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SOLVABILITY OF GROUPS OF ORDER $2^a p^b$

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1. Introduction

At the beginning of this century Burnside proved his famous $p^a q^b$-theorem by the help of character theory. Group-theoretic proof of the theorem was given by Goldschmidt [2] for odd primes in 1970.

The object of this paper is to give a simple group-theoretic proof of the following

Theorem.[1] *Groups of order $2^a p^b$ are solvable.*

Lemma 1, 4 and 5 are due to Goldschmidt [2]. Notation used here follows Gorenstein [3].

2. Preliminaries

Lemma 1. *Suppose $\mathfrak{P}$ is a p-subgroup of the p-solvable group $\mathfrak{G}$. Then* $0_{p'}(N_{\mathfrak{G}}(\mathfrak{P}))\subseteqq 0_{p'}(\mathfrak{G})$

Proof. See Goldschmidt [2], lemma 2.
Next lemma plays an important role in this paper.

Lemma 2. *Suppose $\mathfrak{G}$ is a p-group and $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$. Then $\mathfrak{H}\trianglelefteq\mathfrak{G}$ or $N_{\mathfrak{G}}(\mathfrak{H})\supseteqq\mathfrak{H}^X(\neq\mathfrak{H})$ for some $X\in\mathfrak{G}$.*

Proof. Let Δ be a $\mathfrak{G}$-conjugate class containing $\mathfrak{H}$. If $|\Delta|\neq 1$, then $\mathfrak{H}$ acts on $\Delta-\{\mathfrak{H}\}$ by conjugation. Since $p\nmid|\Delta-\{\mathfrak{H}\}|$, $\mathfrak{H}$ fixes some element $\mathfrak{H}^{X^{-1}}$. Then $\mathfrak{H}\subseteqq N_{\mathfrak{G}}(\mathfrak{H}^{X^{-1}})$ and hence $\mathfrak{H}^X\subseteqq N_{\mathfrak{G}}(\mathfrak{H})$.

Lemma 3. (Suzuki-Thompson) *Suppose Δ is a conjugate class of a group $\mathfrak{G}$. If any two elements of Δ generate a p-group, then $\Delta\subseteqq 0_p(\mathfrak{G})$.*

Proof. See [3], 3.8.2.

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1) After finishing this work the author has found that Bender [1] has also obtained a group-theoretic proof of the theorem in the general case.

3. The Minimal counter example

In this section let $\mathfrak{G}$ be a minimal counter example to the theorem. It is immediate to show that $\mathfrak{G}$ is simple and any proper subgroup of $\mathfrak{G}$ is solvable.

Let r be either prime divisor of $|\mathfrak{G}|$.

Lemma 4. *A sylow r-subgroup of $\mathfrak{G}$ normalizes no non-identity r'-subgroup of $\mathfrak{G}$.*

Proof. See Goldschmidt [2], Lemma 3.

Lemma 5. (Bender) *Suppose $\mathfrak{M}$ is a maximal subgroup of $\mathfrak{G}$. Then the Fitting subgroup of $\mathfrak{M}$ is an r-group.*

Proof. We set $\mathfrak{F}=F(\mathfrak{M})$, the Fitting subgroup of $\mathfrak{M}$. Let $\mathfrak{F}=\mathfrak{F}_2\times\mathfrak{F}_p$ be the primary decomposition, and $\mathfrak{Z}=Z(\mathfrak{F})=\mathfrak{Z}_2\times\mathfrak{Z}_p$, the center of $\mathfrak{F}$.

Suppose lemma 5 is false, then $\mathfrak{F}_2\neq 1$, $\mathfrak{F}_p\neq 1$. We first prove the next assertion $[A]$.

[A] $\mathfrak{F}_r$ has two distinct subgroups of order r, for some $r\in\{2,p\}$.

Suppose $[A]$ is false, then $\mathfrak{F}_p$ is cyclic, and $\mathfrak{F}_2$ is cyclic or a quaternion group.

(i) In the case $\mathfrak{F}_2$ is cyclic.

Let $\mathfrak{P}$ be a Sylow p-subgroup of $\mathfrak{M}$. Since $\mathfrak{P}/C_{\mathfrak{P}}(\mathfrak{F}_2)$ is a 2-group, $\mathfrak{P}=C_{\mathfrak{P}}(\mathfrak{F}_2)$. Then $Z(\mathfrak{P})\subseteqq C_{\mathfrak{M}}(\mathfrak{F})$, and hence $Z(\mathfrak{P})\subseteqq\mathfrak{F}_p$ by Fitting's theorem. (See [3], 6.1.3.) Since $\mathfrak{F}_p$ is cyclic, $Z(\mathfrak{P})$ is a characteristic subgroup of $\mathfrak{F}_p$. Then $\mathfrak{M}=N_{\mathfrak{G}}(Z(\mathfrak{P}))$ and $\mathfrak{P}$ is a Sylow p-subgroup of $\mathfrak{G}$, contrary to lemma 4.

