

Solvability of Nonstationary Problems for Nonhomogeneous Incompressible Fluids and the Convergence with Vanishing Viscosity

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1. Introduction.

Let Ω be a bounded or unbounded domain in \mathbf{R}^3 with a smooth boundary S . We consider the system of equations

$$(1.1) \quad \begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ \rho[v_t + (v \cdot \nabla)v] + \nabla p = \mu \Delta v + \rho f, \\ \operatorname{div} v = 0 \end{cases}$$

in $Q_T = \Omega \times [0, T]$, $T > 0$, where $f(x, t)$ is a given vector field of external forces, while the density $\rho(x, t)$, the velocity vector $v(x, t)$ and the pressure $p(x, t)$ are the unknowns. The viscosity coefficient μ is assumed to be a nonnegative constant.

This paper consists of two parts. In the first part, Part 1, we solve (1.1) under the following initial-boundary conditions:

If $\mu > 0$,

$$(1.2) \quad \begin{cases} v|_{S_T} = 0, \\ \rho|_{t=0} = \rho_0(x), \\ v|_{t=0} = v_0(x), \end{cases}$$

and if $\mu = 0$,

$$(1.3) \quad \begin{cases} v \cdot n|_{S_T} = 0, \\ \rho|_{t=0} = \rho_0(x), \\ v|_{t=0} = v_0(x), \end{cases}$$

where n is the unit outward normal to S , and $S_T = S \times [0, T]$.

In the second part, Part 2, when $\Omega = \mathbf{R}^3$, we consider the Cauchy problem (1.1) and

$$(1.4) \quad \begin{cases} \rho|_{t=0} = \rho_0(x), \\ v|_{t=0} = v_0(x), \end{cases}$$

and establish the uniform convergence of the solution of (1.1) and (1.4) with $\mu > 0$ to the one with $\mu = 0$ as $\mu \rightarrow 0$.

We will use the classical notations and results of the Sobolev spaces. For $k=0, 1, 2, \dots$ and $1 \leq p \leq \infty$,

$$W_p^k(\Omega) = \left\{ u \in L_p(\Omega); \sum_{|\alpha| \leq k} \|D_x^\alpha u\|_{L_p(\Omega)} < \infty \right\},$$

$$W_p^{2,1}(Q_T) = \left\{ u \in L_p(Q_T); \|u\|_{W_p^{2,1}(Q_T)} = \|u_t\|_{L_p(Q_T)} + \sum_{|\alpha| \leq 2} \|D_x^\alpha u\|_{L_p(Q_T)} < \infty \right\},$$

where $D_x^\alpha = (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} (\partial/\partial x_3)^{\alpha_3}$.

If $u(x, t) \in W_p^{2,1}(Q_T)$ and $p > 3$, then for any fixed $t \in [0, T]$, the value of $u(x, t)$ belongs to the Slobodetskii-Besov space $W_p^{2-2/p}(\Omega)$ in which the norm is given by

$$\|u\|_{W_p^{2-2/p}(\Omega)} = \left(\sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{L_p(\Omega)}^p + \sum_{|\alpha|=1} \int_{\Omega} \int_{\Omega} \frac{|D_x^\alpha u(x) - D_x^\alpha u(y)|^p}{|x-y|^{1+p}} dx dy \right)^{1/p}.$$

Moreover, we have the inequality

$$\|u(\cdot, t)\|_{W_p^{2-2/p}(\Omega)} \leq \|u(\cdot, 0)\|_{W_p^{2-2/p}(\Omega)} + \hat{c} \|u\|_{W_p^{2,1}(Q_t)},$$

where the constant \hat{c} does not depend on t (cf. Ladyzhenskaya-Solonnikov-Ural'ceva [7]).

Our theorems related to the unique solvability are the following.

THEOREM 1.1. *Let $p > 3$ and $\mu > 0$. Assume that*

$$(1.5) \quad \rho_0(x) \in C^0(\bar{\Omega}), \quad \nabla \rho_0(x) \in W_p^1(\Omega), \quad 0 < m \leq \rho_0(x) \leq M < \infty,$$

$$(1.6) \quad v_0(x) \in W_p^{2-2/p}(\Omega), \quad v_0|_S = 0, \quad \operatorname{div} v_0 = 0,$$

$$(1.7) \quad f(x, t) \in L_p(Q_T).$$

Then there exists $T_1 \in (0, T]$ such that problem (1.1), (1.2) has a unique solution $(\rho, v, p)(x, t)$ which satisfies

$$(1.8) \quad \begin{aligned} \rho(x, t) &\in C^0(\bar{Q}_{T_1}), \quad \nabla \rho(x, t) \in C^0([0, T_1]; W_p^1(\Omega)), \\ 0 < m &\leq \rho(x, t) \leq M < \infty, \end{aligned}$$

$$(1.9) \quad v(x, t) \in W_p^{2,1}(Q_{T_1}),$$

$$(1.10) \quad \nabla p(x, t) \in L_p(Q_{T_1}).$$

THEOREM 1.2. *Let $p > 3$ and $\mu = 0$. Assume that*

$$(1.5) \quad \rho_0(x) \in C^0(\bar{\Omega}), \quad \nabla \rho_0(x) \in W_p^1(\Omega), \quad 0 < m \leq \rho_0(x) \leq M < \infty,$$

$$(1.11) \quad v_0(x) \in W_p^2(\Omega), \quad v_0 \cdot n|_S = 0, \quad \operatorname{div} v_0 = 0,$$

$$(1.12) \quad f(x, t) \in C^0([0, T]; W_p^2(\Omega)).$$

Then there exists $T_2 \in (0, T]$ such that problem (1.1), (1.3) has a unique solution $(\rho, v, p)(x, t)$ which satisfies

$$(1.13) \quad \rho(x, t) \in C^0(\bar{Q}_{T_2}), \quad \nabla \rho(x, t) \in C^0([0, T_2]; W_p^1(\Omega)), \\ 0 < m \leq \rho(x, t) \leq M < \infty,$$

$$(1.14) \quad v(x, t) \in C^0([0, T_2]; W_p^2(\Omega)),$$

$$(1.15) \quad \nabla p(x, t) \in C^0([0, T_2]; W_p^2(\Omega)).$$

REMARK. In the case that Ω is bounded, Theorems 1.1 and 1.2 were proved by Ladyzhenskaya-Solonnikov [6] and Valli-Zajaczkowski [13], respectively. See also [2-4, 8, 9]. However, in the case that Ω is unbounded, it seems to the authors that the rigorous proofs for these theorems have not been given yet.

The next theorem is concerned with the vanishing viscosity. Analogous result was obtained in [5] in the Sobolev spaces of Hilbert type.

THEOREM 1.3. Let $p \geq 4$ and $0 \leq \mu \leq 1$, and assume that

$$(1.16) \quad \rho_0(x) \in C^0(\mathbf{R}^3), \quad \nabla \rho_0(x) \in W_p^1(\mathbf{R}^3), \quad 0 < m \leq \rho_0(x) \leq M < \infty,$$

$$(1.17) \quad v_0(x) \in W_p^2(\mathbf{R}^3), \quad \operatorname{div} v_0 = 0,$$

$$(1.18) \quad f(x, t) \in C^0([0, T]; W_p^2(\mathbf{R}^3)).$$

Then there exists $T_0 \in (0, T]$ independent of μ such that problem (1.1), (1.4) has a unique solution $(\rho, v, p)(x, t)$ which satisfies

$$(1.19) \quad \rho(x, t) \in C^0(\mathbf{R}^3 \times [0, T_0]), \quad \nabla \rho(x, t) \in C^0([0, T_0]; W_p^1(\mathbf{R}^3)), \\ 0 < m \leq \rho(x, t) \leq M < \infty,$$

$$(1.20) \quad v(x, t) \in C^0([0, T_0]; W_p^2(\mathbf{R}^3)),$$

$$(1.21) \quad \nabla p(x, t) \in C^0([0, T_0]; W_p^1(\mathbf{R}^3)).$$

Furthermore, let (ρ^0, v^0, p^0) be the solution of problem (1.1), (1.4) with $\mu = 0$ and (ρ^μ, v^μ, p^μ) the one with $\mu > 0$, then we have

$$(1.22) \quad \sup_{0 \leq t \leq T_0} [\|(\rho^0 - \rho^\mu)(t)\|_{W_p^1(\mathbf{R}^3)} + \|(v^0 - v^\mu)(t)\|_{W_p^1(\mathbf{R}^3)} \\ + \|\nabla(p^0 - p^\mu)(t)\|_{W_p^1(\mathbf{R}^3)}] \rightarrow 0 \quad \text{as } \mu \rightarrow 0.$$

Part 1 is divided into two sections: In sections 2 and 3, we shall prove Theorems 1.1 and 1.2, respectively. Finally, Theorem 1.3 will be established in Part 2, section 4.

Part I. Existence Theorems

2. The case $\mu > 0$.

In this section, we prove Theorem 1.1 by dividing into three subsections.

2.1. Auxiliary problems. By $C^{\alpha,\beta}(\bar{Q}_T)$ ($0 < \alpha < 1$, $0 < \beta < 1$), we mean the space of functions which are defined in \bar{Q}_T and Hölder continuous with exponent α with respect to x and with exponent β with respect to t . The norm is

$$\|u\|_{C^{\alpha,\beta}(\bar{Q}_T)} = \sup_{\bar{Q}_T} |u(x, t)| + [u]_{C^{\alpha,\beta}(\bar{Q}_T)},$$

where

$$[u]_{C^{\alpha,\beta}(\bar{Q}_T)} = \sup_{(x,t),(y,s) \in \bar{Q}_T, x \neq y, t \neq s} \frac{|u(x, t) - u(y, s)|}{|x - y|^\alpha + |t - s|^\beta}.$$

LEMMA 2.1. *Let $\rho(x, t) \in C^{\alpha,\beta}(\bar{Q}_T)$, $\alpha, \beta \in (0, 1)$ such that $0 < m \leq \rho(x, t) \leq M < \infty$. Then for any $g(x, t) \in L_p(Q_T)$ and $v_0(x) \in W_p^{2-2/p}(\Omega)$ with $v_0|_S = 0$ and $\operatorname{div} v_0 = 0$, problem*

$$(2.1) \quad \begin{cases} \rho v_t - \mu \Delta v + \nabla p = g, \\ \operatorname{div} v = 0, \\ v|_{S_T} = 0, \\ v|_{t=0} = v_0(x) \end{cases}$$

has a unique solution $v(x, t) \in W_p^{2,1}(Q_T)$ and $\nabla p(x, t) \in L_p(Q_T)$, satisfying

$$(2.2) \quad \|v\|_{W_p^{2,1}(Q_T)} + \|\nabla p\|_{L_p(Q_T)} \leq K_1(\|\rho\|_{C^{\alpha,\beta}(\bar{Q}_T)}, T)(\|v_0\|_{W_p^{2-2/p}(\Omega)} + \|g\|_{L_p(Q_T)}),$$

where K_1 is an increasing function of $\|\rho\|_{C^{\alpha,\beta}(\bar{Q}_T)}$ and T , depending on m and M .

