\parallel))$$
$$= \mathrm{Re}((A_{00}f_0+A_{0\parallel}f_\parallel, f_0)+(A_{\parallel 0}f_0+A_{\parallel\parallel}f_\parallel, f_\parallel)) = \mathrm{Re}(Af, f).$$

The identity $\hat{\hat{A}}{=}A$ is straightforward to verify, and this shows that the above argument is reversible. $\square$

We are now in a position to analyze the operator T_A. Due to (11), it suffices to study operators of the form DB in $L_2(\mathbf{R}^n; \mathbf{C}^N)$, where D is a self-adjoint homogeneous first order differential operator with constant coefficients, and B is a bounded multiplication operator which is strictly accretive on $\mathsf{R}(D)$, i.e. there exists $\varkappa>0$ such that

$$(12) \qquad\qquad \mathrm{Re}(BDu, Du) \geq \varkappa\|Du\|_2 \quad \text{for all } u \in \mathsf{D}(D).$$

The applications we have in mind are the specific operators D and $B{=}\hat{A}$ from Definition 3.1, in which case $\overline{\mathsf{R}(D)}{=}\mathsf{N}(\mathrm{curl}_\parallel)$, as well as generalizations of these in Section 6.

Define closed and open sectors and double sectors in the complex plane by

$$S_{\omega+} := \{z \in \mathbf{C}\,;\, |\arg z| \leq \omega\}\cup\{0\}, \quad S_\omega := S_{\omega+}\cup(-S_{\omega+}),$$
$$S_{\nu+}^o := \{z \in \mathbf{C}\,;\, z \neq 0 \text{ and } |\arg z| < \nu\}, \quad S_\nu^o := S_{\nu+}^o\cup(-S_{\nu+}^o),$$

and define the *angle of accretivity* of B to be

$$\omega := \sup_{\substack{f\in\mathsf{R}(D)\\ f\neq 0}} |\arg(Bf, f)| < \pi/2.$$

Proposition 3.3. *Let D be a self-adjoint operator and let B be a bounded operator in $L_2(\mathbf{R}^n; \mathbf{C}^N)$ which satisfies* (12).

(i) *The operator DB is a closed and densely defined ω-bisectorial operator, i.e. $\sigma(DB){\subset}S_\omega$, where ω is the angle of accretivity of B. Moreover, there are resolvent bounds $\|(\lambda I{-}DB)^{-1}\| \lesssim 1/\mathrm{dist}(\lambda, S_\omega)$ when $\lambda \notin S_\omega$.*

(ii) *The operator DB has range $\mathsf{R}(DB){=}\mathsf{R}(D)$ and null space $\mathsf{N}(DB){=}B^{-1}\mathsf{N}(D)$, where B^{-1} denotes the inverse image, such that*

$$L_2(\mathbf{R}^n;\mathbf{C}^N){=}\overline{\mathsf{R}(DB)}{\oplus}\mathsf{N}(DB)$$

topologically (but in general non-orthogonally).

(iii) *The restriction of DB to $\overline{\mathsf{R}(D)}{=}\overline{\mathsf{R}(DB)}$ is a closed and injective operator with dense range in $\overline{\mathsf{R}(D)}$, with estimates on spectrum and resolvents as in* (i).

Proof. As a consequence of (12), it is verified that DB and B^*D are closed and densely defined adjoint unbounded operators, and the topological splitting

$$L_2(\mathbf{R}^n;\mathbf{C}^m){=}\mathsf{N}(D){\oplus}B^*\overline{\mathsf{R}(D)}$$

follows from perturbing the orthogonal splitting $L_2{=}\mathsf{N}(D){\oplus}\overline{\mathsf{R}(D)}$ with B^*, which satisfies (12) as well. Since $\overline{\mathsf{R}(DB)}{=}\mathsf{N}(B^*D)^{\perp}{=}\mathsf{N}(D)^{\perp}{=}\overline{\mathsf{R}(D)}$ and $\mathsf{N}(DB){=}\mathsf{R}(B^*D)^{\perp}{=}(B^*\mathsf{R}(D))^{\perp}$, the splitting (ii) for DB follows, by stability of splittings under taking orthogonal complements.

Since $DB{=}(DB)|_{\overline{\mathsf{R}(D)}}{\oplus}0$ in the splitting (ii), it suffices to prove resolvent bounds for DB on $\overline{\mathsf{R}(D)}$. To this end, let $u{\in}\overline{\mathsf{R}(D)}{\cap}\mathsf{D}(DB)$ and $f{=}(\lambda I{-}DB)u$. Then

$$\mathrm{Im}(Bu,\lambda u{-}DBu){=}\mathrm{Im}(Bu,f).$$

Since D is self-adjoint, $(Bu,DBu){\in}\mathbf{R}$ and we get an estimate $|\mathrm{Im}(\overline{\lambda}(Bu,u))|{\lesssim}\|u\|\|f\|$, from which the resolvent bound $\|u\|{\lesssim}\|f\|/\mathrm{dist}(\lambda,S_\omega)$ follows. $\square$

These properties of closed operators of the form DB have been known for some time, see for example [1] and [14], at least in the case when B is strongly accretive. The following theorem has also been known for some time in the case when D is injective, as it derives from the special case $D{=}{-}i\,d/dx$ developed in [31] (see also Lecture 8 of [1]). In this case DB is similar to the operator $d/dz|_\gamma$ of differentiation on a Lipschitz graph γ, and the boundedness of $\mathrm{sgn}(DB)$ is equivalent to the boundedness of the Cauchy singular integral on γ, proved originally by Calderón when $B{-}I$ is sufficiently small [12], and in general by Coifman, McIntosh and Meyer [13].

The proof of the following theorem however is more involved when D is not injective. In the general case it was proved in [10, Theorem 3.1(iii)], building on results for the Kato problem by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [5]. It is also possible to give a direct proof, as shown in [4].

Theorem 3.4. *Let D be a self-adjoint homogeneous first order differential operator with constant coefficients such that*

$$\|Df\|{\gtrsim}\|\nabla f\|\quad\text{for all }f{\in}\overline{\mathsf{R}(D)}{\cap}\mathsf{D}(D),$$

and let B be a bounded multiplication operator in $L_2(\mathbf{R}^n; \mathbf{C}^N)$ which satisfies (12).

(i) *The operator DB satisfies the quadratic estimates*

$$\int_0^\infty \left\| tDB(1+(tDB)^2)^{-1}f \right\|^2 \frac{dt}{t} \approx \|f\|^2 \quad \text{for all } f \in \overline{\mathsf{R}(D)}.$$

(ii) *The operator DB has a bounded holomorphic functional calculus in* $\overline{\mathsf{R}(D)}$, *i.e. for each bounded holomorphic function $b(z)$ on a double sector S_ν^o, $\omega < \nu < \pi/2$, the operator $b(DB)$ in* $\overline{\mathsf{R}(D)}$ *is bounded with estimates*

$$\|b(DB)\|_{L_2 \to L_2} \lesssim \|b\|_{L_\infty(S_\nu^o)}.$$

For the precise definition of the operators $b(DB)$ in the functional calculus of DB we refer to [1]. Note that the map $H_\infty(S_\nu^o) \ni b \mapsto b(DB) \in \mathcal{L}(\overline{\mathsf{R}(D)})$ is a continuous algebra homomorphism.

We now return to the operator T_A of Definition 3.1. Note that the isomorphism $\bar{A}$ in (11) maps the subspace $\overline{\mathsf{R}(D)}$ onto itself.

Definition 3.5. Let $\mathcal{H}$ denote the closed subspace

$$\mathcal{H} := \overline{\mathsf{R}(D)} = \mathsf{N}(\mathrm{curl}_\parallel) = \{ f \in L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}) \, ; \mathrm{curl}_x \, f_\parallel = 0 \}$$

of $L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m})$.

In this notation, Proposition 3.3 and Theorem 3.4 with $B = \hat{A}$, thus have the following corollary.

Corollary 3.6. *The operator T_A from Definition 3.1 is an ω-bisectorial operator in $L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m})$, where ω is the angle of accretivity of $\hat{A}$. Furthermore we have a splitting*

$$L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}) = \overline{\mathsf{R}(T_A)} \oplus \mathsf{N}(T_A) = \mathcal{H} \oplus \{(0, f_\parallel)^t \, ; \mathrm{div}_x \, A_{\parallel\parallel} f_\parallel = 0\}$$

in which $T_A = T_A|_{\mathcal{H}} \oplus 0$. The restriction of T_A to $\mathcal{H}$ is an injective operator with dense range in $\mathcal{H}$, which satisfies quadratic estimates and has a bounded holomorphic functional calculus.

Note that the restriction of T_A to $\mathcal{H}$, which we continue to denote by T_A, coincides with the operator T_A used in [3] for $m=1$.

Of importance to us are the following operators, which are bounded operators in $\mathcal{H}$ because they belong to the functional calculus of T_A.