(ii) In the case $\mathfrak{F}_2$ is a quaternion group.

Let $\mathfrak{Q}$ be a Sylow 2-group of $\mathfrak{M}$. Since $\mathfrak{Q}/C_{\mathfrak{Q}}(\mathfrak{F}_p)$ is abelian, $\mathfrak{Q}'\subseteqq C_{\mathfrak{Q}}(\mathfrak{F}_p)$. Then $Z(\mathfrak{Q})\cap\mathfrak{Q}'\subseteqq\mathfrak{F}_2$. $Z(\mathfrak{Q})\cap\mathfrak{Q}'$ contains a unique subgroup $\mathfrak{H}$ of order 2. So $\mathfrak{H}$ is a chatacteristic subgroup of $\mathfrak{Q}$. Since $\mathfrak{M}=N_{\mathfrak{G}}(\mathfrak{H})\supseteqq N_{\mathfrak{G}}(\mathfrak{Q})$, it follows that $\mathfrak{Q}$ is a Sylow 2-subgroup of $\mathfrak{G}$. A contradiction.

By (i) and (ii), we have [A].

Next we prove the following statement $[B]$.

[B] Let $\overline{\mathfrak{M}}$ be a maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Z}$. Then $\overline{\mathfrak{M}}=\mathfrak{M}$

Let $\overline{\mathfrak{F}}=F(\overline{\mathfrak{M}})=\overline{\mathfrak{F}}_2\times\overline{\mathfrak{F}}_p$ be the Fitting subgroup of $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{Z}}=\overline{\mathfrak{Z}}_2\times\overline{\mathfrak{Z}}_p$ be the centre of $\overline{\mathfrak{F}}$. Since $\mathfrak{Z}_2\times\mathfrak{Z}_p$ is contained in $\overline{\mathfrak{M}}$, $O_p(N_{\overline{\mathfrak{M}}}(\mathfrak{Z}_2))\subseteqq\overline{\mathfrak{F}}_p=O_p(\overline{\mathfrak{M}})$ by lemma 1. Now $\mathfrak{Z}_p$ is a normal subgroup of $N_{\overline{\mathfrak{M}}}(\mathfrak{Z}_2)$ we have $\mathfrak{Z}_p\subseteqq O_p(N_{\overline{\mathfrak{M}}}(\mathfrak{Z}_2))$. Then $[\mathfrak{Z}_p,\overline{\mathfrak{F}}_2]=1$. So $\overline{\mathfrak{F}}_2\subseteqq N_{\mathfrak{G}}(\mathfrak{Z}_p)=\mathfrak{M}$. In the same way, we have $\overline{\mathfrak{F}}_p\subseteqq\mathfrak{M}$. Then in the same way as above we have $\overline{\mathfrak{F}}_2\subseteqq O_2(N_{\mathfrak{M}}(\overline{\mathfrak{F}}_p))\subseteqq\mathfrak{F}_2$. Interchanging $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ in the above argument, we obtain $\mathfrak{F}_2\subseteqq\overline{\mathfrak{F}}_2$. Then $\mathfrak{F}_2=\overline{\mathfrak{F}}_2$ and we have $\overline{\mathfrak{M}}=\mathfrak{M}$. Thus [B] holds.

Now we prove lemma 5. By [A] we may assume that $\mathfrak{F}_r$ contains an abelian subgroup $\mathfrak{A}$ of type (r,r). Let $\mathfrak{R}$ be a Sylow r-subgroup of $\mathfrak{M}$. If $\mathfrak{K}$ is an r'-subgroup of $\mathfrak{G}$ normailized by $\mathfrak{R}$, then $\mathfrak{K}=\prod_{X\in A-\{1\}} C_{\mathfrak{K}}(X)$. (See [3], 5.3.16.) Since

$C_{\mathfrak{R}}(X) \subseteq C_{\mathfrak{G}}(X)$ and $C_{\mathfrak{G}}(X) \supseteq \mathfrak{Z}$, $C_{\mathfrak{R}}(X) \subseteq \mathfrak{M}$ by [B]. It follows $\mathfrak{R} \subseteq \mathfrak{M}$. Then $\mathfrak{F}_{r'}$ is the unique maximal r'-subgroup of $\mathfrak{G}$ normalized by $\mathfrak{R}$. So $N_{\mathfrak{G}}(\mathfrak{R}) \subseteq N_{\mathfrak{G}}(\mathfrak{F}_{r'}) = \mathfrak{M}$. Then $\mathfrak{R}$ is a Sylow r-subgroup of $\mathfrak{G}$. A contradiction.

q.e.d.

Lemma 6. *$\mathfrak{G}$ contains a maximal subgroup $\mathfrak{M}$ which satisfies the following condition;*

$$\mathfrak{M} \cap Z(\mathfrak{P}) \neq 1, \quad \mathfrak{M} \cap Z(\mathfrak{Q}) \neq 1$$

for some Sylow p-subgroup $\mathfrak{P}$ and Sylow 2-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$.