PROOF. Let us seek the solution of (2.1) in the form $v(x, t) = u(x, t) + w(x, t)$ and $p(x, t) = r(x, t) + s(x, t)$, where (u, r) and (w, s) satisfy the following systems, respectively:

$$(2.3) \quad \begin{cases} u_t - \mu \Delta u + \nabla r = g, \\ \operatorname{div} u = 0, \\ u|_{S_T} = 0, \\ u|_{t=0} = v_0(x), \end{cases}$$

and

$$(2.4) \quad \begin{cases} \rho w_t - \mu \Delta w + \nabla s = (1 - \rho)u_t \equiv g', \\ \operatorname{div} w = 0, \\ w|_{S_T} = 0, \\ w|_{t=0} = 0. \end{cases}$$

Problem (2.3) was solved by Solonnikov [11] so that it has a unique solution $u(x, t) \in W_p^{2,1}(Q_T)$ and $\nabla r(x, t) \in L_p(Q_T)$, satisfying

$$(2.5) \quad \|u\|_{W_p^{2,1}(Q_T)} + \|\nabla r\|_{L_p(Q_T)} \leq c_1(1 + e^{c_1 T})(\|v_0\|_{W_p^{2-2/p}(\Omega)} + \|g\|_{L_p(Q_T)}),$$

where c_1 is a constant independent of T . Therefore it is sufficient to establish the unique solvability of (2.4). It is proved by the method of regularizer. To this end, we introduce two systems of covering $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ such that

1. $\omega^{(k)} \subset \Omega^{(k)} \subset \bar{\Omega}$ and $\bigcup_k \omega^{(k)} = \bigcup_k \Omega^{(k)} = \bar{\Omega}$,
2. for any x , there exists $\omega^{(k)}$ such that $x \in \omega^{(k)}$ and $\text{dist}(x, \bar{\Omega} - \omega^{(k)}) \geq \delta_1 > 0$,
3. for any $\lambda > 0$, there exists a number N_0 independent of λ such that $\bigcap_{k=1}^{N_0+1} \Omega^{(k)} = \emptyset$,
4. (a) if $\Omega^{(k)} \cap S = \emptyset$ (we denote the set of indices k by \mathcal{M}), then $\omega^{(k)}$ and $\Omega^{(k)}$ are the cubes with the same center and with the length of their edges equal to $\lambda/2$ and λ , respectively,
- (b) if $\omega^{(k)} \cap S \neq \emptyset$ (we denote the set of indices k by \mathcal{N}), then for a local rectangular coordinate system $\{y\}$ with the center $\xi^{(k)} \in S$,

$$\omega^{(k)} = \left\{ |y_i| \leq \frac{1}{2} \delta_2 \lambda \ (i=1, 2), \ 0 \leq y_3 - F(y'; \xi^{(k)}) \leq \delta_2 \lambda \right\},$$

$$\Omega^{(k)} = \{ |y_i| \leq \delta_2 \lambda \ (i=1, 2), \ 0 \leq y_3 - F(y'; \xi^{(k)}) \leq 2\delta_2 \lambda \},$$

where $F(y'; \xi^{(k)})$ ($y' = (y_1, y_2)$) is a function describing the boundary S in the neighborhood of $\xi^{(k)}$ and δ_2 is a positive constant independent of λ .

By changing the variables in such a way that $z_i = y_i$ ($i=1, 2$) and $z_3 = y_3 - F(y')$, $\Omega^{(k)}$ ($k \in \mathcal{N}$) and the boundary in $\Omega^{(k)}$ are, respectively, transformed into a standard cube

$$K = \{ |z_i| \leq \delta_2 \lambda \ (i=1, 2), \ 0 \leq z_3 \leq 2\delta_2 \lambda \},$$

$$K' = \{ |z_i| \leq \delta_2 \lambda \ (i=1, 2), \ z_3 = 0 \}.$$

Furthermore, it is well known that there exist the smooth functions $\{\zeta^{(k)}(x)\}$ and $\{\eta^{(k)}(x)\}$ such that

$$\left\{ \begin{array}{l} \zeta^{(k)}(x) = \begin{cases} 1 & \text{if } x \in \omega^{(k)}, \\ 0 & \text{if } x \in \bar{\Omega} - \Omega^{(k)}, \end{cases} \quad 0 \leq \zeta^{(k)}(x) \leq 1, \\ \eta^{(k)}(x) = 0 \quad \text{if } x \in \bar{\Omega} - \Omega^{(k)}, \quad \sum_k \zeta^{(k)}(x) \eta^{(k)}(x) = 1, \\ |D_x^\alpha \zeta^{(k)}(x)| \leq c_\alpha \lambda^{-|\alpha|}, \quad |D_x^\alpha \eta^{(k)}(x)| \leq c_\alpha \lambda^{-|\alpha|}. \end{array} \right.$$

Now, let us construct regularizer. For $k \in \mathcal{M}$, let $(\bar{w}^{(k)}, \bar{s}^{(k)})(x, t)$ be the solution of problem

$$(2.6) \quad \begin{cases} \rho(\xi^{(k)}, 0)\bar{w}_t^{(k)} - \mu\Delta\bar{w}^{(k)} + \nabla\bar{s}^{(k)} = \zeta^{(k)}(x)g', \\ \operatorname{div}\bar{w}^{(k)} = 0, \\ \bar{w}^{(k)}|_{t=0} = 0. \end{cases}$$

Further, for $k \in \mathcal{N}$, let $(\bar{w}^{(k)}, \bar{s}^{(k)})(z, t)$ be the solution of problem

$$(2.7) \quad \begin{cases} \rho(\xi^{(k)}, 0)\bar{w}_t^{(k)} - \mu\Delta\bar{w}^{(k)} + \nabla\bar{s}^{(k)} = \Pi_z^x \zeta^{(k)}(x)g', \\ \operatorname{div}\bar{w}^{(k)} = 0, \\ \bar{w}^{(k)}|_{z_n=0} = 0, \\ \bar{w}^{(k)}|_{t=0} = 0, \end{cases}$$

where Π_z^x is the transformation from x to z .

These problems were also solved in [11] to have a unique solution satisfying

$$(2.8) \quad \|\bar{w}_t^{(k)}\|_{L_p(\mathbf{R}_T^3)} + \sum_{|\alpha|=2} \|D_x^\alpha \bar{w}^{(k)}\|_{L_p(\mathbf{R}_T^3)} + \|\nabla\bar{s}^{(k)}\|_{L_p(\mathbf{R}_T^3)} \leq c_2 \|\zeta^{(k)}g'\|_{L_p(\mathbf{R}_T^3)},$$

$$(2.9) \quad \|\nabla\bar{w}^{(k)}\|_{L_p(\mathbf{R}_T^3)} + \|\bar{s}^{(k)}\|_{L_p(\mathbf{R}_T^3)} \leq c_2 \sqrt{T} \|\zeta^{(k)}g'\|_{L_p(\mathbf{R}_T^3)},$$

$$(2.10) \quad \|\bar{w}^{(k)}\|_{L_p(\mathbf{R}_T^3)} \leq c_2 T \|\zeta^{(k)}g'\|_{L_p(\mathbf{R}_T^3)},$$

where $\mathbf{R}_T^3 = \begin{cases} \mathbf{R}^3 \times [0, T] & \text{for } k \in \mathcal{M} \\ \mathbf{R}_+^3 \times [0, T] & \text{for } k \in \mathcal{N} \end{cases}$ and $\mathbf{R}_+^3 = \{x \in \mathbf{R}^3; x_3 > 0\}$.

Defining the regularizer R by the formula

$$(2.11) \quad Rh = \sum_k \eta^{(k)}(x)(w^{(k)}, s^{(k)})(x, t) = \sum_k \eta^{(k)}(x) \Pi_x^z(\bar{w}^{(k)}, \bar{s}^{(k)})(z, t)$$

for $h = (g', 0, 0)$, and the operator A by

$$A(w, s) = (\rho w_t - \mu\Delta w + \nabla s, \operatorname{div} w, w|_{S_T}),$$

we obtain $ARh = h + Mh$. Here $Mh = (M_1h, M_2h, 0)$,

$$\begin{aligned} M_1h &= \sum_k [\{\rho(x, t)(\eta^{(k)}w^{(k)})_t - \mu\Delta(\eta^{(k)}w^{(k)}) + \nabla(\eta^{(k)}s^{(k)})\} \\ &\quad - \eta^{(k)}\{\rho(x, t)w_t^{(k)} - \mu\Delta w^{(k)} + \nabla s^{(k)}\}] \\ &\quad + \sum_k \eta^{(k)}[\{\rho(x, t)w_t^{(k)} - \mu\Delta w^{(k)} + \nabla s^{(k)}\} - \{\rho(\xi^{(k)}, 0)w_t^{(k)} - \mu\Delta w^{(k)} + \nabla s^{(k)}\}] \\ &\quad + \sum_{k \in \mathcal{N}} \eta^{(k)} \Pi_x^z [\{\rho(\xi^{(k)}, 0)\bar{w}_t^{(k)} - \mu(\nabla - \nabla F \cdot \nabla_3)^2 \bar{w}^{(k)} + (\nabla - \nabla F \cdot \nabla_3)\bar{s}^{(k)}\} \\ &\quad - \{\rho(\xi^{(k)}, 0)\bar{w}_t^{(k)} - \mu\Delta\bar{w}^{(k)} + \nabla\bar{s}^{(k)}\}] \\ &= \sum_k [-2\mu\nabla\eta^{(k)}\nabla w^{(k)} - \mu\Delta\eta^{(k)}w^{(k)} + \nabla\eta^{(k)}s^{(k)}] \end{aligned}$$

$$\begin{aligned}
& + \sum_k \eta^{(k)} [\rho(x, t) - \rho(\xi^{(k)}, 0)] w_t^{(k)} \\
& + \sum_{k \in \mathcal{N}} \eta^{(k)} \Pi_x^z [-\mu \{ (\nabla F)^2 \nabla_3^2 \bar{w}^{(k)} - 2 \nabla F \nabla_3 \nabla \bar{w}^{(k)} - \nabla^2 F \nabla_3 \bar{w}^{(k)} \} - \nabla F \nabla_3 \nabla_3 \bar{s}^{(k)}], \\
M_2 h &= \sum_k [\nabla \cdot (\eta^{(k)} w^{(k)}) - \eta^{(k)} (\nabla \cdot w^{(k)})] + \sum_{k \in \mathcal{N}} \eta^{(k)} \Pi_x^z [(\nabla - \nabla F \nabla_3) \cdot \bar{w}^{(k)} - \nabla \cdot \bar{w}^{(k)}] \\
&= \sum_k \nabla \eta^{(k)} \cdot w^{(k)} - \sum_{k \in \mathcal{N}} \eta^{(k)} \Pi_x^z \nabla F \nabla_3 \cdot \bar{w}^{(k)}.
\end{aligned}$$

It is easily seen that M is a bounded operator in the space

$$\mathcal{B}_{p,T} \equiv L_p(Q_T) \times L_p(0, T; W_p^1(\Omega)) \times W_p^{2-1/p}(S_T)$$

with norm

$$\|h\|_{\mathcal{B}_{p,T}} \equiv \|h_1\|_{L_p(Q_T)} + \|h_2\|_{L_p(0,T;W_p^1(\Omega))} + \|h_3\|_{W_p^{2-1/p}(S_T)}$$

for $h = (h_1, h_2, h_3) \in \mathcal{B}_{p,T}$.