- The characteristic functions

$$\chi^{\pm}(z) = \begin{cases} 1, & \text{if } \pm \operatorname{Re} z > 0, \\ 0, & \text{if } \pm \operatorname{Re} z < 0 \end{cases}$$

which give the generalized *Hardy projections* $E_A^{\pm} := \chi^{\pm}(T_A)$.

- The signum function $\operatorname{sgn}(z) = \chi^+(z) - \chi^-(z)$ which gives the generalized *Cauchy operator* $E_A := \operatorname{sgn}(T_A) = E_A^+ - E_A^-$.

- The exponential functions $e^{-t|z|}$, $t > 0$, which give the operators $e^{-t|T_A|}$. Note that $|z| := z \operatorname{sgn}(z)$ does not denote absolute value for non-real z, but $z \mapsto |z|$ is holomorphic on $S_{\pi/2}^o$.

Note that the quadratic estimates in Corollary 3.6 can be written as

$$(13) \qquad \int_0^\infty \|\psi(tT_A)f\|^2 \frac{dt}{t} \approx \|f\|^2, \quad f \in \overline{\mathsf{R}(D)},$$

where $\psi(z) := z/(1+z^2)$. The estimate $\lesssim$ remains valid for any holomorphic $\psi(z)$ on S_ν^o such that $|\psi(z)| \lesssim \min(|z|^\alpha, |z|^{-\alpha})$ for some $\alpha > 0$. If furthermore $\psi|_{S_{\nu_+}^o} \neq 0$ and $\psi|_{S_{\nu_-}^o} \neq 0$, then the estimate $\approx$ holds. See [1].

Proof of Theorem 2.3. By Corollary 3.6, the infinitesimal generator has a bounded holomorphic functional calculus. With the notation introduced above, define Hardy type subspaces $E_A^{\pm}\mathcal{H} := \{E_A^{\pm}f; f \in \mathcal{H}\}$, so that

$$\mathcal{H} = E_A^+\mathcal{H} \oplus E_A^-\mathcal{H}.$$

In terms of the operator T_A, the vector fields f and $F^{\pm}$ are related as

$$f = f^+ + f^- \longleftrightarrow E_A^{\pm}f = f^{\pm} = F^{\pm}|_{\mathbf{R}^n} \longleftrightarrow e^{\mp t|T_A|}f^{\pm} = F^{\pm}.$$

Indeed, differentiating $F^{\pm} = e^{\mp t|T_A|}f^{\pm}$ at $(t,x) \in \mathbf{R}_{\pm}^{1+n}$, we have

$$\partial_t F^{\pm} = \mp |T_A|F^{\pm} = -T_A e^{\mp t|T_A|}(\pm E_A f^{\pm}) = -T_A e^{\mp t|T_A|}f^{\pm} = -T_A F^{\pm}$$

since $f^{\pm} \in E_A^{\pm}\mathcal{H}$. Thus $F^{\pm}$ satisfies (10), which we have seen at the beginning of this section is equivalent to (3). Conversely, each vector field

$$F \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}))$$

which satisfies (10) for $t > 0$ and has limits $\lim_{t \to \infty} F_t = 0$ and $\lim_{t \to 0} F_t = f$ in L_2 norm, is of the form $F(t,x) = e^{-t|T_A|}f$, where $f \in E_A^+\mathcal{H}$. To see this, split $F_t = F_t^+ + F_t^-$, where $F_t^{\pm} \in E_A^{\pm}\mathcal{H}$. Applying $E_A^{\pm}$ to (10) gives the equations $\partial_t F_t^{\pm} \pm |T_A|F_t^{\pm} = 0$. Multiplying with suitable exponentials shows that, for fixed $t > 0$, $e^{(s-t)|T_A|}F_s^+$ is constant for $s \in (0,t)$ and $e^{(t-s)|T_A|}F_s^-$ is constant for $s \in (t, \infty)$. Thus $F_t^- = 0$ and $F_t^+ = e^{-t|T_A|}f$, where $f = \lim_{t \to 0} F_t$. The corresponding result for F^- is proved similarly.

The equivalences of norms follow from the quadratic estimates for T_A as follows. Boundedness of the complementary projections $E_A^\pm$ shows that $\|f\| \approx \|f^+\| + \|f^-\|$, the uniform boundedness of $e^{-t|T_A|}$, $t>0$, shows that $\|f^\pm\| \approx \sup_{\pm t>0} \|F_t^\pm\|$, and the quadratic estimates for T_A shows that $\|t\partial_t F_{\pm t}^\pm\| \approx \|f^\pm\|$. Finally $\|f^\pm\| \approx \|\widetilde{N}_*(F^\pm)\|$ was proved in [3, Proposition 2.56] for $m=1$. The extension to divergence form systems is straightforward.

To verify Lipschitz continuity, one shows that the operators

$$\text{(14)} \qquad f \longmapsto b(T_A)f : \mathcal{H} \longrightarrow \mathcal{H},$$

$$\text{(15)} \qquad f \longmapsto (b(tT_A)f)_{t>0} : \mathcal{H} \longrightarrow L_2((a,b); \mathcal{H}),$$

$$\text{(16)} \qquad f \longmapsto (\psi(tT_A)f)_{t>0} : \mathcal{H} \longrightarrow L_2(\mathbf{R}_+, dt/t; \mathcal{H})$$

depend analytically on A, where b and ψ are bounded holomorphic functions on S_ν^o, ψ decays at 0 and ∞ as in (13), and the interval (a,b) is finite. One proceeds similar to [10, Theorem 6.4] and [3, Lemma 2.41], starting from the analyticity of resolvents $A \mapsto (\lambda I - T_A)^{-1}$, using the quadratic estimates from Corollary 3.6 for the operator T_A and the fact that uniform limits of analytic functions are analytic.

Lipschitz continuity can now be deduced from analyticity. Given A_1 and $A_2 \in L_\infty(\mathbf{R}^n; \mathbf{C}^{(1+n)m})$ which are strictly accretive on $\mathcal{H}$, define

$$A(\zeta) := A_1 + \frac{\zeta(A_2 - A_1)}{\|A_2 - A_1\|_\infty}$$

so that $\zeta \mapsto A(\zeta)$ is analytic in a neighborhood Ω of the interval $[0, \|A_2 - A_1\|_\infty]$. Consider the analytic function $\zeta \mapsto A(\zeta) \mapsto F_t^\pm = b(tT_{A(\zeta)})f$, where $b(z) := e^{-t|z|}\chi^\pm(z)$, which has bounds $\|F_t^\pm\|_2 \lesssim \|f\|_2$ in Ω. Thus $\|dF_t^\pm/d\zeta\|_2 \lesssim \|f\|_2$ in Ω, from which Lipschitz continuity of $A \mapsto F_t$ follows, uniformly for all $t>0$.

Lipschitz continuity of $A \mapsto F_t$ for the square function norm and the norm of the non-tangential maximal function are proved similarly, using analyticity of (16), with $\psi(z) = ze^{-|z|}\chi^\pm(z)$, and (15), with $b(z) = e^{-t|z|}\chi^\pm(z)$, respectively. For the non-tangential maximal function, we refer to the proof of [3, Theorem 1.1] for further details. $\square$

4. Dirichlet and Neumann boundary value problems

In this section, we return to the Dirichlet and Neumann BVPs and use Theorem 2.3 to prove Theorem 2.2. We start by translating (Dir-A) to an auxiliary Neumann problem (Neu$^\perp$-A), which consists in finding V solving (4) with boundary condition.

• Neumann problem (Neu$^\perp$-A): $-\partial_t V(0,x)=\varphi(x)$, where $\varphi \in L_2(\mathbf{R}^n;\mathbf{C}^m)$ is given.

More precisely, we use the following first order formulation of well-posedness.

Definition 4.1. We say that the boundary value problem (Neu$^\perp$-A) is *well posed* if for each boundary data $\varphi \in L_2(\mathbf{R}^n;\mathbf{C}^m)$, there exists a unique vector field

$$F(t,x)=F_t(x)=\nabla_{t,x}V(t,x) \in C^1(\mathbf{R}_+;L_2(\mathbf{R}^n;\mathbf{C}^{(1+n)m}))$$

which satisfies (3) for $t>0$, and has limits $\lim_{t\to\infty}F_t=0$ and $\lim_{t\to 0}F_t=f$ in L_2 norm, where the full boundary trace f satisfies the boundary condition $-f_0=\varphi$.