Proof. Let $\mathfrak{Q}$ be a Sylow 2-subgroup of $\mathfrak{G}$ and X be an invloution contained in $Z(\mathfrak{Q})$. Suppose Δ is a conjugate class of $\mathfrak{G}$ containing X. By lemma 3, Δ contains two elements X_1, X_2 such that $\langle X_1, X_2\rangle$ is not a 2-group. Since $\langle X_1, X_2\rangle$ is a dihedıal group, $|X_1 \cdot X_2|$ is not a power of 2. Then $\langle X_1 \cdot X_2\rangle$ contains a unique subgroup $\mathfrak{H}$ of order P. Let $\mathfrak{M}$ be a maximal subgroup containing $N_{\mathfrak{G}}(\mathfrak{H})$. It is immediate to show that $\mathfrak{M}$ satisfies the condition of the lemma. q.e.d.

Proof of the theorem. Let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{G}$ which satisfies the condition of lemma 6. By lemma 5 $F(\mathfrak{M})$ is an r-group. Let G be an element of $\mathfrak{M}$ contained in the centre of some Sylow r'-subgroup $\overline{\mathfrak{R}}$ of $\mathfrak{G}$, and let $\mathfrak{R}$ be a Sylow r-subgroup of $\mathfrak{G}$ containing $\mathfrak{F}_r = F(\mathfrak{M})$. Since $\mathfrak{M} = N_{\mathfrak{G}}(\mathfrak{F}_r)$, it follows $Z(\mathfrak{R}) \subseteq \mathfrak{F}_r$ by Fitting. Then $\mathfrak{N}_0 = \langle Z(\mathfrak{R})^X \colon X \in \langle G\rangle\rangle \subseteq \mathfrak{F}_r$ and hence it is an r-group normalized by G. Let Ω be a complete $\mathfrak{G}$-conjugate class containing $Z(\mathfrak{R})$ and $\Omega = \Omega_1 + \cdots + \Omega_s$ be a disjoint sum of $\langle G\rangle$-orbits. Let $\mathfrak{N}_i$ be a group generated by Ω_i. For some element $Y \in \mathfrak{R}$, $Z(\mathfrak{R})^Y \in \Omega_i$, then $\Omega_i = \langle Z(\mathfrak{R})^{YX} \colon X \in \langle G\rangle\rangle = \langle Z(\mathfrak{R})^{XY}; X \in \langle G\rangle\rangle$. It follows that $\mathfrak{N}_i = \mathfrak{N}_0^Y$. Then $\mathfrak{N}_i$ is an r-group normalized by $G^Y = G$ for $i = 1, \cdots, S$. So there exist $\Omega_{i_1}, \cdots, \Omega_{i_l}$ such that the group generated by $\Omega_{i_1} \cup \cdots \cup \Omega_{i_l}$ is an r-group normalized by G. $(l \geqq 1)$ Let l be maximal. We may assume $\{i_1 \cdots i_l\} = \{1, \cdots, l\}$ and $\mathfrak{N} = \langle \Omega_1 \cup \cdots \cup \Omega_l\rangle$. It is trivial to show that $N_{\mathfrak{G}}(\mathfrak{N}) \ni G$. Let $\mathfrak{R}_0$ be a Sylow r-subgroup of $\mathfrak{G}$ containing $\mathfrak{N}$. By lemma 2, $\mathfrak{N} \trianglelefteq \mathfrak{R}_0$ or $N_{\mathfrak{G}}(\mathfrak{N}) \supseteq \mathfrak{N}^X (\neq \mathfrak{N})$ for some $X \in \mathfrak{R}_0$. If $\mathfrak{N} \trianglelefteq \mathfrak{R}_0$, then $N_{\mathfrak{G}}(\mathfrak{N})$ contains a complete conjugate class of $\mathfrak{G}$ containing G. A contradiction. If $N_{\mathfrak{G}}(\mathfrak{N}) \supseteq \mathfrak{N}^X (\neq \mathfrak{N})$, then since $\Omega_1^X \cup \cdots \cup \Omega_l^X \not\subseteq \mathfrak{N}$, there exists some element Y of $\mathfrak{R}$ such that $Z(\mathfrak{R})^Y \subseteq \mathfrak{N}^X$ and $Z(\mathfrak{R})^Y \not\subseteq \mathfrak{N}$. Suppose $Z(\mathfrak{R})^Y$ is an element of Ω_i. $(i > l)$, then $\mathfrak{N}_i \subseteq N_{\mathfrak{G}}(\mathfrak{N})$ from $N_{\mathfrak{G}}(\mathfrak{N}) \ni G$ and $N_{\mathfrak{G}}(\mathfrak{N}) \supseteq Z(\mathfrak{R})^Y$. Now $\mathfrak{N} \cdot \mathfrak{N}_i$ is an r-group normalized by G and generated by $\Omega_1 \cup \cdots \cup \Omega_l \cup \Omega_i$, contrary to our choice of $\mathfrak{N}$. Thus we proved the theorem. q.e.d.

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