If $\tau = \kappa \lambda^2$ with $\kappa \leq 1$ and $\kappa \lambda^2 \leq T$, then by (2.8), (2.9) and (2.10), we get

$$\begin{aligned}
\|M_1 h\|_{L_p(Q_\tau)} &\leq c_3 \left(\frac{\sqrt{\tau}}{\lambda} + \frac{\tau}{\lambda^2} + \lambda^\alpha + \lambda^\beta + \lambda^2 + \lambda + \sqrt{\tau} \right) \|g'\|_{L_p(Q_\tau)} \\
&\leq c_4 (\sqrt{\kappa} + \lambda^\alpha + \lambda^\beta) \|g'\|_{L_p(Q_\tau)},
\end{aligned}$$

$$\begin{aligned}
\|M_2 h\|_{L_p(0,\tau;W_p^1(\Omega))} &\leq c_5 \left(\frac{\sqrt{\tau}}{\lambda} + \frac{\tau}{\lambda} + \frac{\tau}{\lambda^2} + \lambda \sqrt{\tau} + \lambda + \sqrt{\tau} \right) \|g'\|_{L_p(Q_\tau)} \\
&\leq c_6 (\sqrt{\kappa} + \lambda) \|g'\|_{L_p(Q_\tau)},
\end{aligned}$$

where c_4 and c_6 are independent of κ and λ . Therefore we have for sufficiently small κ and λ ,

$$\|Mh\|_{\mathcal{B}_{p,\tau}} \leq \frac{1}{2} \|h\|_{\mathcal{B}_{p,\tau}}.$$

Whence the solution (w, s) on the interval $[0, \tau]$ can be found in the form

$$(w, s) = R(I + M + M^2 + \dots)h = R(I - M)^{-1}h.$$

Moreover, (2.8), (2.9) and (2.10) imply

$$\|w\|_{W_p^{2,1}(Q_\tau)} + \|\nabla s\|_{L_p(Q_\tau)} \leq c_7 \|g'\|_{L_p(Q_\tau)}.$$

Next we prove the solvability on the interval $[\tau, 2\tau]$. Put

$$w^*(x, t) = \begin{cases} w(x, t) & \text{for } 0 \leq t \leq \tau, \\ w(x, 2\tau - t) & \text{for } \tau \leq t \leq 2\tau, \end{cases}$$

$$s^*(x, t) = \begin{cases} s(x, t) & \text{for } 0 \leq t \leq \tau, \\ s(x, 2\tau - t) & \text{for } \tau \leq t \leq 2\tau. \end{cases}$$

Let $(\tilde{w}, \tilde{s})(x, t)$ be a solution of problem

$$\begin{cases} \rho \tilde{w}_t - \mu \Delta \tilde{w} + \nabla \tilde{s} = g'(x, t) - g'(x, 2\tau - t) \\ \quad + \rho(x, t)w_t(x, 2\tau - t) + \rho(x, 2\tau - t)w_t(x, 2\tau - t), \\ \operatorname{div} \tilde{w} = 0, \\ \tilde{w}|_{S_\tau} = 0, \\ \tilde{w}|_{t=\tau} = 0 \end{cases}$$

for $\tau \leq t \leq 2\tau$, where $\tilde{S}_\tau = S \times [\tau, 2\tau]$, and $(\tilde{w}, \tilde{s}) = (0, 0)$ for $0 \leq t \leq \tau$. Then it is easy to verify that $w = w^* + \tilde{w}$ and $s = s^* + \tilde{s}$ is the solution of the problem

$$\begin{cases} \rho w_t - \mu \Delta w + \nabla s = g', \\ \operatorname{div} w = 0, \\ w|_{S_{2\tau}} = 0, \\ w|_{t=0} = 0. \end{cases}$$

Repeating this argument, we can obtain the solution on $[0, T]$ satisfying

$$(2.12) \quad \|w\|_{W_p^{2,1}(Q_T)} + \|\nabla s\|_{L_p(Q_T)} \leq c_8 \|g'\|_{L_p(Q_T)}.$$

From (2.5) and (2.12), we can derive the estimate (2.2). \square

LEMMA 2.2. *If $v(x, t)$ satisfies $\operatorname{div} v = 0$, $v|_{S_T} = 0$ and*

$$(2.13) \quad \|v\|_{L_\infty(Q_T)} + \int_0^T \|\nabla v(t)\|_{L_\infty(\Omega)} dt < \infty,$$

then for any $\rho_0(x) \in C^1(\bar{\Omega})$ such that $0 < m \leq \rho_0(x) \leq M < \infty$, problem

$$(2.14) \quad \begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ \rho|_{t=0} = \rho_0(x), \end{cases}$$

has a unique solution $\rho(x, t) \in C^{1,1}(\bar{Q}_T)$, which satisfies

$$(2.15) \quad m \leq \rho(x, t) \leq M,$$

$$(2.16) \quad \|\nabla \rho\|_{L_\infty(Q_T)} \leq \sqrt{3} \|\nabla \rho_0\|_{L_\infty(\Omega)} \exp\left(\int_0^T \|\nabla v(t)\|_{L_\infty(\Omega)} dt\right),$$

$$(2.17) \quad \|\rho_t\|_{L_\infty(Q_T)} \leq \sqrt{3} \|v\|_{L_\infty(Q_T)} \|\nabla \rho_0\|_{L_\infty(\Omega)} \exp\left(\int_0^T \|\nabla v(t)\|_{L_\infty(\Omega)} dt\right).$$

Moreover, if $\nabla \rho_0(x) \in W_p^1(\Omega)$ and $v(x, t) \in L_1(0, T; W_p^2(\Omega))$, then

$$(2.18) \quad \frac{d}{dt} \|\nabla \rho(t)\|_{W_p^1(\Omega)} \leq c_9 \|v(t)\|_{W_p^2(\Omega)} \|\nabla \rho(t)\|_{W_p^1(\Omega)}.$$

PROOF. It is well-known that, according to the classical method of characteristics, the solution of problem (2.14) is given by $\rho(x, t) = \rho_0(y(\tau, x, t)|_{\tau=0})$, where $y(\tau, x, t)$ is the solution of the Cauchy problem

$$\begin{cases} \frac{dy}{d\tau} = v(y, \tau), \\ y|_{\tau=t} = x. \end{cases}$$

From this the estimate (2.15) results. For the estimates (2.16) and (2.17), we refer to Lemma 1.3 in [6].

Next let us establish (2.18). Apply the operator D_x^α on each side of (2.14)₁. Multiplying the result by $D_x^\alpha \rho |D_x^\alpha \rho|^{p-2}$, integrating over Ω and summing over $|\alpha|=1, 2$, we have the equality

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\nabla \rho(t)\|_{W_p^1(\Omega)}^p &= - \sum_{|\alpha|=1}^2 \int_{\Omega} (v \cdot \nabla D_x^\alpha \rho) D_x^\alpha \rho |D_x^\alpha \rho|^{p-2} dx \\ &\quad - \sum_{|\alpha|=1}^2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \int_{\Omega} (D_x^\beta v \cdot \nabla D_x^{\alpha-\beta} \rho) D_x^\alpha \rho |D_x^\alpha \rho|^{p-2} dx. \end{aligned}$$

The first term of the right hand side is zero, by integration by parts, since $\operatorname{div} v = 0$ and $v|_{S_T} = 0$. The second term can be estimated as follows:

$$\begin{aligned} \left| \sum_{|\alpha|=1}^2 \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \int_{\Omega} (D_x^\beta v \cdot \nabla D_x^{\alpha-\beta} \rho) D_x^\alpha \rho |D_x^\alpha \rho|^{p-2} dx \right| \\ \leq c_{10} \|\nabla v(t)\|_{W_p^1(\Omega)} \|\nabla \rho(t)\|_{W_p^1(\Omega)}. \end{aligned}$$

Hence we get the estimate (2.18). \square

The next lemma is directly proved by means of the method of characteristics.

LEMMA 2.3. *Let $v(x, t)$ be the same as in Lemma 2.2. If $\rho(x, t) \in C^{1,1}(\bar{Q}_T)$ satisfies*

$$\begin{cases} \rho_t + v \cdot \nabla \rho = \tilde{g} \in L_1(0, T; L_\infty(\Omega)), \\ \rho|_{t=0} = 0, \end{cases}$$

then we have

$$\|\rho(t)\|_{L_\infty(\Omega)} \leq \int_0^t \|\tilde{g}(s)\|_{L_\infty(\Omega)} ds.$$

2.2. Successive approximations. We construct approximate solutions inductively:

$$(2.19) \quad v^{(0)} = 0,$$

and for $k=1, 2, 3, \dots$, $\rho^{(k)}$ and $(v^{(k)}, p^{(k)})$ are, respectively, the solutions of problems

$$(2.20) \quad \begin{cases} \rho_t^{(k)} + v^{(k-1)} \cdot \nabla \rho^{(k)} = 0, \\ \rho^{(k)}|_{t=0} = \rho_0(x), \end{cases}$$

and

$$(2.21) \quad \begin{cases} \rho^{(k)}[v_t^{(k)} + (v^{(k-1)} \cdot \nabla)v^{(k-1)}] + \nabla p^{(k)} = \mu \Delta v^{(k)} + \rho^{(k)} f, \\ \operatorname{div} v^{(k)} = 0 \\ v^{(k)}|_{S_T} = 0, \\ v^{(k)}|_{t=0} = v_0(x). \end{cases}$$

In this subsection, we show their boundedness.