Lemma 4.2. *Given $u=-\varphi \in L_2(\mathbf{R}^n;\mathbf{C}^m)$, we have a one-to-one correspondence*

$$U(t,x)=F_0(t,x) \longleftrightarrow F(t,x)=-\int_t^\infty \nabla_{s,x}U(s,x)\,ds$$

between solutions $U(t,x)$ to (Dir-A) *and solutions $F(t,x)$ to* (Neu$^\perp$-A). *In particular* WP(Dir)=WP(Neu$^\perp$), *where*

$$\mathrm{WP}(\mathrm{Neu}^\perp):=\{A\,;(\mathrm{Neu}^\perp\text{-}A)\text{ is well posed}\} \subset L_\infty(\mathbf{R}^n;\mathbf{C}^{(1+n)m}).$$

Proof. Assume that F solves (Neu$^\perp$-A) with boundary condition $f_0=u$, and let $U:=F_0$. Then $U_t \in C^1(\mathbf{R}_+;L_2)$ and $\nabla_x U=\partial_t F_\parallel \in C^0(\mathbf{R}_+;L_2)$. The limits $\lim_{t\to 0}U_t=u$, $\lim_{t\to\infty}U_t=0$ and $\lim_{t\to\infty}\nabla_{t,x}U_t=0$ are direct consequences of the limits $\lim_{t\to 0}F_t=f$, $\lim_{t\to\infty}F_t=\lim_{t\to\infty}t\partial_t F_t=0$, whereas

$$\int_{t_0}^{t_1}\nabla_x U_s\,ds=\int_{t_0}^{t_1}\partial_s F_\parallel\,ds=F_\parallel(t_1)-F_\parallel(t_0)\to -f_\parallel, \quad \text{as }(t_0,t_1)\to(0,\infty).$$

The function U satisfies (4) since

$$\int_t^\infty(\nabla_x v,(A\nabla_{s,x}U_s)_\parallel)\,ds=\int_t^\infty \partial_s(\nabla_x v,(AF_s)_\parallel)\,ds=-(\nabla_x v,(AF_t)_\parallel)$$

$$=(v,\mathrm{div}_x(AF_t)_\parallel)=-(v,\partial_t(AF_t)_0)=-(v,(A\nabla_{t,x}U_t)_0).$$

Conversely, assume that U solves (Dir-A) with boundary condition $U|_{\mathbf{R}^n}=-\varphi$, and let $F(t,x):=-\int_t^\infty \nabla_{s,x}U(s,x)\,ds$. This gives a well-defined function since $F=U-\lim_{t_1\to\infty}\int_t^{t_1}\nabla_x U_s\,ds$, and $F \in C^1(\mathbf{R}_+;L_2)$. Clearly, we have $\lim_{t\to\infty}F_t=0$ and $\lim_{t\to 0}F_t=-\varphi-h$, where $h:=\int_0^\infty \nabla_x U_s\,ds$. The vector field F satisfies (3) as $\mathrm{curl}_x \int_t^\infty \nabla_x U(s,x)\,ds=0$, $\partial_t F_\parallel=\nabla_x U=\nabla_x F_0$ and

$$(v, \operatorname{div}_x(AF_t)_\parallel) = \left(\nabla_x v, \int_t^\infty (A\nabla_{s,x}U_s)_\parallel \, ds \right)$$

$$= \int_t^\infty (\nabla_x v, (A\nabla_{s,x}U_s)_\parallel) \, ds = -(v, (A\nabla_{t,x}U_t)_0) = -(v, \partial_t(AF_t)_0),$$

for all $v \in C_0^\infty(\mathbf{R}^n; \mathbf{C}^m)$, so that $\partial_t(AF_t)_0 = -\operatorname{div}_x(AF_t)_\parallel$. $\square$

As in the proof of Theorem 2.3, we denote the upper Hardy type subspace of $\mathcal{H}$ by

$$E_A^+ \mathcal{H} = \left\{ f \, ; F \in C^1(\mathbf{R}_+^{1+n}; \mathcal{H}) \text{ solves (10) in } \mathbf{R}_+^{1+n}, \lim_{t \to 0^+} \|F_t - f\|_2 = \lim_{t \to \infty} \|F_t\|_2 = 0 \right\}.$$

Proof of Theorem 2.2. By Theorem 2.3, the spectral projection $E_A^+ = \chi_+(T_A)$ onto this subspace is bounded and depends Lipschitz continuously on A. We now observe that (Reg-A), (Neu-A) and (Neu$^\perp$-A) are well posed if and only if

$$E_A^+ \mathcal{H} \longrightarrow \{g \in L_2(\mathbf{R}^n; \mathbf{C}^{nm}) \, ; \operatorname{curl}_x g = 0\} : f \longmapsto f_\parallel,$$

$$E_A^+ \mathcal{H} \longrightarrow L_2(\mathbf{R}^n; \mathbf{C}^m) : f \longmapsto (Af)_0,$$

$$E_A^+ \mathcal{H} \longrightarrow L_2(\mathbf{R}^n; \mathbf{C}^m) : f \longmapsto f_0,$$

are isomorphisms, respectively. Since E_A^+ depends continuously on A, by Theorem 2.3, the following lemma shows that the sets of well-posedness are open.

Lemma 4.3. *Let P_t be bounded projections in a Hilbert space $\mathcal{H}$ which depend continuously on a parameter $t \in (-\delta, \delta)$, and let $S: \mathcal{H} \to \mathcal{K}$ be a bounded operator into a Hilbert space $\mathcal{K}$. If $S: P_0\mathcal{H} \to \mathcal{K}$ is an isomorphism, then there exists $0 < \varepsilon < \delta$, such that $S: P_t\mathcal{H} \to \mathcal{K}$ is an isomorphism when $|t| < \varepsilon$.*

Proof. Consider the family of operators $P_0\mathcal{H} \ni f \mapsto SP_t f \in \mathcal{K}$ between fixed spaces. By assumption and continuous dependence, they are invertible when $|t|$ is small. Thus it suffices to prove that $P_t: P_0\mathcal{H} \to P_t\mathcal{H}$ is invertible when $|t|$ is small. This holds since $(I - P_0(P_0 - P_t))^{-1}P_0, P_0(I - P_t(P_t - P_0))^{-1}: P_t\mathcal{H} \to P_0\mathcal{H}$ are seen to be left and right inverses respectively. $\square$

What remains to be proved is that the three maps are isomorphisms when A is either Hermitean, block or constant. In fact, it suffices to prove this for (Reg-A) and (Neu-A), due to the following result proved in [3, Proposition 2.52].

Proposition 4.4. *The boundary value problem* (Neu$^\perp$-A) *is well posed if and only if* (Reg-A^*) *is well posed.*

That (Reg-A) and (Neu-A) are well posed for Hermitean, block and constant coefficients, follows from [3, Section 3] when $m=1$. For Hermitean and block form coefficients, these proofs are readily adapted to systems, but to be self contained, we give simplified proofs below.

Define the operator $N:=\left(\begin{smallmatrix} -1 & 0 \\ 0 & I \end{smallmatrix}\right)$, which reflects a vector in $\mathbf{R}^n$. Write $N^+=\frac{1}{2}(1+N)=\left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$ for the tangential projection and $N^-=\frac{1}{2}(1-N)=\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for the normal projection, so that $N=N^+-N^-$.

4.1. Hermitean matrices

Let $f\in E_A^+\mathcal{H}$. This means that there is a vector field F_t in $\mathbf{R}_+^{1+n}$ such that $\partial_t F_t=-T_A F_t$, $\lim_{t\to\infty} F_t=0$ and $\lim_{t\to 0} F_t=f$. Recall that

$$T_A = \bar{A}^{-1} D\underline{A}$$

and note that $DN+ND=0$. Furthermore, assuming that $A^*=A$, it is seen from the definition of $\hat{A}$ that the Hermitean condition translates to $(\hat{A})^*=N\hat{A}N$. The Rellich type identity which is useful here is the following.

$$
\begin{aligned}
(N\underline{A}f, \bar{A}f) &= -\int_0^\infty \partial_t(N\underline{A}F_t, \bar{A}F_t)\, dt \\
&= \int_0^\infty (N\underline{A}T_A F_t, \bar{A}F_t)+(N\underline{A}F_t, \bar{A}T_A F_t)\, dt \\
&= \int_0^\infty (N\hat{A}D\underline{A}F_t, \bar{A}F_t)+(N\underline{A}F_t, D\underline{A}F_t)\, dt \\
&= \int_0^\infty ((ND+DN)\underline{A}F_t, \underline{A}F_t)\, dt \\
&= 0.
\end{aligned}
$$

Thus $((\underline{A}f)_0, (\bar{A}f)_0)=((\underline{A}f)_\parallel, (\bar{A}f)_\parallel)$, or in view of (9),

$$(17) \qquad\qquad ((f)_0, (Af)_0) = ((Af)_\parallel, (f)_\parallel).$$

Consider first the Neumann problem. From (17) it follows that

$$\|f\|^2 \approx \mathrm{Re}(Af, f) = \mathrm{Re}((Af)_0, f_0)+((Af)_\parallel, f_\parallel)) = 2\,\mathrm{Re}((Af)_0, f_0) \lesssim \|(Af)_0\|\|f\|.$$

This shows that $\|f\|\lesssim\|(Af)_0\|$ holds for the Neumann map $E_A^+\mathcal{H}\ni f\mapsto (Af)_0$, which implies that this map is injective with closed range.

It remains for us to prove surjectivity of this map. Note that the above estimates also show that $E_{A_t}^+\mathcal{H}\ni f\mapsto (A_t f)_0$ is injective with closed range when

$A_t := (1-t)I + tA$, $0 \le t \le 1$. It follows from the proof of Lemma 4.3 and the method of continuity that all these maps have the same index. Since $I = A_0 \in \mathrm{WP}$, it follows that $A \in \mathrm{WP}$.