LEMMA 2.4. *For sufficiently small $T_1 \in (0, T]$, the sequence $\{v^{(k)}, \nabla p^{(k)}\}_k$ is bounded in $W_p^{2,1}(Q_{T_1}) \times L_p(Q_{T_1})$.*

PROOF. Let

$$(2.22) \quad V^{(k)}(T) = \|v^{(k)}\|_{W_p^{2,1}(Q_T)} + \|\nabla p^{(k)}\|_{L_p(Q_T)}.$$

From the consequences in subsection 2.1, we have

$$m \leq \rho^{(k)} \leq M,$$

$$\|\nabla \rho^{(k)}\|_{L_\infty(Q_T)} \leq \sqrt{3} \|\nabla \rho_0\|_{L_\infty(\Omega)} \exp\left(\int_0^T \|\nabla v^{(k-1)}(t)\|_{L_\infty(\Omega)} dt\right),$$

$$\|\rho_t^{(k)}\|_{L_\infty(Q_T)} \leq \sqrt{3} \|v^{(k-1)}\|_{L_\infty(Q_T)} \|\nabla \rho_0\|_{L_\infty(\Omega)} \exp\left(\int_0^T \|\nabla v^{(k-1)}(t)\|_{L_\infty(\Omega)} dt\right),$$

$$(2.23) \quad \begin{aligned} \|v^{(k)}\|_{W_p^{2,1}(Q_T)} + \|\nabla p^{(k)}\|_{L_p(Q_T)} &\leq K_1(\|\rho^{(k)}\|_{C^{1,1}(\bar{Q}_T)}, T) \\ &\quad \times (\|v_0\|_{W_p^{2-2/p}(\Omega)} + \|\rho^{(k)} f\|_{L_p(Q_T)} + \|\rho^{(k)}(v^{(k-1)} \cdot \nabla)v^{(k-1)}\|_{L_p(Q_T)}). \end{aligned}$$

Let us estimate the right hand side of (2.23).

First, we can get the inequality

$$(2.24) \quad \|v^{(k-1)}\|_{L_\infty(Q_T)} \leq c_{11}(\|v_0\|_{W_p^{2-2/p}(\Omega)} + T^{(1-1/p)(1-3/p)} V^{(k-1)}(T)).$$

Indeed, since imbedding theorems imply the inequalities

$$\|v^{(k-1)}(t) - v_0\|_{L_\infty(\Omega)} \leq c_{12} \|v^{(k-1)}(t) - v_0\|_{W_p^{3/p}(\Omega)} \|v^{(k-1)}(t) - v_0\|_{L_p(\Omega)}^{1-3/p},$$

$$\|v_0\|_{L_\infty(\Omega)} \leq c_{12} \|v_0\|_{W_p^1(\Omega)} \leq c_{13} \|v_0\|_{W_p^{2-2/p}(\Omega)},$$

$$\sup_{0 \leq t \leq T} \|v^{(k-1)}\|_{W_p^1(\Omega)} \leq \sup_{0 \leq t \leq T} \|v^{(k-1)}\|_{W_p^{2-2/p}(\Omega)}$$

$$\leq c_{14} \|v^{(k-1)}\|_{W_p^{2,1}(Q_T)} + \|v_0\|_{W_p^{2-2/p}(\Omega)},$$

$$\begin{aligned}
\|v^{(k-1)}(t) - v_0\|_{L_p(\Omega)}^p &= \int_{\Omega} |v^{(k-1)}(t) - v_0|^p dx = \int_{\Omega} \left| \int_0^t v_t^{(k-1)}(s) ds \right|^p dx \\
&\leq \int_{\Omega} \left| \left(\int_0^t ds \right)^{1/q} \left(\int_0^t |v_t^{(k-1)}(s)|^p ds \right)^{1/p} \right|^p dx \\
&\leq t^{p/q} \|v^{(k-1)}\|_{W_p^{2,1}(Q_T)}^p
\end{aligned}$$

for $p^{-1} + q^{-1} = 1$, the estimate (2.24) is easily derived.

Secondly, we have

$$\begin{aligned}
(2.25) \quad \int_0^T \|\nabla v^{(k-1)}(t)\|_{L_{\infty}(\Omega)} dt &\leq c_{15} \int_0^T \|v^{(k-1)}(t)\|_{W_p^2(\Omega)} dt \\
&\leq c_{15} \left(\int_0^T dt \right)^{1/q} \left(\int_0^T \|v^{(k-1)}(t)\|_{W_p^2(\Omega)}^p dt \right)^{1/p} \\
&\leq c_{15} T^{1/q} \|v^{(k-1)}\|_{W_p^{2,1}(Q_T)} \leq c_{15} T^{1/q} V^{(k-1)}(T).
\end{aligned}$$

Thirdly, it is obvious that

$$(2.26) \quad \|\rho^{(k)} f\|_{L_p(Q_T)} \leq M \|f\|_{L_p(Q_T)}.$$

Finally, we have the estimate

$$\begin{aligned}
(2.27) \quad \|\rho^{(k)}(v^{(k-1)} \cdot \nabla)v^{(k-1)}\|_{L_p(Q_T)} &\leq c_{16} M T^{(2-3/p)/p} V^{(k-1)}(T)^{(p-3)/(2p-3)} \\
&\quad \times (\|v_0\|_{W_p^{2-2/p}(\Omega)} + T^{(1-1/p)(1-3/p)} V^{(k-1)}(T))^{3(p-1)/(2p-3)} \\
&\leq c_{16} M [\|v_0\|_{W_p^{2-2/p}(\Omega)}^2 + T^{\delta} V^{(k-1)}(T)^2]
\end{aligned}$$

with some positive constants δ and $c_{16} \geq M^{-1} + 1$. Indeed, from the inequality

$$\|\nabla v^{(k-1)}(t)\|_{L_p(\Omega)} \leq \|v^{(k-1)}(t)\|_{W_p^1(\Omega)} \leq \|v^{(k-1)}(t)\|_{W_p^2(\Omega)}^a \|v^{(k-1)}(t)\|_{L_{\infty}(\Omega)}^{1-a}$$

with $a = (p-3)/(2p-3)$, it follows that

$$\begin{aligned}
\|\rho^{(k)}(v^{(k-1)} \cdot \nabla)v^{(k-1)}\|_{L_p(Q_T)}^p &\leq M^p \|v^{(k-1)}\|_{L_{\infty}(Q_T)}^p \int_0^T \|\nabla v^{(k-1)}(t)\|_{L_p(\Omega)}^p dt \\
&\leq c_{17} M^p \|v^{(k-1)}\|_{L_{\infty}(Q_T)}^{p(2-a)} \int_0^T \|v^{(k-1)}(t)\|_{W_p^2(\Omega)}^{ap} dt \\
&\leq c_{17} M^p \|v^{(k-1)}\|_{L_{\infty}(Q_T)}^{p(2-a)} T^{1-a} \|v^{(k-1)}\|_{W_p^{2,1}(Q_T)}^{ap}.
\end{aligned}$$

This inequality combined with (2.24) leads to (2.27). Moreover, we obtain

$$\begin{aligned}
\|\rho^{(k)}\|_{C^{1,1}(\bar{Q}_T)} &\leq M + \sqrt{3} (1 + \|v^{(k-1)}\|_{L_{\infty}(Q_T)}) \\
&\quad \times \|\nabla \rho_0\|_{L_{\infty}(\Omega)} \exp\left(\int_0^T \|\nabla v^{(k-1)}(t)\|_{L_{\infty}(\Omega)} dt \right)
\end{aligned}$$

$$\begin{aligned}
&\leq M + \sqrt{3} [1 + c_{11}(\|v_0\|_{W_p^{2-2/p}(\Omega)} + T^{(1-1/p)(1-3/p)} V^{(k-1)}(T))] \\
&\quad \times \|\nabla \rho_0\|_{L^\infty(\Omega)} \exp(c_{15} T^{(1-1/p)} V^{(k-1)}(T)) \\
&\equiv K_3(V^{(k-1)}(T), T).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
(2.28) \quad V^{(k)}(T) &\leq K_1(K_3(V^{(k-1)}(T), T), T) \\
&\quad \times [\|v_0\|_{W_p^{2-2/p}(\Omega)} + M\|f\|_{L_p(Q_T)} + c_{16}M(\|v_0\|_{W_p^{2-2/p}(\Omega)}^2 + T^\delta V^{(k-1)}(T)^2)] \\
&\leq c_{18}MK_1(K_3(V^{(k-1)}(T), T), T) \\
&\quad \times [\|v_0\|_{W_p^{2-2/p}(\Omega)} + \|v_0\|_{W_p^{2-2/p}(\Omega)}^2 + \|f\|_{L_p(Q_T)} + T^\delta V^{(k-1)}(T)^2].
\end{aligned}$$

We choose

$$\begin{aligned}
A_1 &\geq K_1(M + \sqrt{3}e[1 + c_{11}(1 + \|v_0\|_{W_p^{2-2/p}(\Omega)})]\|\nabla \rho_0\|_{L^\infty(\Omega)}, T) \\
&\quad \times c_{18}M[\|v_0\|_{W_p^{2-2/p}(\Omega)} + \|v_0\|_{W_p^{2-2/p}(\Omega)}^2 + \|f\|_{L_p(Q_T)} + 1],
\end{aligned}$$

and define

$$T_1 = \min\{A_1^{-(1-1/p)^{-1}(1-3/p)^{-1}}, A_1^{-2/\delta}, (c_{15}A_1)^{-(1-1/p)^{-1}}\}.$$

Then it is easily seen that $V^{(k)}(T_1) \leq A_1$ holds provided that $V^{(k-1)}(T_1) \leq A_1$. Since

$$V^{(1)}(T_1) \leq K_1(M + \|\nabla \rho_0\|_{L^\infty(\Omega)}, T_1)(\|v_0\|_{W_p^{2-2/p}(\Omega)} + M\|f\|_{L_p(Q_{T_1})}) \leq A_1,$$

the assertion of the lemma comes out. \square

Furthermore, we can immediately get

LEMMA 2.5. *For any $k = 1, 2, 3, \dots$,*

$$(2.29) \quad \|\nabla \rho^{(k)}\|_{L^\infty(Q_{T_1})} + \|\rho_t^{(k)}\|_{L^\infty(Q_{T_1})} \leq K_3(A_1, T_1) \equiv A_2,$$

$$(2.30) \quad \sup_{0 \leq t \leq T_1} \|\nabla \rho^{(k)}(t)\|_{W_p^1(\Omega)} \leq \|\nabla \rho_0\|_{W_p^1(\Omega)} \exp(c_{19}A_1(T_1)^{(p-1)/p}) \equiv A_3.$$

2.3. Proof of Theorem 1.1. Setting $\sigma^{(k)} = \rho^{(k)} - \rho^{(k-1)}$, $w^{(k)} = v^{(k)} - v^{(k-1)}$ and $q^{(k)} = p^{(k)} - p^{(k-1)}$, we have

$$(2.31) \quad \begin{cases} \sigma_t^{(k)} + v^{(k-1)} \cdot \nabla \sigma^{(k)} = -w^{(k-1)} \cdot \nabla \rho^{(k-1)}, \\ \sigma^{(k)}|_{t=0} = 0, \end{cases}$$

$$(2.32) \quad \begin{cases} \rho^{(k)} w_t^{(k)} - \mu \Delta w^{(k)} + \nabla q^{(k)} = g^{(k)}, \\ \operatorname{div} w^{(k)} = 0, \\ w^{(k)}|_{S_T} = 0, \\ w^{(k)}|_{t=0} = 0, \end{cases}$$

where

$$g^{(k)} = -\sigma^{(k)}[v_t^{(k-1)} + (v^{(k-1)} \cdot \nabla)v^{(k-1)} + f] \\ - \rho^{(k-1)}[(w^{(k-1)} \cdot \nabla)v^{(k-2)} + (v^{(k-1)} \cdot \nabla)w^{(k-1)}].$$

Let

$$W^{(k)}(t) = \|w^{(k)}\|_{W_p^{2,1}(Q_t)} + \|\nabla q^{(k)}\|_{L_p(Q_t)}.$$

Then, from Lemma 2.3, it follows that for $t \in (0, T_1]$,

$$(2.33) \quad \begin{aligned} \|\sigma^{(k)}(t)\|_{L_\infty(\Omega)} &\leq A_2 \int_0^t \|w^{(k-1)}(s)\|_{L_\infty(\Omega)} ds \\ &\leq c_{20} A_2 \int_0^t \|w^{(k-1)}\|_{W_p^{2,1}(Q_s)} ds \\ &\leq c_{21} \int_0^t W^{(k-1)}(s) ds. \end{aligned}$$

Furthermore, we have

$$\|\sigma^{(k)}(t)\|_{W_p^1(\Omega)} \leq c_{22} A_3 \int_0^t \|w^{(k-1)}(s)\|_{W_p^1(\Omega)} ds.$$

Here we used the imbeddings

$$\begin{aligned} \|u\|_{L_\infty(\Omega)} &\leq c_{23} \|u\|_{W_p^1(\Omega)} \leq c_{23} \|u\|_{W_p^{2-2/p}(\Omega)} \\ &\leq c_{24} (\|u(x, 0)\|_{W_p^{2-2/p}(\Omega)} + \|u\|_{W_p^{2,1}(Q_t)}). \end{aligned}$$

Next, using Lemma 2.4, we can estimate each term in $g^{(k)}$:

$$\begin{aligned} &\|\sigma^{(k)}[v_t^{(k-1)} + (v^{(k-1)} \cdot \nabla)v^{(k-1)} + f]\|_{L_p(Q_t)}^p \\ &\leq \|\sigma^{(k)}\|_{L_\infty(Q_t)}^p (\|v_t^{(k-1)}\|_{L_p(Q_t)}^p + \|v^{(k-1)}\|_{L_\infty(Q_t)}^p \|\nabla v^{(k-1)}\|_{L_p(Q_t)}^p + \|f\|_{L_p(Q_t)}^p) \\ &\leq \|\sigma^{(k)}\|_{L_\infty(Q_t)}^p (A_1^p + c_{25}(1 + \|v_0\|_{W_p^{2-2/p}(\Omega)})^p A_1^p + \|f\|_{L_p(Q_t)}^p), \\ &\|\rho^{(k-1)}[(w^{(k-1)} \cdot \nabla)v^{(k-2)} + (v^{(k-1)} \cdot \nabla)w^{(k-1)}]\|_{L_p(Q_t)}^p \\ &\leq M^p \int_0^t ds \int_\Omega (|\nabla v^{(k-2)}|^p |w^{(k-1)}|^p + |v^{(k-1)}|^p |\nabla w^{(k-1)}|^p) dx \end{aligned}$$

$$\begin{aligned}
&\leq M^p \left(\sup_{0 \leq s \leq t} \|\nabla v^{(k-2)}\|_{L_p(\Omega)}^p \int_0^t \|w^{(k-1)}\|_{L_\infty(\Omega)}^p ds \right. \\
&\quad \left. + \|v^{(k-1)}\|_{L_\infty(Q_t)}^p \int_0^t \|\nabla w^{(k-1)}\|_{L_p(\Omega)}^p ds \right) \\
&\leq c_{26} M^p \left(\|v^{(k-2)}\|_{W_p^{2,1}(Q_t)}^p \int_0^t \|w^{(k-1)}\|_{W_p^{2,1}(Q_s)}^p ds \right. \\
&\quad \left. + (1 + \|v_0\|_{W_p^{2-2/p}(\Omega)})^p \int_0^t \|w^{(k-1)}\|_{W_p^{2,1}(Q_s)}^p ds \right).
\end{aligned}$$

Hence, Lemma 2.1 yields

$$\begin{aligned}
(2.34) \quad W^{(k)}(t) &\leq c_{27} \left[\int_0^t W^{(k-1)}(s) ds + \left(\int_0^t W^{(k-1)}(s)^p ds \right)^{1/p} \right] \\
&\leq c_{28} \left(\int_0^t W^{(k-1)}(s)^p ds \right)^{1/p},
\end{aligned}$$

consequently,

$$(2.35) \quad W^{(k)}(T_1) \leq c_{28}^{k-1} \frac{(T_1)^{(k-1)/p}}{\Gamma(k)^{1/p}} W^{(1)}(T_1).$$

Therefore we find that

$$\sum_{k=1}^{\infty} W^{(k)}(T_1) < \infty,$$

which implies that the sequence $(\rho^{(k)}, v^{(k)}, p^{(k)})(x, t)$ converges to the desired solution $(\rho, v, p)(x, t)$ as $k \rightarrow \infty$.

The uniqueness is proved by making use of the estimates analogous to (2.33) and (2.34).

3. The case $\mu=0$.

In this section, we prove Theorem 1.2.

3.1. Auxiliary problems. We assume that $v(x, t) \in C^0([0, T]; W_p^2(\Omega))$ is a given function such that $\operatorname{div} v = 0$ and $v \cdot n|_{S_T} = 0$.

LEMMA 3.1. *Let $\rho(x, t) \in C^{1,1}([0, T] \times \bar{\Omega})$ such that $0 < m \leq \rho(x, t) \leq M < \infty$, $\nabla \rho(x, t) \in C^0([0, T]; W_p^1(\Omega))$ and $f(x, t) \in C^0([0, T]; W_p^2(\Omega))$. Then problem*

$$(3.1) \quad \begin{cases} \operatorname{div}(\rho^{-1}\nabla p) = \operatorname{div} f - \sum_{i,j=1}^3 v_{x_j}^i v_{x_i}^j \equiv F, \\ \rho^{-1} \frac{\partial p}{\partial n} \Big|_S = f \cdot n + \sum_{i,j=1}^3 v^i v^j \phi^{ij} \equiv G, \quad \phi^{ij} = n_{x_i}^j \end{cases}$$

has a unique solution $\nabla p(x, t) \in C^0([0, T]; W_p^2(\Omega))$, satisfying

$$(3.2) \quad \|\nabla p(t)\|_{W_p^2(\Omega)} \leq K_4(\|\nabla \rho(t)\|_{W_p^1(\Omega)})(\|f(t)\|_{W_p^2(\Omega)} + \|v(t)\|_{W_p^2(\Omega)}^2),$$

where K_4 is a nondecreasing function of $\|\nabla \rho(t)\|_{W_p^1(\Omega)}$, depending on m and M . Hereafter, K_j 's are functions, having the same properties as K_4 .

PROOF. We first note that (3.1)₁ comes from applying the divergence operator on both sides of (1.1)₂, and (3.1)₂ from taking the scalar product of each side of (1.1)₂ with n (cf. Temam [12]). It is well-known from the result of Agmon, Douglis and Nirenberg [1] that problem (3.1) is solvable in $W_p^2(\Omega)$ and the estimate

$$\|\nabla p(t)\|_{W_p^1(\Omega)} \leq K_5(\|\rho^{-1}\|_{C^1(\bar{\Omega})})(\|F\|_{L_p(\Omega)} + \|G\|_{W_p^{1-1/p}(S)})$$

is valid. Writing the problem in the form

$$\begin{cases} \Delta p = \rho F - \rho \nabla(\rho^{-1}) \cdot \nabla p, \\ \frac{\partial p}{\partial n} \Big|_S = \rho G, \end{cases}$$

we get

$$\begin{aligned} \|\nabla p(t)\|_{W_p^2(\Omega)} &\leq c_{29}(\|\rho F - \rho \nabla(\rho^{-1}) \cdot \nabla p\|_{W_p^1(\Omega)} + \|\rho G\|_{W_p^{2-1/p}(S)}) \\ &\leq K_6(\|\nabla \rho(t)\|_{W_p^1(\Omega)})(\|F\|_{W_p^1(\Omega)} + \|G\|_{W_p^{2-1/p}(S)}) \\ &\leq K_7(\|\nabla \rho(t)\|_{W_p^1(\Omega)})(\|f(t)\|_{W_p^2(\Omega)} + \|v(t)\|_{W_p^2(\Omega)}^2). \quad \square \end{aligned}$$

LEMMA 3.2. Let $\rho(x, t)$ and $f(x, t)$ be the same as in Lemma 3.1, and $\nabla p(x, t) \in C^0([0, T]; W_p^2(\Omega))$ be the unique solution of (3.1) guaranteed in Lemma 3.1. Then problem

$$(3.3) \quad \begin{cases} u_t + v \cdot \nabla u = -\rho^{-1}\nabla p + f, \\ u|_{t=0} = v_0(x), \end{cases}$$

has a unique solution $u(x, t) \in C^0([0, T]; W_p^2(\Omega))$. Moreover, $u(x, t)$ satisfies

$$(3.4) \quad \begin{aligned} \frac{d}{dt} \|u(t)\|_{W_p^2(\Omega)} &\leq c_{30} \|v(t)\|_{W_p^2(\Omega)} \|u(t)\|_{W_p^2(\Omega)} \\ &\quad + K_8(\|\nabla \rho(t)\|_{W_p^1(\Omega)})(\|f(t)\|_{W_p^2(\Omega)} + \|v(t)\|_{W_p^2(\Omega)}^2). \end{aligned}$$

PROOF. Referring to the proof of Lemma 2.2, we should only estimate the term

$$\sum_{|\alpha|=0}^2 \int_{\Omega} D_x^\alpha (f - \rho^{-1} \nabla p) \cdot D_x^\alpha u |D_x^\alpha u|^{p-2} dx.$$