Well-posedness for (Reg-A) and (Neu$^\perp$-A) is proved in a similar way, by keeping the factor $f_\parallel$ and f_0 respectively from (17).

4.2. Block matrices

Note that in this case the Neumann problems (Neu-A) and (Neu$^\perp$-A) coincide, and that the operator T_A has the form

$$T_A = \begin{pmatrix} 0 & A_{00}^{-1} \operatorname{div}_x A_{\parallel\parallel} \\ -\nabla_x & 0 \end{pmatrix}.$$

Note further that in this case, the accretivity condition (5) splits into the two independent assumptions $\operatorname{Re}(A_{00}u, u) \gtrsim \|u\|_2^2$ and $\operatorname{Re}(A_{\parallel\parallel}\nabla_x v, \nabla_x v) \gtrsim \|\nabla_x v\|_2^2$ for all $u \in L_2(\mathbf{R}^n; \mathbf{C}^m)$ and $\nabla_x v \in L_2(\mathbf{R}^n; \mathbf{C}^{nm})$. Since the diagonal elements in T_A are zero, so are the diagonal elements of E_A as

$$(18) \qquad E_A = T_A (T_A^2)^{-1/2} = (T_A^2)^{-1/2} T_A = \begin{pmatrix} 0 & L^{-1/2} A_{00} \operatorname{div}_x A_{\parallel\parallel} \\ -\nabla_x L^{-1/2} & 0 \end{pmatrix},$$

where $L := -A_{00} \operatorname{div}_x A_{\parallel\parallel} \nabla_x$. Another way to see this is from the calculation

$$N E_A = N \operatorname{sgn}(T_A) N^{-1} N = \operatorname{sgn}(N T_A N^{-1}) N = \operatorname{sgn}(-T_A) N = -E_A N.$$

To prove well-posedness, we need to prove that $N^+ : E_A^+ \mathcal{H} \to N^+ \mathcal{H}$ and $N^- : E_A^+ \mathcal{H} \to N^- \mathcal{H}$ are isomorphisms. From $E_A N + N E_A = 0$, we obtain explicit inverses as

$$2E_A^+ : N^+ \mathcal{H} \longrightarrow E_A^+ \mathcal{H} \quad \text{and} \quad 2E_A^+ : N^- \mathcal{H} \longrightarrow E_A^+ \mathcal{H}.$$

For example, to see that $N^+(2E_A^+ g) = g$ when $g \in N^+ \mathcal{H}$, we calculate

$$2N^+ E_A^+ g = N^+(I + E_A)g = g + N^+ E_A g$$
$$= g + \tfrac{1}{2}(1+N)E_A g = g + \tfrac{1}{2}(E_A g - E_A N g) = g.$$

In fact well-posedness of the Neumann and regularity problems for block coefficients is equivalent to the Kato square root estimate

$$\|\sqrt{L}u\|_2 \approx \|\nabla_x u\|_2,$$

as was first observed by Kenig [25, Remark 2.5.6]. To see this, we deduce from (18) that $f \in E_A^+ \mathcal{H}$, i.e. $f = E_A f$, if and only if $f_\parallel = -\nabla_x L^{-1/2} f_0$, or inversely

$$f_0 = L^{-1/2} A_{00} \operatorname{div}_x A_{\parallel\parallel} f_\parallel$$

and $\operatorname{curl}_x f_\parallel = 0$. This can be used to construct $f = f_0 + E_A f_0 = E_A f_\parallel + f_\parallel$ from either f_0 or $f_\parallel$. Note that the Kato estimate translates to $\|f_0\| \approx \|f_\parallel\|$.

4.3. Constant matrices

If A is a constant matrix, we can make use of the Fourier transform.

(i) First consider the simpler case when $m=1$. In this case, the solutions to the BVPs can be explicitly computed on the Fourier transform side, since the problem reduces to an eigenvector calculation. At the frequency point $\xi \neq 0$, the space $\mathcal{H}$ corresponds to the two-dimensional space $\mathcal{H}_\xi := \{ze_0 + w\xi; z, w \in \mathbf{C}\}$ of vectors with tangential part parallel to ξ. The operator D corresponds to $D_\xi := \left(\begin{smallmatrix} 0 & i\xi^t \\ -i\xi & 0 \end{smallmatrix} \right)$. Compressing the constant matrix $\hat{A}$ to $\mathcal{H}_\xi$, we define the 2×2 strictly accretive matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \hat{A}_{00} & \hat{A}_{0\parallel}\xi \\ \xi^t \hat{A}_{\parallel 0} & \xi^t \hat{A}_{\parallel\parallel}\xi \end{pmatrix}$$

so that

$$D_\xi \hat{A} \sim i \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}.$$

Computing eigenvalues and vectors shows that $ze_0 + w\xi \in \chi_\pm(D_\xi\hat{A})$ if and only if

$$(19) \qquad az + bw = \left(\tfrac{1}{2}(c-b) \pm i\sqrt{ad - \tfrac{1}{4}(b+c)^2} \right) w.$$

Applying the similarities in (11), we characterize well-posedness as follows. That (Neu-A), (Reg-A) and (Neu$^\perp$-A) are well posed means that $ze_0 + w\xi \in \chi_\pm(D_\xi\hat{A})$ is determined by z, w and $az + bw$, respectively. This is straightforward to verify using (19).

(ii) Next consider the case $m > 1$. In this case, we perform a Rellich type argument on the Fourier symbol, or rather we make a "reverse Rellich estimate".

The space $\mathcal{H}_\xi$ is now isomorphic to $\mathbf{C}^{2m}$ since $z, w \in \mathbf{C}^m$. In view of Proposition 4.4, it suffices to prove a-priori estimates $\|f\| \lesssim \|f_\parallel\|$ and $\|f\| \lesssim \|(Af)_0\|$ for $f \in \chi_+(T_\xi)\mathcal{H}_\xi^k$ uniformly for almost all $\xi \in \mathbf{R}^n$, where $T_\xi := \bar{A}^{-1}D_\xi\underline{A}$. However, since $\chi_+(tT_\xi) = \chi_+(T_\xi)$ for $t > 0$, it suffices to consider the unit sphere $|\xi| = 1$. By continuity and compactness, we need only verify that no non-zero vector f such that $f_\parallel = 0$ or $(Af)_0 = 0$ can be in the Hardy space, i.e. be of the form $f = F(0)$, where $F: \mathbf{R}_+ \to \mathcal{H}_\xi^k$ satisfies $\partial_t F = -T_\xi F$ and $\lim_{t \to \infty} F = 0$. To prove this, we use the fact that $D_\xi^2 = I$ to obtain

$$(D_\xi\bar{A}f, \bar{A}f) = -\int_0^\infty \partial_t(D_\xi\bar{A}F(t), \bar{A}F(t))\, dt$$

$$= 2\int_0^\infty \mathrm{Re}(\underline{A}F(t), \bar{A}F(t))\, dt = 2\int_0^\infty \mathrm{Re}(AF(t), F(t))\, dt.$$

We now observe that the left-hand side vanishes if $f_\parallel = 0$ or $(Af)_0 = 0$, and from the right-hand side we then see that $F = 0$ identically, and therefore $f = 0$. The method of continuity, perturbing A to I now shows that the maps $f \mapsto f_\parallel$ and $f \mapsto (Af)_0$ are surjective, and thus isomorphisms.

We have now completed the proof of Theorem 2.2. $\square$

Proof of Corollary 2.4. As in Lemma 4.2 a function U solving (4), with properties as in Definition 2.1, is the normal part of a vector field $F = U + F_\parallel$ solving (3), with properties as in Definition 2.1. Theorem 2.3 shows that $F = e^{-t|T_A|}f$, where $f = F|_{\mathbf{R}^n} \in E_A^+ \mathcal{H}$, and that estimates

$$\|f\|_2 \approx \sup_{t>0} \|F_t\|_2 \approx \|\widetilde{N}_*(F)\|_2 \approx \|t\partial_t F_t\|$$

hold. If $A \in \mathrm{WP}(\mathrm{Dir}) = \mathrm{WP}(\mathrm{Neu}^\perp)$, then $\|f\|_2 \approx \|u\|_2$ and $\|F_t\|_2 \approx \|U_t\|_2$ for all $t > 0$, since $F_t \in E_A^+ \mathcal{H}$. For the square function norm we observe that $\partial_t F = \nabla_{t,x} U$, and for the non-tangential maximal function clearly $\|\widetilde{N}_*(F)\|_2 \gtrsim \|\widetilde{N}_*(U)\|_2$ holds. As $\|u\|_2 \approx \sup_{t>0} \|U_t\|_2$ for solutions to (4) has been shown, we have $\|U_t\|_2 \lesssim \|U_s\|_2$ when $t > s$. The reverse estimate $\|\widetilde{N}_*(F)\|_2 \lesssim \|\widetilde{N}_*(U)\|_2$ now follows from

$$\|\widetilde{N}_*(U)\|^2 \gtrsim \sup_{t>0} \int_{\mathbf{R}^n} \fint_{|y-x|<c_1 t} \fint_{|s-t|<c_0 t} |U(s,y)|^2 \, ds \, dy \, dx$$

$$= \sup_{t>0} \fint_{|s-t|<c_0 t} \|U_s\|^2 ds \gtrsim \sup_{t>0} \|U_{(1+c_0)t}\|^2 \approx \|u\|^2.$$

This proves the corollary. $\square$

5. Uniqueness of solutions

In this section we compare the solutions to the BVPs (Neu-A), (Dir-A) and (Reg-A) in the sense of Definition 2.1, with the standard solutions obtained from the Lax–Milgram theorem. This uses the homogeneous Sobolev space $\dot{H}^1(\mathbf{R}_+^{1+n}; \mathbf{C}^m)$, equipped with the norm $\|U\|_{\dot{H}^1}^2 := \int_{\mathbf{R}_+^{1+n}} |\nabla_{t,x} U|^2$, and the subspace of functions with vanishing trace. Continuing our first order approach to BVPs via (3), we make the following definition.