Since

$$\begin{aligned} & \sum_{|\alpha|=0}^2 \int_{\Omega} |D_x^\alpha (\rho^{-1} \nabla p)| |D_x^\alpha u|^{p-1} dx \\ & \leq m^{-1} \|\nabla p\|_{W_p^2(\Omega)} \|u\|_{W_p^2(\Omega)}^{p-1} + \|\nabla \rho^{-1}\|_{W_p^1(\Omega)} \|\nabla p\|_{W_p^1(\Omega)} \|\nabla u\|_{W_p^1(\Omega)}^{p-1} \\ & \leq K_9 (\|\nabla \rho\|_{W_p^1(\Omega)}) (\|f\|_{W_p^2(\Omega)} + \|v\|_{W_p^2(\Omega)}) \|u\|_{W_p^2(\Omega)}^{p-1}, \end{aligned}$$

the desired estimate is obtained. \square

3.2. Successive approximations. In order to prove Theorem 1.2, we use the method of successive approximations in the following form:

$$(3.5) \quad v^{(0)} = 0,$$

and for $k = 1, 2, 3, \dots$, $\rho^{(k)}$, $p^{(k)}$ and $u^{(k)}$ are, respectively, the solutions of problems

$$(3.6) \quad \begin{cases} \rho_t^{(k)} + v^{(k-1)} \cdot \nabla \rho^{(k)} = 0, \\ \rho^{(k)}|_{t=0} = \rho_0(x), \end{cases}$$

$$(3.7) \quad \begin{cases} \operatorname{div} \left(\frac{1}{\rho^{(k)}} \nabla p^{(k)} \right) = \operatorname{div} f - \sum_{i,j=1}^3 v_{x_j}^{(k-1),i} v_{x_i}^{(k-1),j}, \\ \frac{1}{\rho^{(k)}} \frac{\partial p^{(k)}}{\partial n} \Big|_S = f \cdot n + \sum_{i,j=1}^3 v^{(k-1),i} v^{(k-1),j} \phi^{ij}, \end{cases}$$

and

$$(3.8) \quad \begin{cases} u_t^{(k)} + v^{(k-1)} \cdot \nabla u^{(k)} = -\frac{1}{\rho^{(k)}} \nabla p^{(k)} + f, \\ u^{(k)}|_{t=0} = v_0(x). \end{cases}$$

Finally, let

$$(3.9) \quad v^{(k)} = u^{(k)} - \nabla \psi^{(k)},$$

where $\psi^{(k)}$ is the solution of problem

$$(3.10) \quad \begin{cases} \Delta \psi^{(k)} = \operatorname{div} u^{(k)}, \\ \frac{\partial \psi^{(k)}}{\partial n} \Big|_S = u^{(k)} \cdot n. \end{cases}$$

LEMMA 3.3. *The sequence $\{v^{(k)}\}_k$ is bounded in $C^0([0, T_2]; W_p^2(\Omega))$ for a sufficiently small $T_2 \in (0, T]$.*

PROOF. From the consequences in the previous subsections, we can derive

$$\begin{aligned}
m &\leq \rho^{(k)}(x, t) \leq M, \\
\|\nabla \rho^{(k)}(t)\|_{W_p^1(\Omega)} &\leq \|\nabla \rho_0\|_{W_p^1(\Omega)} \exp\left(c_9 \int_0^t \|v^{(k-1)}(s)\|_{W_p^2(\Omega)} ds\right), \\
\|\nabla p^{(k)}(t)\|_{W_p^2(\Omega)} &\leq K_4(\|\nabla \rho^{(k)}(t)\|_{W_p^1(\Omega)})(\|f(t)\|_{W_p^2(\Omega)} + \|v^{(k-1)}(t)\|_{W_p^2(\Omega)}^2), \\
\|u^{(k)}(t)\|_{W_p^2(\Omega)} &\leq \exp\left(c_{30} \int_0^t \|v^{(k-1)}(s)\|_{W_p^2(\Omega)} ds\right) [\|v_0\|_{W_p^2(\Omega)} \\
&\quad + \int_0^t K_8(\|\nabla \rho^{(k)}(s)\|_{W_p^1(\Omega)})(\|f(s)\|_{W_p^2(\Omega)} + \|v^{(k-1)}(s)\|_{W_p^2(\Omega)}^2) ds].
\end{aligned}$$

Since

$$\|v^{(k)}(t)\|_{W_p^2(\Omega)} \leq \|u^{(k)}(t)\|_{W_p^2(\Omega)} + \|\nabla \psi^{(k)}(t)\|_{W_p^2(\Omega)} \leq c_{31} \|u^{(k)}(t)\|_{W_p^2(\Omega)},$$

ultimately, we get

$$\begin{aligned}
(3.11) \quad \|v^{(k)}(t)\|_{W_p^2(\Omega)} &\leq c_{31} \exp\left(c_{30} \int_0^t \|v^{(k-1)}(s)\|_{W_p^2(\Omega)} ds\right) [\|v_0\|_{W_p^2(\Omega)} \\
&\quad + \int_0^t K_8(\|\nabla \rho^{(k)}(s)\|_{W_p^1(\Omega)})(\|f(s)\|_{W_p^2(\Omega)} + \|v^{(k-1)}(s)\|_{W_p^2(\Omega)}^2) ds].
\end{aligned}$$

Let us choose

$$A_4 \geq 2c_{31} [\|v_0\|_{W_p^2(\Omega)} + K_8(2\|\nabla \rho_0\|_{W_p^1(\Omega)})(T\|f\|_{C^0([0, T]; W_p^2(\Omega))} + 1)],$$

and define

$$T_2 = \min\{(c_{30}A_4)^{-1} \log 2, A_4^{-2}, (c_9A_4)^{-1} \log 2\}.$$

Then we find that

$$\sup_{0 \leq t \leq T_2} \|v^{(k)}(t)\|_{W_p^2(\Omega)} \leq A_4$$

provided that

$$\sup_{0 \leq t \leq T_2} \|v^{(k-1)}(t)\|_{W_p^2(\Omega)} \leq A_4.$$

Therefore by induction we have the assertion of the lemma. \square

By the direct calculation, we get

LEMMA 3.4. For $k=1, 2, 3, \dots$, the estimates

$$\sup_{0 \leq t \leq T_2} \|\nabla \rho^{(k)}(t)\|_{W_p^1(\Omega)} \leq 2 \|\nabla \rho_0\|_{W_p^1(\Omega)} \equiv A_5,$$

$$\sup_{0 \leq t \leq T_2} \|\nabla p^{(k)}(t)\|_{W_p^2(\Omega)} \leq K_4(A_5)(\|f\|_{C^0([0, T_1]; W_p^2(\Omega))} + A_4^2) \equiv A_6,$$

$$\sup_{0 \leq t \leq T_2} \|\rho_t^{(k)}(t)\|_{W_p^1(\Omega)} \leq A_4 A_5 \equiv A_7$$

$$\sup_{0 \leq t \leq T_2} \|u_t^{(k)}(t)\|_{W_p^1(\Omega)} \leq A_4^2 + m^{-1} A_6(1 + m^{-1} A_5) + \|f\|_{C^0([0, T_1]; W_p^2(\Omega))} \equiv A_8$$

hold.

3.3. Proof of Theorem 1.2. Set $\sigma^{(k)} = \rho^{(k)} - \rho^{(k-1)}$, $h^{(k)} = u^{(k)} - u^{(k-1)}$, $q^{(k)} = p^{(k)} - p^{(k-1)}$ and $w^{(k)} = v^{(k)} - v^{(k-1)}$. Then we have

$$(3.12) \quad \begin{cases} \sigma_t^{(k)} + v^{(k-1)} \cdot \nabla \sigma^{(k)} = -w^{(k-1)} \cdot \nabla \rho^{(k-1)}, \\ \sigma^{(k)}|_{t=0} = 0, \end{cases}$$

$$(3.13) \quad \begin{cases} \operatorname{div}\left(\frac{1}{\rho^{(k)}} \nabla q^{(k)}\right) = \operatorname{div}\left(\frac{\sigma^{(k)}}{\rho^{(k-1)} \rho^{(k)}} \nabla p^{(k-1)}\right) \\ \quad - \sum_{i,j=1}^3 (w_{x_j}^{(k-1),i} v_{x_i}^{(k-1),j} + v_{x_j}^{(k-2),i} w_{x_i}^{(k-1),j}), \\ \frac{1}{\rho^{(k)}} \frac{\partial q^{(k)}}{\partial n} \Big|_S = \sum_{i,j=1}^3 (w^{(k-1),i} v^{(k-1),j} + v^{(k-2),i} w^{(k-1),j}) \phi^{ij} \\ \quad - \frac{\sigma^{(k)}}{\rho^{(k-1)} \rho^{(k)}} \frac{\partial p^{(k-1)}}{\partial n} \Big|_S, \end{cases}$$

$$(3.14) \quad \begin{cases} h_t^{(k)} + (v^{(k-1)} \cdot \nabla) h^{(k)} + \frac{1}{\rho^{(k)}} \nabla q^{(k)} = -(w^{(k-1)} \cdot \nabla) u^{(k-1)} + \frac{\sigma^{(k)}}{\rho^{(k-1)} \rho^{(k)}} \nabla p^{(k-1)}, \\ h^{(k)}|_{t=0} = 0. \end{cases}$$

In the same way used for getting the estimates of ρ , p and u , we get

$$\|\sigma^{(k)}(t)\|_{W_p^1(\Omega)} \leq c_{32} A_5 \exp(c_{32} A_4 T_2) \int_0^t \|w^{(k-1)}(s)\|_{W_p^1(\Omega)} ds$$

$$\equiv A_9 \int_0^t \|w^{(k-1)}(s)\|_{W_p^1(\Omega)} ds,$$

$$\|\nabla q^{(k)}(t)\|_{W_p^1(\Omega)} \leq A_{10} (\|\sigma^{(k)}(t)\|_{W_p^1(\Omega)} + \|w^{(k-1)}(t)\|_{W_p^1(\Omega)}),$$

$$\|h^{(k)}(t)\|_{W_p^1(\Omega)} \leq A_{11} \int_0^t (\|\sigma^{(k)}(s)\|_{W_p^1(\Omega)} + \|\nabla q^{(k)}(s)\|_{W_p^1(\Omega)} + \|w^{(k-1)}(s)\|_{W_p^1(\Omega)}) ds.$$

From these inequalities, since

$$\|w^{(k)}(t)\|_{W_p^1(\Omega)} \leq c_{33} \|h^{(k)}(t)\|_{W_p^1(\Omega)},$$

it follows that

$$\begin{aligned} \|w^{(k)}(t)\|_{W_p^1(\Omega)} &\leq A_{12} \int_0^t \|w^{(k-1)}(s)\|_{W_p^1(\Omega)} ds \\ &\leq A_{12}^{k-1} \frac{t^{k-1}}{(k-1)!} \sup_{0 \leq s \leq t} \|w^{(1)}(s)\|_{W_p^1(\Omega)}, \end{aligned}$$

consequently,

$$\sup_{0 \leq t \leq T_2} \|w^{(k)}(t)\|_{W_p^1(\Omega)} \leq A_4 A_{12}^{k-1} \frac{(T_2)^{k-1}}{(k-1)!}.$$

Therefore we find that

$$\sum_{k=1}^{\infty} \|w^{(k)}\|_{C^0([0, T_2]; W_p^1(\Omega))} < \infty.$$

This implies that $(\rho^{(k)}, p^{(k)}, u^{(k)}, v^{(k)})(x, t) \rightarrow (\rho, p, u, v)(x, t)$ as $k \rightarrow \infty$, which satisfies equations

$$(3.15) \quad \begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ \operatorname{div}((v \cdot \nabla)v + \rho^{-1} \nabla p - f) = 0, \\ u_t + (v \cdot \nabla)u + \rho^{-1} \nabla p = f, \\ \Delta \psi = \operatorname{div} u, \\ v = u - \nabla \psi, \end{cases}$$

$$(3.16) \quad \begin{cases} ((v \cdot \nabla)v + \rho^{-1} \nabla p - f) \cdot n|_{S_{T_2}} = 0, \\ (u - \nabla \psi) \cdot n|_{S_{T_2}} = 0, \\ \rho|_{t=0} = \rho_0(x), \\ u|_{t=0} = v_0(x). \end{cases}$$

Now let us show that $u = v$. Applying the divergence operator on both sides of (3.15)₃ and taking into account (3.15)₂, we get

$$(\operatorname{div} u)_t + v \cdot \nabla(\operatorname{div} u) = - \sum_{i,j=1}^3 v_{x_j}^i \psi_{x_i x_j}.$$

If we take the scalar product of each side of (3.15)₃ with n , we obtain

$$(u \cdot n)_t + v \cdot \nabla(u \cdot n) = \sum_{i,j=1}^3 v^i \psi_{x_j} \phi^{ij}.$$

Noting that $\operatorname{div} v = 0$, $v \cdot n|_S = 0$ and

$$\|\psi\|_{W_p^2(\Omega)} \leq c_{34} (\|\operatorname{div} u\|_{L_p(\Omega)} + \|u \cdot n\|_{W_p^{1-1/p}(S)}),$$

we have the inequality

$$\|\operatorname{div} u\|_{L^p(\Omega)} + \|u \cdot n\|_{W_p^{1-1/p}(S)} \leq c_{35} \int_0^t (\|\operatorname{div} u\|_{L^p(\Omega)} + \|u \cdot n\|_{W_p^{1-1/p}(S)}) ds,$$

which means $\operatorname{div} u = 0$ and $u \cdot n|_S = 0$.

This completes the proof of Theorem.

Part II. Vanishing Viscosity

4. The convergence problem as $\mu \rightarrow 0$.

In this section, we shall prove Theorem 1.3.

4.1. A priori estimates. Let $(\rho, v, p)(x, t)$ be a sufficiently regular solution. Hereafter C stands for the generic constant independent of μ .

LEMMA 4.1. *For $\rho(x, t)$, the estimates*

$$(4.1) \quad m \leq \rho(x, t) \leq M,$$

$$(4.2) \quad \frac{d}{dt} \|\nabla \rho(t)\|_{W_p^1(\mathbb{R}^3)} \leq C \|v(t)\|_{W_p^2(\mathbb{R}^3)} \|\nabla \rho(t)\|_{W_p^1(\mathbb{R}^3)}$$

hold. Moreover, if we put $\xi(x, t) = \rho(x, t)^{-1}$, then the estimates

$$(4.3) \quad M^{-1} \leq \xi(x, t) \leq m^{-1},$$

$$(4.4) \quad \frac{d}{dt} \|\nabla \xi(t)\|_{W_p^1(\mathbb{R}^3)} \leq C \|v(t)\|_{W_p^2(\mathbb{R}^3)} \|\nabla \xi(t)\|_{W_p^1(\mathbb{R}^3)}$$

are valid.

PROOF. Quite similarly to the proof of Lemma 2.2, we can obtain (4.1) and (4.2). If we note that $\xi(x, t)$ satisfies the equation

$$\begin{cases} \xi_t + v \cdot \nabla \xi = 0, \\ \xi|_{t=0} = \rho_0(x)^{-1} \equiv \xi_0(x), \end{cases}$$

the estimates (4.3) and (4.4) directly follow from (4.1) and (4.2). \square

LEMMA 4.2. *Let $\omega(x, t) = \operatorname{rot} v(x, t)$. Then the estimate*

$$(4.5) \quad \frac{d}{dt} \|\omega(t)\|_{W_p^1(\mathbb{R}^3)} \leq C(1 + \|\operatorname{rot} f\|_{C^0([0, T]; W_p^1(\mathbb{R}^3))}) \\ \times (1 + \|\omega(t)\|_{W_p^1(\mathbb{R}^3)} + \|\nabla \xi(t)\|_{W_p^1(\mathbb{R}^3)} + \|\nabla p(t)\|_{W_p^1(\mathbb{R}^3)})^3$$

is valid.

PROOF. By applying the rotation operator on both sides of (1.1)₂, we obtain

$$(4.6) \quad \omega_t + (v \cdot \nabla)\omega - (\omega \cdot \nabla)v + \nabla \xi \times \nabla p = \mu \xi \Delta \omega + \mu \nabla \xi \times \Delta v + \text{rot } f.$$

If we apply the operator D_x^α to (4.6), multiply the result by $D_x^\alpha \omega |D_x^\alpha \omega|^{p-2}$, integrate over \mathbf{R}^3 and sum over $|\alpha|=0, 1$, then we have the equality

$$(4.7) \quad \frac{1}{p} \frac{d}{dp} \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^p = \sum_{|\alpha|=0,1} \left[- \int_{\mathbf{R}^3} D_x^\alpha ((v \cdot \nabla)\omega) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \right. \\ + \int_{\mathbf{R}^3} D_x^\alpha ((\omega \cdot \nabla)v) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx - \int_{\mathbf{R}^3} D_x^\alpha (\nabla \xi \times \nabla p) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \\ + \mu \int_{\mathbf{R}^3} D_x^\alpha (\xi \Delta \omega) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx + \mu \int_{\mathbf{R}^3} D_x^\alpha (\nabla \xi \times \Delta v) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \\ \left. + \int_{\mathbf{R}^3} D_x^\alpha \text{rot } f \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \right] \equiv \sum_{j=1}^6 I_j.$$

Let us estimate each I_j by making use of $\text{div } v = 0$ and $\mu \leq 1$.

$$|I_1| = \left| \sum_{|\alpha|=0,1} \int_{\mathbf{R}^3} (v \cdot \nabla) D_x^\alpha \omega \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \right. \\ \left. + \sum_{|\alpha|=1} \int_{\mathbf{R}^3} (D_x^\alpha v \cdot \nabla) \omega \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \right| \\ = \left| \sum_{|\alpha|=1} \int_{\mathbf{R}^3} (D_x^\alpha v \cdot \nabla) \omega \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \right| \leq \sum_{|\alpha|=1} \int_{\mathbf{R}^3} |D_x^\alpha v| |D_x^\alpha \omega|^p dx \\ \leq C \left(\sum_{|\alpha|=1} \|D_x^\alpha v(t)\|_{W_p^1(\mathbf{R}^3)} \right) \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^p.$$

$$|I_2| = \left| \sum_{|\alpha|=0,1} \int_{\mathbf{R}^3} (D_x^\alpha \omega \cdot \nabla) v \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \right. \\ \left. + \sum_{|\alpha|=1} \int_{\mathbf{R}^3} (\omega \cdot \nabla) D_x^\alpha v \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \right| \\ \leq \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\beta v| |D_x^\alpha \omega|^p dx \\ + \sum_{|\alpha|=1} \sum_{|\beta|=2} \int_{\mathbf{R}^3} |D_x^\beta v| |\omega| |D_x^\alpha \omega|^{p-1} dx \\ \leq C \left(\sum_{|\alpha|=1} \|D_x^\alpha v(t)\|_{W_p^1(\mathbf{R}^3)} \right) \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^p.$$

$$\begin{aligned}
|I_3| &= \left| \sum_{|\alpha|=0,1} \int_{\mathbf{R}^3} (\nabla \xi \times \nabla D_x^\alpha p) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \right. \\
&\quad \left. + \sum_{|\alpha|=1} \int_{\mathbf{R}^3} (\nabla D_x^\alpha \xi \times \nabla p) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \right| \\
&\leq 2 \sum_{|\alpha|=0,1} \int_{\mathbf{R}^3} |\nabla \xi| |\nabla D_x^\alpha p| |D_x^\alpha \omega|^{p-1} dx \\
&\quad + 2 \sum_{|\alpha|=1} \int_{\mathbf{R}^3} |\nabla D_x^\alpha \xi| |\nabla p| |D_x^\alpha \omega|^{p-1} dx \\
&\leq C \|\nabla \xi(t)\|_{W_p^1(\mathbf{R}^3)} \|\nabla p(t)\|_{W_p^1(\mathbf{R}^3)} \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^{p-1}. \\
I_4 &= \mu \sum_{|\alpha|=0,1} \int_{\mathbf{R}^3} \xi \Delta D_x^\alpha \omega \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \\
&\quad + \mu \sum_{|\alpha|=1} \int_{\mathbf{R}^3} D_x^\alpha \xi \Delta \omega \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \equiv I_{4,1} + I_{4,2}, \\
I_{4,1} &= -\mu \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} \xi |D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-2} dx \\
&\quad - \mu(p-2) \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} \xi |D_x^\alpha \omega \cdot D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-4} dx \\
&\quad - \mu \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} D_x^\beta \xi D_x^\alpha D_x^\beta \omega \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx.
\end{aligned}$$

Let the third term of the right hand side be $I_{4,3}$, then

$$\begin{aligned}
|I_{4,2} + I_{4,3}| &\leq C\mu \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\beta \xi| |D_x^\alpha D_x^\beta \omega| |D_x^\alpha \omega|^{p-1} dx \\
&\leq \frac{\mu}{4M} \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-2} dx \\
&\quad + C \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\beta \xi|^2 |D_x^\alpha \omega|^p dx \\
&\leq \frac{\mu}{4M} \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-2} dx \\
&\quad + C \|\nabla \xi(t)\|_{W_p^1(\mathbf{R}^3)}^2 \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^p.
\end{aligned}$$

$$\begin{aligned}
I_5 &= \mu \int_{\mathbf{R}^3} (\nabla \xi \times \Delta v) \cdot \omega |\omega|^{p-2} dx \\
&\quad + \mu \sum_{|\alpha|=1} \int_{\mathbf{R}^3} D_x^\alpha (\nabla \xi \times \Delta v) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \equiv I_{5,1} + I_{5,2},
\end{aligned}$$

$$\begin{aligned}
|I_{5,1}| &\leq 2\mu \int_{\mathbf{R}^3} |\nabla \xi| |\Delta v| |\omega|^{p-1} dx \\
&\leq C \|\nabla \xi(t)\|_{W_p^1(\mathbf{R}^3)} \|\Delta v(t)\|_{L_p(\mathbf{R}^3)} \|\omega(t)\|_{L_p(\mathbf{R}^3)}^{p-1}.
\end{aligned}$$