Definition 5.1. Introduce spaces of vector fields

$$L_2^\nabla(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)m}) := \{F \in L_2(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)m}) \, ; \mathrm{curl}_{\mathbf{R}_+^{1+n}} F = 0\} \quad \text{and}$$

$$L_2^{\nabla_0}(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)m}) := \{F \in L_2(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)m}) \, ; \mathrm{curl}_{\mathbf{R}^{1+n}}(F_z) = 0\}.$$

The condition $\mathrm{curl}_{\mathbf{R}^{1+n}}(F_z)=0$ here means that the extension by zero F_z, of F to $\mathbf{R}^{1+n}$, is curl free, or formally: $\mathrm{curl}_{\mathbf{R}^{1+n}_+} F=0$ and the boundary trace of F is normal to $\mathbf{R}^n$. If $F\in L_2^\nabla(\mathbf{R}^{1+n}_+;\mathbf{C}^{(1+n)m})$, it is seen that there exists $U\in L_2^{\mathrm{loc}}(\mathbf{R}^{1+n}_+;\mathbf{C}^m)$, unique up to constants among the distributions on $\mathbf{R}^{1+n}_+$, such that $\nabla_{t,x}U=F$. Define Hilbert spaces

$$\dot{H}^1(\mathbf{R}^{1+n}_+;\mathbf{C}^m) := \{U\in L_2^{\mathrm{loc}}(\mathbf{R}^{1+n}_+;\mathbf{C}^m)\,;\nabla_{t,x}U\in L_2^\nabla(\mathbf{R}^{1+n}_+;\mathbf{C}^{(1+n)m})\},$$

$$\dot{H}^1_0(\mathbf{R}^{1+n}_+;\mathbf{C}^m) := \{U\in L_2^{\mathrm{loc}}(\mathbf{R}^{1+n}_+;\mathbf{C}^m)\,;\nabla_{t,x}U\in L_2^{\nabla_0}(\mathbf{R}^{1+n}_+;\mathbf{C}^{(1+n)m})\},$$

with norms so that the correspondence $U\leftrightarrow F=\nabla_{t,x}U$ is an isometry.

It is straightforward to verify that a function $U\in\dot{H}^1(\mathbf{R}^{1+n}_+;\mathbf{C}^m)$ belongs to the subspace $\dot{H}^1_0(\mathbf{R}^{1+n}_+;\mathbf{C}^m)$ if and only if there exists a constant C such that U extended by C to $\mathbf{R}^{1+n}$ belongs to $\dot{H}^1(\mathbf{R}^{1+n};\mathbf{C}^m)$.

Functions $U\in\dot{H}^1(\mathbf{R}^{1+n}_+;\mathbf{C}^m)$ are well defined only up to constants, whereas for $U\in\dot{H}^1_0(\mathbf{R}^{1+n}_+;\mathbf{C}^m)$, we will choose the constant so that $U|_{\mathbf{R}^n}=0$. It is not true that $\dot{H}^1_0(\mathbf{R}^{1+n}_+;\mathbf{C}^m)\subset L_2(\mathbf{R}^{1+n}_+;\mathbf{C}^m)$, as a scaling argument readily shows. However, Poincaré's inequality shows that

$$\iint_{\mathbf{R}^{1+n}_+} |U(t,x)|^2\frac{dt\,dx}{1+t^2+|x|^2} \lesssim \iint_{\mathbf{R}^{1+n}_+} |\nabla_{t,x}U(t,x)|^2\,dt\,dx,\quad U\in\dot{H}^1_0(\mathbf{R}^{1+n}_+;\mathbf{C}^m).$$

If $F=\nabla_{t,x}U$ solves (3), then we formally have

$$J_A(U,V):=\iint_{\mathbf{R}^{1+n}_+}(A\nabla_{t,x}U,\nabla_{t,x}V)\,dt\,dx = -\int_{\mathbf{R}^n}((Af)_0,v)\,dx,$$

where $f=F|_{\mathbf{R}^n}$ and $v=V|_{\mathbf{R}^n}$. As pointed out in Section 2, the standing assumption that A is strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$, i.e. (5), implies that the Gårding inequality (8) in $\mathbf{R}^{n+1}_+$ holds, i.e. $|J_A(U,U)|\gtrsim\|U\|^2_{\dot{H}^1}$.

Note that $V\mapsto\int_{\mathbf{R}^n}(\phi(x),v(x))\,dx$ in (20) below defines a bounded functional on $\dot{H}^1(\mathbf{R}^{1+n}_+;\mathbf{C}^m)$ if $\phi=\mathrm{div}_x\,w$, where $\phi\in L_2(\mathbf{R}^n;\mathbf{C}^m)$ and $w\in L_2(\mathbf{R}^n;\mathbf{C}^{nm})$, since $\int_{\mathbf{R}^n}|\hat{\phi}(\xi)|^2\max(|\xi|^{-1},1)\,d\xi<\infty$ and the trace map $V\mapsto v$ maps $\dot{H}^1(\mathbf{R}^{1+n}_+;\mathbf{C}^m)\to\dot{H}^{1/2}(\mathbf{R}^n;\mathbf{C}^m)$. Furthermore $V\mapsto J_A(P_tu,V)$ defines a bounded functional on $\dot{H}^1_0(\mathbf{R}^{1+n}_+;\mathbf{C}^m)$ if $u\in H^1(\mathbf{R}^n;\mathbf{C}^m)$ since $\iint_{\mathbf{R}^{1+n}_+}|\xi e^{-t|\xi|}\hat{u}(\xi)|^2\,dt\,d\xi\approx\|u\|^2_{H^1(\mathbf{R}^n)}<\infty$, where P_t denotes the Poisson extension

$$P_tu(x):=\frac{\Gamma((1+n)/2)}{\pi^{(1+n)/2}}\int_{\mathbf{R}^n}\frac{tu(y)\,dy}{(t^2+|x-y|^2)^{(1+n)/2}}.$$

The Lax–Milgram theorem proves the existence and uniqueness of the following $\dot{H}^1$ solutions U.

Definition 5.2. We say that ϕ is *good boundary data* for (Neu-A) if

$$L_2(\mathbf{R}^n; \mathbf{C}^m) \ni \phi = \operatorname{div}_x w,$$

where $w \in L_2(\mathbf{R}^n; \mathbf{C}^{nm})$. If ϕ is good, we define *the $\dot{H}^1$ solution* to the Neumann problem to be the unique function $U \in \dot{H}^1(\mathbf{R}^{1+n}_+; \mathbf{C}^m)$ such that

$$(20) \qquad J_A(U, V) = \int_{\mathbf{R}^n} (\phi(x), v(x)) \, dx \quad \text{for all } V \in \dot{H}^1(\mathbf{R}^{1+n}_+; \mathbf{C}^m).$$

We say that u is *good boundary data* for (Dir-A), or equivalently that $\nabla_x u$ is *good boundary data* for (Reg-A), if $u \in H^1(\mathbf{R}^n; \mathbf{C}^m)$. If u is good, we define *the $\dot{H}^1$ solution* to the Dirichlet (regularity) problem to be the unique function $U \in \dot{H}^1(\mathbf{R}^{1+n}_+; \mathbf{C}^m)$ such that

$$(21) \qquad J_A(U, V) = 0 \quad \text{for all } V \in \dot{H}^1_0(\mathbf{R}^{1+n}_+; \mathbf{C}^m),$$

and $U(t, x) - P_t u(x) \in \dot{H}^1_0(\mathbf{R}^{1+n}_+; \mathbf{C}^m)$.

The goal in this section is to prove the following uniqueness result.

Theorem 5.3. *Assume that A belongs to the connected component of* WP(Neu)/WP(Reg)/WP(Dir) *that contains I. If the boundary data is good, then the solutions to* (Neu-A)/(Reg-A)/(Dir-A) *in the sense of Definition* 2.1, *coincide with the $\dot{H}^1$ solutions.*

Remark 5.4. For general A in the set of well-posedness, the solutions constructed in this paper using the boundary equation method do not necessarily coincide with the $\dot{H}^1$ solutions. Examples of this were shown in [8]. Note that these examples combined with Theorem 5.3 proves the existence of many coefficients that do not have well posed BVPs (even when $n = m = 1$ with real A), sufficiently many to disconnect these A with non-$\dot{H}^1$ solutions from the identity.

The proof of Theorem 5.3 uses the following lemma with $A_0 = I$.

Lemma 5.5. *Let A_0 be a block matrix. Then there exists $\varepsilon > 0$, such that if $\|A - A_0\|_\infty < \varepsilon$ and the boundary data is good, then the solutions to the BVPs in the sense of Definition* 2.1, *coincide with the $\dot{H}^1$ solutions.*

Proof for (Neu-A). Let $F = \nabla_{t,x} U = e^{-t|T_A|} f$ be the boundary equation solution to (Neu-A) in $\mathbf{R}^{1+n}_+$ with data $\phi = \operatorname{div}_x w = -(Af)_0$. Using the isomorphism $\bar{A}$ from (11), we define the similar Hardy function $\tilde{f} := \bar{A}f \in \chi_+(D\hat{A})\mathcal{H}$.

Let N^- and N^+ be the normal and tangential projections from Section 4. The boundary condition on $\tilde{f}$ can be written $N^-\tilde{f}=-\phi$. We solve for $\tilde{f}$ by making the ansatz $\tilde{f}=2\chi_+(D\hat{A})h$, where $h\in N^-\mathcal{H}$ (i.e. $h_{\|}=0$). This yields the equation

$$-\phi = N^-\tilde{f} = 2N^-\chi_+(D\hat{A})h = (I+N^-\operatorname{sgn}(D\hat{A}))h$$

for h, in the normal subspace $N^-\mathcal{H}$. We note the following properties of "the double layer type operator" $K_A:=N^-\operatorname{sgn}(D\hat{A})N^-$. (See [7] for explanations of this terminology.) When $A=A_0$, then as in Section 4.2 it follows that $K_{A_0}=0$, since the diagonal entries of $\operatorname{sgn}(D\hat{A})$ are zero in the normal/tangential splitting of the space. Theorem 2.3 shows that K_A depends continuously on A. Moreover $K_A(\mathsf{R}(D))\subset\mathsf{R}(D)$ since

$$K_A Dg = N^-\operatorname{sgn}(D\hat{A})DN^+g = N^-D\operatorname{sgn}(\hat{A}D)N^+g = D(N^+\operatorname{sgn}(\hat{A}D)N^+g)$$

when $g\in\mathsf{D}(D)$. Therefore, when $\|A-A_0\|_\infty$ is small, we can expand $(I+K_A)^{-1}$ in a Neumann series and deduce that $h\in\mathsf{R}(D)$ since $-\phi=-Dw\in\mathsf{R}(D)$. Indeed

$$\sum_{k=0}^{N}(-K_A)^k(-Dw) = D\left(-\sum_{k=0}^{N}(-N^+\operatorname{sgn}(\hat{A}D)N^+)^k w\right) =: Dw_N,$$

where w_N and $Dw_N\to h$ converges in L_2. Since D is closed, $h\in\mathsf{R}(D)$. This shows that $\tilde{f}=2\chi_+(D\hat{A})h\in\mathsf{R}(D\hat{A})$, and thus $f\in\mathsf{R}(T_A)$. In particular $f=|T_A|^{1/2}f_0$ for some $f_0\in\mathcal{H}$. Quadratic estimates for the operator T_A now shows that

$$\iint_{\mathbf{R}_+^{1+n}}|\nabla_{t,x}U|^2\,dt\,dx = \int_0^\infty \|F_t\|^2\,dt$$

$$= \int_0^\infty \|(t|T_A|)^{1/2}e^{-t|T_A|}f_0\|^2\frac{dt}{t} \approx \|f_0\|^2 < \infty.$$

Thus $U\in\dot{H}^1(\mathbf{R}_+^{1+n};\mathbf{C}^m)$. To verify (20), let $V\in\dot{H}^1(\mathbf{R}_+^{1+n};\mathbf{C}^m)$ and consider the function

$$g(t) := \int_{\mathbf{R}^n}((AF_t)_0,V_t)\,dx, \quad t>0,$$

where we view $t\mapsto(AF_t)_0$ as a C^∞ curve in $\mathsf{R}(\operatorname{div}_x;L_2)$ and $t\mapsto V_t$ as a continuous curve in $\dot{H}^{1/2}(\mathbf{R}^n;\mathbf{C}^m)$. If $V_t\in C^1(\mathbf{R}_+;\dot{H}^{1/2})$, then

$$g'(t) = \int_{\mathbf{R}^n}((-\operatorname{div}_x AF_t,V_t)+(AF_t,\partial_t V_t))\,dx = \int_{\mathbf{R}^n}(AF_t,\nabla_{t,x}V)\,dx.$$

Therefore $g(T)-g(\varepsilon)=\iint_{\varepsilon<t<T}(AF_t,\nabla_{t,x}V)\,dt\,dx$. This also holds for general $V\in\dot{H}^1(\mathbf{R}_+^{1+n};\mathbf{C}^m)$, which can be shown by mollifying $t\mapsto V_t$. Taking limits $(\varepsilon,T)\to(0,\infty)$ proves (20). $\quad\square$

Proof for (Reg-A). Similar to the proof for the Neumann problem, we consider the equation

$$\nabla_x u = N^+ \tilde{f} = 2N^+ \chi_+(D\hat{A})h = (I + N^+ \operatorname{sgn}(D\hat{A}))h$$

for $h \in N^+ \mathcal{H}$, in the tangential subspace. We deduce that the trace f of the solution is in the range of T_A, and therefore $U \in \dot{H}^1(\mathbf{R}^{1+n}_+; \mathbf{C}^m)$ and (21) follows as in the proof for the Neumann problem.

To prove that $U - P_t u \in \dot{H}^1_0(\mathbf{R}^{1+n}_+; \mathbf{C}^m)$, it suffices to show that $\partial_0 H_j = \partial_j H_0$ on $\mathbf{R}^{1+n}$, for $j = 1, ..., n$, when H is $\nabla_{t,x}(U - P_t u)$ extended by zero. To this end, let $\Phi \in C_0^\infty(\mathbf{R}^{1+n})$ and consider the function

$$g_j(t) := \int_{\mathbf{R}^n} H_j(t, x) \Phi(t, x) \, dx, \quad t > 0.$$

Since H is curl-free on $\mathbf{R}^{1+n}_+$, we have $\partial_0 g_j = \int_{\mathbf{R}^n} H_0(-\partial_j \Phi) \, dx + H_j \partial_0 \Phi$. As we have $L_2(\mathbf{R}^n)$ convergence

$$H_j = \partial_j U_t - \partial_j P_t u \to \partial_j u - \partial_j u = 0, \quad \text{as } t \to 0,$$

integration of $\partial_0 g_j$ over $t \in (0, \infty)$ shows that

$$\iint_{\mathbf{R}^{1+n}_+} H_j \partial_0 \Phi \, dt \, dx = \iint_{\mathbf{R}^{1+n}_+} H_0 \partial_j \Phi \, dt \, dx$$

for all $\Phi \in C_0^\infty(\mathbf{R}^{1+n})$, i.e. $\partial_0 H_j = \partial_j H_0$.   □

Proof for (Dir-A). Let $F = \nabla_{t,x} U$ be the boundary equation solution to (Neu$^\perp$-A) with data $\varphi = -u \in H^1(\mathbf{R}^n; \mathbf{C}^m)$, and recall that $U = F_0$ is the solution to (Dir-A). Consider the boundary trace $f = F|_{\mathbf{R}^n} \in E_A^+ \mathcal{H}$. We now instead use the isomorphism $\underline{A}$ from (11), and define the similar Hardy function $\tilde{f} := \underline{A}f \in \mathsf{R}(\chi_+(\hat{A}D))$. Note that $\underline{A}$ is an isomorphism from $\overline{\mathsf{R}(T_A)} = \mathcal{H}$ onto $\overline{\mathsf{R}(\hat{A}D)} = \underline{A}\mathcal{H}$ since $\bar{A}$ is an isomorphism on $\mathcal{H}$ and $\hat{A}$ is strictly accretive on $\mathcal{H}$.