Since

$$\begin{aligned}
I_{5,2} &= -\mu \sum_{|\alpha|=1} \int_{\mathbf{R}^3} (\nabla \xi \times \Delta v) \cdot D_x^{2\alpha} \omega |D_x^\alpha \omega|^{p-2} dx \\
&\quad - \mu(p-2) \sum_{|\alpha|=1} \int_{\mathbf{R}^3} [(\nabla \xi \times \Delta v) \cdot D_x^\alpha \omega] D_x^{2\alpha} \omega \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-4} dx,
\end{aligned}$$

we get

$$\begin{aligned}
|I_{5,2}| &\leq C\mu \sum_{|\alpha|=1} \int_{\mathbf{R}^3} |\nabla \xi| |\Delta v| |D_x^{2\alpha} \omega| |D_x^\alpha \omega|^{p-2} dx \\
&\leq \frac{\mu}{4M} \sum_{|\alpha|=1} \int_{\mathbf{R}^3} |D_x^{2\alpha} \omega|^2 |D_x^\alpha \omega|^{p-2} dx \\
&\quad + C \sum_{|\alpha|=1} \int_{\mathbf{R}^3} |\nabla \xi|^2 |\Delta v|^2 |D_x^\alpha \omega|^{p-2} dx \\
&\leq \frac{\mu}{4M} \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-2} dx \\
&\quad + C \|\nabla \xi(t)\|_{W_p^1(\mathbf{R}^3)}^2 \|\Delta v(t)\|_{L_p(\mathbf{R}^3)}^2 \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^{p-2}. \\
|I_6| &\leq \|\operatorname{rot} f(t)\|_{W_p^1(\mathbf{R}^3)} \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^{p-1}.
\end{aligned}$$

Hence, using the inequality

$$\sum_{|\alpha|=1} \|D_x^\alpha v(t)\|_{W_p^1(\mathbf{R}^3)} \leq C \|\omega(t)\|_{W_p^1(\mathbf{R}^3)},$$

we have

$$\begin{aligned}
(4.8) \quad &\frac{1}{p} \frac{d}{dt} \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^p + \frac{\mu}{2M} \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-2} dx \\
&\quad + \frac{\mu(p-2)}{M} \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\alpha \omega \cdot D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-4} dx \\
&\leq C [\|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^2 + \|\nabla \xi(t)\|_{W_p^1(\mathbf{R}^3)} \|\nabla p(t)\|_{W_p^1(\mathbf{R}^3)} + \|\nabla \xi(t)\|_{W_p^1(\mathbf{R}^3)} \|\omega(t)\|_{W_p^1(\mathbf{R}^3)} \\
&\quad + \|\nabla \xi(t)\|_{W_p^1(\mathbf{R}^3)}^2 \|\omega(t)\|_{W_p^1(\mathbf{R}^3)} + \|\operatorname{rot} f(t)\|_{W_p^1(\mathbf{R}^3)}] \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^{p-1}. \quad \square
\end{aligned}$$

LEMMA 4.3. *There exists $T_0 \in (0, T]$ independent of μ such that*

$$(4.9) \quad \sup_{0 \leq t \leq T_0} (\|\nabla \rho(t)\|_{W_p^1(\mathbb{R}^3)} + \|v(t)\|_{W_p^2(\mathbb{R}^3)} + \|\nabla p(t)\|_{W_p^1(\mathbb{R}^3)}) \leq C.$$

PROOF. Similarly to getting the estimate ω , we first obtain

$$\frac{1}{p} \frac{d}{dt} \|v(t)\|_{L_p(\mathbb{R}^3)} \leq C [\|\nabla p(t)\|_{L_p(\mathbb{R}^3)} + \|\nabla \xi(t)\|_{W_p^1(\mathbb{R}^3)}^2 \|v(t)\|_{L_p(\mathbb{R}^3)} + \|f(t)\|_{L_p(\mathbb{R}^3)}].$$

Next, from the equation

$$\operatorname{div}(\xi \nabla p) = \operatorname{div} f - \sum_{i,j=1}^3 v_{x_j}^i v_{x_i}^j + \mu \nabla \xi \cdot \Delta v \equiv F,$$

we have

$$\begin{aligned} \|\nabla p(t)\|_{W_p^1(\mathbb{R}^3)} &\leq K_5 (\|\nabla \xi(t)\|_{W_p^1(\mathbb{R}^3)}) \|F\|_{L_p(\mathbb{R}^3)} \\ &\leq CK_5 (\|\nabla \xi(t)\|_{W_p^1(\mathbb{R}^3)}) \\ &\quad \times (\|\operatorname{div} f\|_{L_p(\mathbb{R}^3)} + \|\omega(t)\|_{W_p^1(\mathbb{R}^3)}^2 + \|\nabla \xi(t)\|_{W_p^1(\mathbb{R}^3)} \|\omega(t)\|_{W_p^1(\mathbb{R}^3)}). \end{aligned}$$

Therefore, if we set

$$\begin{aligned} Y(t) &= 1 + \|\nabla \xi(t)\|_{W_p^1(\mathbb{R}^3)} + \|v(t)\|_{L_p(\mathbb{R}^3)} + \|\omega(t)\|_{W_p^1(\mathbb{R}^3)}, \\ B &= (1 + \|f\|_{C^0([0, T_1]; W_p^2(\mathbb{R}^3))})^2, \end{aligned}$$

then the above lemmas imply a differential inequality

$$\frac{d}{dt} Y(t) \leq CBH(Y(t)),$$

where H is a increasing function of $Y(t)$ independent of μ .

Hence we conclude that $Y(t) \leq Z(t)$, where $Z(t)$ is the solution of the problem

$$\begin{cases} \frac{d}{dt} Z(t) = CBH(Z(t)), \\ Z(0) = Y(0), \end{cases}$$

and exists as a continuous function on an interval $[0, T_0]$ with $T_0 > 0$.

Since T_0 is obviously independent of μ , we obtain the desired result. \square

4.2. Proof of Theorem 1.3. First, it follows from Theorems 1.1, 1.2 and Lemma 4.3 that the existence of a unique solution on $[0, T_0]$ with $T_0 > 0$ independent of μ .

Next we prove (1.22). Subtracting (1.1) with $\mu > 0$ from (1.1) with $\mu = 0$, we get the following linear system of equations for $\sigma = \rho^0 - \rho^\mu$, $w = v^0 - v^\mu$ and $q = p^0 - p^\mu$:

$$(4.10) \quad \begin{cases} \sigma_t + v^\mu \cdot \nabla \sigma = -w \cdot \nabla \rho^0, \\ \rho^\mu [w_t + (v^\mu \cdot \nabla)w] + \nabla q = -\rho^\mu (w \cdot \nabla)v^0 + (\nabla p^0 / \rho^0)\sigma - \mu \Delta v^\mu, \\ \operatorname{div} w = 0, \\ \sigma|_{t=0} = 0, \\ w|_{t=0} = 0. \end{cases}$$

In the same way for getting a priori estimates, we have, from (4.9),

$$\begin{aligned} \|\sigma(t)\|_{L_p(\mathbf{R}^3)} &\leq C \int_0^t \|w(s)\|_{L_p(\mathbf{R}^3)} ds, \\ \|w(t)\|_{L_p(\mathbf{R}^3)} &\leq C \int_0^t (\|\sigma(s)\|_{L_p(\mathbf{R}^3)} + \|w(s)\|_{L_p(\mathbf{R}^3)}) ds + \mu C T_0. \end{aligned}$$

Hence, by Gronwall's inequality, we find that

$$(4.11) \quad \|\sigma(t)\|_{L_p(\mathbf{R}^3)} + \|w(t)\|_{L_p(\mathbf{R}^3)} \leq \mu C T_0 \exp(CT_0).$$

Furthermore, making use of the interpolation inequalities, we have

$$\begin{aligned} \|\nabla \sigma(t)\|_{L_p(\mathbf{R}^3)} &\leq C \|\sigma(t)\|_{L_p(\mathbf{R}^3)}^{1/2} \|\nabla \sigma(t)\|_{W_p^1(\mathbf{R}^3)}^{1/2} \\ &\leq C \|\sigma(t)\|_{L_p(\mathbf{R}^3)}^{1/2} (\|\nabla \rho^0(t)\|_{W_p^1(\mathbf{R}^3)} + \|\nabla \rho^\mu(t)\|_{W_p^1(\mathbf{R}^3)})^{1/2} \\ &\leq C \|\sigma(t)\|_{L_p(\mathbf{R}^3)}^{1/2}, \\ \|\nabla w(t)\|_{L_p(\mathbf{R}^3)} &\leq C \|w(t)\|_{L_p(\mathbf{R}^3)}^{1/2} \|\nabla w(t)\|_{W_p^1(\mathbf{R}^3)}^{1/2} \\ &\leq C \|w(t)\|_{L_p(\mathbf{R}^3)}^{1/2} (\|\nabla v^0(t)\|_{W_p^1(\mathbf{R}^3)} + \|\nabla v^\mu(t)\|_{W_p^1(\mathbf{R}^3)})^{1/2} \\ &\leq C \|w(t)\|_{L_p(\mathbf{R}^3)}^{1/2}. \end{aligned}$$

On the other hand, since $q(x, t)$ satisfies the equation

$$\begin{aligned} \operatorname{div}(\xi^\mu \nabla q) &= -\mu \nabla \xi^\mu \cdot \Delta v^\mu - \sum_{i,j=1}^3 (v_{x_j}^{\mu,i} w_{x_i}^j + v_{x_j}^{0,i} w_{x_i}^j) \\ &\quad + \Delta p^0 \xi^0 \xi^\mu \sigma + \nabla p^0 \cdot \nabla \xi^0 \xi^\mu \sigma + \nabla p^0 \cdot \nabla \xi^\mu \xi^0 \sigma + \nabla p^0 \cdot \nabla \sigma \xi^0 \xi^\mu, \end{aligned}$$

we get

$$\|\nabla q(t)\|_{W_p^1(\mathbf{R}^3)} \leq C (\|\sigma(t)\|_{W_p^1(\mathbf{R}^3)} + \|w(t)\|_{W_p^1(\mathbf{R}^3)} + \mu).$$

Thus, owing to (4.11), the proof of Theorem 1.3 is completed.

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