The boundary condition on $\tilde{f}$ can be written $N^- \tilde{f} = N^- f = u$. We solve for $\tilde{f}$ using the ansatz $\tilde{f} = 2\chi_+(\hat{A}D)h$, where $h \in N^- \mathcal{H}$. This yields the equation

$$u = N^- \tilde{f} = 2N^- \chi_+(\hat{A}D)h = (I + N^- \operatorname{sgn}(\hat{A}D))h$$

for h. We note that the double layer type operator $K_A := N^- \operatorname{sgn}(\hat{A}D)N^-$ maps $K_A(\mathsf{D}(D)) \subset \mathsf{D}(D)$ since

$$DK_A g = N^+ D \operatorname{sgn}(\hat{A}D)N^- g = N^+ \operatorname{sgn}(D\hat{A})DN^- g = N^+ \operatorname{sgn}(D\hat{A})N^+(Dg)$$

when $g \in \mathsf{D}(D)$. As above, since $\|K_A\|$ is small when $\|A - A_0\|_\infty$ is small, we can expand $(I + K_A)^{-1}$ in a Neumann series and deduce that $h \in \mathsf{D}(D)$ since $u \in \mathsf{D}(D)$. This

shows that $\tilde{f}=2\chi_+(\hat{A}D)h\in\mathsf{D}(\hat{A}D)$, and thus $f\in\mathsf{D}(T_A)$. In particular $f\in\mathsf{D}(|T_A|^{1/2})$. Quadratic estimates for the operator T_A now shows that

$$\iint_{\mathbf{R}^{1+n}_+}|\nabla_{t,x}U|^2\,dt\,dx=\int_0^\infty\|\partial_t F_t\|^2 dt=\int_0^\infty\|T_A F_t\|^2\,dt$$

$$=\int_0^\infty\|(t|T_A|)^{1/2}e^{-t|T_A|}(|T_A|^{1/2}f)\|^2\frac{dt}{t}\approx\left\||T_A|^{1/2}f\right\|^2<\infty.$$

Thus $U\in\dot{H}^1(\mathbf{R}^{1+n}_+;\mathbf{C}^m)$, and (20) follows as in the proof for the Neumann problem. Finally, note that $\nabla_{t,x}U=-T_A F_t=-e^{-t|T_A|}(T_A f)$. This shows L_2 convergence

$$\nabla_x U_t\to-(T_A f)_\parallel=\nabla_x f_0=\nabla_x u.$$

As in the proof for the regularity problem, $U-P_t u\in\dot{H}^1_0(\mathbf{R}^{1+n}_+;\mathbf{C}^m)$ follows. $\quad\square$

Remark 5.6. (i) Note that for any L_2 boundary data, the solutions to the Neumann and regularity problems always satisfy $\int_0^1\|\nabla_{t,x}U\|^2\,dt<\infty$, whereas the solution to the Dirichlet problem always satisfies $\int_1^\infty\|\nabla_{t,x}U\|^2\,dt<\infty$. Thus the problem whether good boundary data give $\dot{H}^1$ solutions concerns large t for the Neumann and regularity problem, and small t for the Dirichlet problem.

(ii) The structure of the problem in Lemma 5.5 is best explained abstractly as follows. Let $\mathcal{H}_1\hookrightarrow\mathcal{H}_0$ be a continuous and dense inclusion of Hilbert spaces. Assume that $T_0\colon\mathcal{H}_0\to\mathcal{H}_0$ is an isomorphism which restricts to a bounded operator $T_1\colon\mathcal{H}_1\to\mathcal{H}_1$. It follows from [29, Theorem 11.1] that we have regularity $\mathcal{H}_1=T_0^{-1}(\mathcal{H}_1)$ if and only if T_1 is a Fredholm operator and has index zero. See also [7, Proposition 3.2.16].

In our situation, $T_0=I+K$, $\mathcal{H}_0=N^\pm\mathcal{H}$ and $\mathcal{H}_1$ is either $\mathsf{R}(D)\cap N^\pm\mathcal{H}$ or $\mathsf{D}(D)\cap N^-\mathcal{H}$. In principle, the technique of Lemma 5.5 could be used to prove regularity for more general A in the component of WP containing I. The problem though is that in general the well-posedness of two different BVPs, for the matrix A, is needed both for the proof that T_0 is an isomorphism and that T_1 is Fredholm. Index zero for T_1 could then be proved by the method of continuity, perturbing A to I.

Proof of Theorem 5.3. Fix good boundary data and let $0<a<b<\infty$. For all A with the assumed properties, let $F=F_A=\nabla_{t,x}U$ be the solutions given by Theorem 2.2, and let $F^0=F_A^0=\nabla_{t,x}U$ denote the standard $\dot{H}^1$ solutions constructed with the Lax–Milgram theorem as in Definition 2.1.

From the quadratic estimates for T_A it follows with arguments as in [3, proof of Theorem 1.1] that $A\mapsto F_A\in L_2(\mathbf{R}^n\times(a,b);\mathbf{C}^{(1+n)m})$ is analytic on WP. The main result this uses is the analyticity of $A\mapsto b(T_A)$ for operators $b(T_A)$ in the functional

calculus of T_A. This was proved in [10, Theorem 6.4]. Moreover, it is straight-forward to verify that $A \mapsto F_A^0 \in L_2(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)m})$ is analytic. This means that whenever $A_0 \in \mathrm{WP}$, $\mathbf{C} \supset D \ni z \mapsto A(z)$ are coefficients depending analytically on a complex variable z, and $A(0) = A_0$ and $H \in L_2(\mathbf{R}^n \times (a,b); \mathbf{C}^{(1+n)m})$, then the scalar function $z \mapsto h(A(z))$, where

$$h(A) := \iint_{(a,b) \times \mathbf{R}^n} (F_A - F_A^0, H) \, dt \, dx,$$

is analytic on D.

Consider one of the BVPs and fix A in the connected component of WP contain-ing I. Pick a sequence of balls $B_k = B(A_k; r_k) \subset \mathrm{WP}$, $k = 0, 1, ..., N$, such that $A_0 = I$, $A_N = A$ and $B_{k-1} \cap B_k \neq \varnothing$. We may take $r_0 < \varepsilon$, so that $h = 0$ on B_0 by Lemma 5.5. Now assume that $h = 0$ on B_{k-1} and pick any $A^1 \in B_k$. Let $A^0 \in B_{k-1} \cap B_k$ and let $A(z) := (1-z)A^0 + zA^1$. Then $h(A(z))$ vanishes on a neighborhood of 0. By analytic continuation $h(A^1) = h(A(1)) = 0$, and since $A^1 \in B_k$ was arbitrary, $h = 0$ on B_k. We arrive at the conclusion that $h(A) = 0$. Since a, b and H are arbitrary, it follows that $F_A = F_A^0$. □

6. Boundary value problems for differential forms

In this section, we demonstrate how Theorems 2.2 and 2.3 generalize to exte-rior/interior differential systems for k-vector fields, i.e. differential forms of order k.

We use the notation from [3, Section 2.1]. In particular, for fixed $k \in \{1, 2, ..., n\}$, we consider functions

$$F(t, x) = \sum F_{\{s_1, ..., s_k\}}(t, x) \, e_{s_1} \wedge ... \wedge e_{s_k},$$

taking values in the space $\bigwedge^k = \bigwedge^k \mathbf{R}^{1+n}$ of complex k-vectors on $\mathbf{R}^{1+n}$. The vector fields in (3) is the special case $k=1$. We point out that we assume the component functions F_s to be scalar-valued here ($m=1$), although the methods apply, mutatis mutandis, to systems of exterior differential systems. A natural generalization of the first order system (3) is the interior/exterior differential system

(22)
$$\begin{cases} d_{t,x}^* BF(t,x) = 0, \\ d_{t,x} F(t,x) = 0, \end{cases}$$

where $F \colon \mathbf{R}_+^{1+n} \to \bigwedge^k$. Here the exterior and interior derivative operators are

$$d_{t,x}F = \nabla_{t,x} \wedge F = \sum_{j=0}^{n} e_j \wedge \partial_j F = \mu \partial_t F + d_x F,$$

$$d_{t,x}^* F = -\nabla_{t,x} \lrcorner F = -\sum_{j=0}^{n} e_j \lrcorner \partial_j F = -\mu^* \partial_t F + d_x^* F,$$

where $\wedge$ denotes exterior product, $\lrcorner$ denotes (left) interior product, $\mu f = e_0 \wedge f$ and $\mu^* f = e_0 \lrcorner f$. The matrix function $B \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$ is assumed to be t-independent and *pointwise strictly accretive* in the sense that

$$\mathrm{Re}(B(x)w, w) \geq \varkappa |w|^2 \quad \text{for all } w \in \bigwedge^k \text{ and a.e. } x \in \mathbf{R}^n.$$

To prove an analogue of Theorem 2.3 for (22), we proceed as in Section 3 and introduce auxiliary matrices

$$\overline{B} := \begin{pmatrix} B_{\perp\perp} & B_{\perp\parallel} \\ 0 & I \end{pmatrix} \text{ and } \underline{B} := \begin{pmatrix} I & 0 \\ B_{\parallel\perp} & B_{\parallel\parallel} \end{pmatrix}, \quad \text{if } B = \begin{pmatrix} B_{\perp\perp} & B_{\perp\parallel} \\ B_{\parallel\perp} & B_{\parallel\parallel} \end{pmatrix}$$

in the normal/tangential splitting of $\bigwedge^k$. Recall that a basis k-vector $e_{s_1} \wedge ... \wedge e_{s_k}$ is normal if one of the factors is e_0, and tangential otherwise. Denote the tangential and normal parts of f by $f_\parallel$ and $f_\perp$. Splitting each of the equations (22) into normal and tangential parts and using the analogue of (9), shows that (22) is equivalent to

$$\begin{cases} \partial_t(\overline{B}F)_\perp - \mu d_x^*(\underline{B}F)_\parallel = 0, \\ \partial_t(\overline{B}F)_\parallel + \mu^* d_x(\underline{B}F)_\perp = 0, \end{cases}$$

together with the constraints $d_x F_\parallel = 0 = d_x^*(BF)_\perp$. These tangential derivatives in the equations define the appropriate function space

$$\mathcal{H}_B^k := \{ f \in L_2(\mathbf{R}^n; \bigwedge^k) \, ; \, d_x f_\parallel = 0 = d_x^*(Bf)_\perp \}$$

generalizing $\mathcal{H}$ from Definition 3.5. Note that when $k \geq 2$, the space $\mathcal{H}_B^k$ depends on B, unlike the case $k=1$.

The normal derivatives in the equation give an equation $\partial_t F + T_B F = 0$, where the infinitesimal generator is

(23) $$T_B := \overline{B}^{-1} D \underline{B}.$$

Here $D := \mu^* d_x - \mu d_x^*$ is a self-adjoint differential operator. The operator T_B has similarities

$$\overline{B}^{-1}(D\widehat{B})\overline{B} = T_B = \underline{B}^{-1}(\widehat{B}D)\underline{B},$$

where $\widehat{B} := \underline{B}\overline{B}^{-1}$ is shown to be pointwise strictly accretive as in Proposition 3.2. Thus Proposition 3.3 applies and proves that $T_B = \overline{B}^{-1}(D\widehat{B})\overline{B}$ is an ω-bisectorial

operator, ω being the angle of accretivity of $\widehat{B}$. Moreover, T_B restricts to an injective ω-bisectorial operator in $\mathcal{H}_B^k = \overline{B}^{-1}\overline{\mathsf{R}(D)}$ with dense range, and in the splitting

$$L_2(\mathbf{R}^n; \textstyle\bigwedge^k) = \overline{\mathsf{R}(T_B)} \oplus \mathsf{N}(T_B) = \mathcal{H}_B^k \oplus \{(f_\perp, f_\parallel)^t \,; d_x f_\perp = 0 = d_x^*(Bf)_\parallel\},$$

we have $T_B = T_B|_{\mathcal{H}_B^k} \oplus 0$.

Similar to the proof of Theorem 2.3, Theorem 3.4 proves the boundedness of the Cauchy operator

$$E_B := \operatorname{sgn}(T_B)$$

and the Hardy projections $E_B^\pm := \chi_\pm(T_B)$. To handle perturbation theory for the variable space $\mathcal{H}_B^k$, we extend this operator to $L_2(\mathbf{R}^n; \bigwedge^k)$ by defining $E_B^\pm f = E_B f = 0$ when $f \in \mathsf{N}(T_B)$.

We obtain the following result on Hardy space splittings of $\mathcal{H}_B^k \subset L_2(\mathbf{R}^n; \bigwedge^k)$.

Theorem 6.1. *Let $B \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$ be a t-independent, complex coefficient matrix function which is pointwise strictly accretive.*

Then each $f \in \mathcal{H}_B^k$ is in one-to-one correspondence with a pair of k-vector fields $F_t^\pm = F^\pm(t, \cdot) \in C^1(\mathbf{R}_\pm; L_2(\mathbf{R}^n; \bigwedge^k))$ in $\mathbf{R}_\pm^{1+n}$ satisfying (22) and having L_2 limits $\lim_{t\to 0^\pm} F_t^\pm = f^\pm$ and $\lim_{t\to\pm\infty} F_t^\pm = 0$, such that

$$f = f^+ + f^-.$$

Under this correspondence, we have equivalences of norms $\|f\|_2 \approx \|f^+\|_2 + \|f^-\|_2$ and

$$(24) \qquad\qquad \|f^\pm\|_2 \approx \sup_{\pm t > 0} \|F_t^\pm\|_2 \approx \|t\partial_t F_{\pm t}^\pm\|.$$

Moreover, the Hardy space projections $L_2(\mathbf{R}^n; \bigwedge^k) \ni f \mapsto F^\pm = F_B^\pm := e^{\mp t|T_B|} E_B^\pm f$ depend locally Lipschitz continuously on B in the sense that

$$\|F_{B_2}^\pm - F_{B_1}^\pm\|_{\mathcal{X}} \le C\|B_2 - B_1\|_{L_\infty(\mathbf{R}^n)}\|f\|_2,$$

where $C = C(\varkappa_{B_1}, \varkappa_{B_2}, \|B_1\|_\infty, \|B_2\|_\infty)$ and where $\|\cdot\|_{\mathcal{X}}$ denotes any of the three norms in (24).

We remark that the proof of the non-tangential maximal estimate $\|\widetilde{N}_*(F)\|_2 \approx \|f\|_2$ from [3, Proposition 2.56] in Theorem 2.3 uses the divergence form structure of the second order system. This technique does not generalize to more general exterior differential systems.

Finally we extend the results in Section 4 and show how Theorem 6.1 gives perturbation results for BVPs for k-vector fields. The natural BVPs are the following. We are looking for a k-vector field $F_t \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \bigwedge^k))$ solving (22) in $\mathbf{R}_+^{1+n}$ with L_2 limits $\lim_{t\to 0^+} F_t = f$ and $\lim_{t\to\infty} F_t = 0$, where the boundary trace f satisfies one of the following conditions:

• Tangential boundary condition (Tan-B): $f_\parallel = g$, where the given boundary data $g \in L_2(\mathbf{R}^n; \bigwedge^k)$ is tangential and satisfies $d_x g = 0$.

• Conormal boundary condition (Nor-B): $(Bf)_\perp = g$, where the given boundary data $g \in L_2(\mathbf{R}^n; \bigwedge^k)$ is normal and satisfies $d_x^* g = 0$.

Note that when $k = 1$, (Tan-B) coincides with the Dirichlet regularity problem (Reg-B) and (Nor-B) coincides with the Neumann problem (Neu-B).

Theorem 6.2. *The sets of well-posedness* WP(Tan) *and* WP(Nor) *are both open subsets of* $L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$ *and each contains*

(i) *all Hermitean matrices* $B(x) = B(x)^*$ (*and in particular all real symmetric matrices*),

(ii) *all block matrices*

$$B(x) = \begin{pmatrix} B_{\perp\perp}(x) & 0 \\ 0 & B_{\parallel\parallel}(x) \end{pmatrix}, \quad and$$

(iii) *all constant matrices* $B(x) = B$.

What is new here as compared with [3], is the perturbation result around Hermitean and constant matrices, as well as the openness of the sets of well-posedness. The proof of Theorem 6.2 is similar to the proof Theorem 2.2. We observe that (Tan-B) and (Nor-B) are well posed if and only if

$$E_B^+ \mathcal{H}_B^k \longrightarrow \{g \in L_2(\mathbf{R}^n; \textstyle\bigwedge^k); \mu^* g = 0 \text{ and } d_x g = 0\} : f \longmapsto f_\parallel,$$

$$E_B^+ \mathcal{H}_B^k \longrightarrow \{g \in L_2(\mathbf{R}^n; \textstyle\bigwedge^k); \mu g = 0 \text{ and } d_x^* g = 0\} : f \longmapsto (Bf)_\perp,$$

are isomorphisms, respectively. Theorem 6.1 shows that E_B depends continuously on B, so we obtain from Lemma 4.3 that WP(Tan) and WP(Nor) are open sets.

That Hermitean and block matrices belong to WP is shown as in Sections 4.1 and 4.2, mutatis mutandis. For constant matrices, the reverse Rellich argument used for second order divergence form elliptic systems $m > 1$ applies. For exterior differential systems, the symbol of the operator is

$$T_\xi := \overline{B}^{-1} D_\xi \underline{B} \quad \text{and} \quad D_\xi := i(\mu^* \mu_\xi + \mu \mu_\xi^*),$$

acting in the $2\binom{n-1}{k-1}$-dimensional space $\mathcal{H}_\xi^k := \{f \in \bigwedge^k; \mu\mu_\xi f = 0 = \mu^* \mu_\xi^* Bf\}$. The proof uses that if $\mu\mu_\xi f = 0 = \mu^* \mu_\xi^* f$, i.e. $f \in \mathsf{R}(D_\xi)$, then

$$(25) \qquad D_\xi^2 f = -(\mu^* \mu_\xi \mu \mu_\xi^* + \mu \mu_\xi^* \mu^* \mu_\xi) f = (\mu_\xi \mu^* \mu \mu_\xi^* + \mu_\xi^* \mu \mu^* \mu_\xi) f$$

$$= (\mu_\xi (I - \mu\mu^*) \mu_\xi^* + \mu_\xi^* (I - \mu^* \mu) \mu_\xi) f = (\mu_\xi \mu_\xi^* + \mu_\xi^* \mu_\xi) f = |\xi|^2 f.$$

For the anticommutation relations, we refer to [3, Lemma 2.3].

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