SOLVABILITY OF OVERDETERMINED PDE SYSTEMS THAT ADMIT A COMPLETE PROLONGATION AND SOME LOCAL PROBLEMS IN CR GEOMETRY

Chong-Kyu Han

Abstract. We study the existence of solutions for overdetermined PDE systems that admit prolongation to a complete system. We reduce the problem to a Pfaffian system on a submanifold of the jet space of unknown functions and then express the integrability conditions in terms of the coefficients of the original system. As possible applications we present some local problems in CR geometry: determining the CR embeddibility into spheres and the existence of infinitesimal CR automorphisms.

Introduction.

Let $u = (u^1, \dots, u^q)$ be a system of real-valued functions of independent variables $x = (x^1, \dots, x^p)$. Consider an over-determined system of partial differential equations of order m:

$$\Delta_\lambda(x, u^{(m)}) = 0, \qquad \lambda = 1, \cdots \ell,$$

where $u^{(m)}$ denotes the partial derivatives of u of order up to m and each Δ_λ is a smooth (C^∞) functions in the arguments. Generically, after differentiating the system sufficiently many times one can solve for all the partial derivatives of u of a certain order, say k, as smooth functions in $(x, u^{(k-1)})$. This is the case where the non-degeneracy condition of the implicit function theorem holds when we solve for all the partial derivatives of u of order k as

$$u^\alpha_K = H^\alpha_K(x, u^{(k-1)}),$$

for all $\alpha = 1, \dots, q$, and for all multi-indices K with $|K| = k$, which we call a complete system or order k, see §2. We shall discuss the existence of

---

2000 Mathematics Subject Classification: 32F25, 35N10, 58A17

Key words and phrases: overdetermined pde system, prolongation, complete system, involutivity, CR embedding, infinitesimal CR automorphism

The author was partially supported by grant 2002-DP0018 of Korea Research Foundation.

solutions of overdetermined PDE systems from which a complete system can be obtained. Recently, K. Yamaguchi and T. Yatsui studied in [YY] the geometry of the plane field defined by a compete system of order k in the space of the $(k-1)$th jets from the viewpoint of Tanaka's theory of graded Lie algebra. In this paper we observe that the solutions of the original system are in one-to-one correspondence with integral manifolds of n-dimensional plane field defined on a submanifold of the $(k-1)$th jet space. We investigate the existence of solutions by a process of repeated application of the Frobenius theorem. This paper consists of the following sections:

1. Pfaffian system with independence condition.
2. Local solvability for generic over-determined PDE systems.
3. Some local problems in CR geometry.

§**1. Pfaffian system with independence condition.** In this section we review some of the classical existence theory for the exterior differential systems. For the details we refer to [BCGGG] and [GJ]. Let M be a smooth manifold of dimension n. Let

$$\theta^1, \dots, \theta^s, \omega^1, \dots, \omega^p, \quad s + p \leq n$$

be smooth 1-forms that are linearly independent. We are concerned with the problem of finding a submanifold of dimension p on which

$$\theta^\alpha = 0, \quad \alpha = 1, \dots, s \quad \text{(Pfaffian system)} \tag{1.1}$$

and

$$\Omega := \omega^1 \wedge \dots \wedge \omega^p \neq 0 \quad \text{(independence condition).} \tag{1.2}$$

A submanifold N of M on which (1.1) holds is called an integral manifold. On an integral manifold N we have $\theta^\alpha = 0$, which implies that $d\theta^a = 0$ for each $\alpha = 1, \dots, s$. We want to construct a sequence of integral manifolds

$$\{x\} = N^0 \subset N^1 \subset \cdots \subset N^{p-1} \subset N^p,$$

so that (1.2) holds on N^p. For each $q = 1, \dots, p-1$, N^q may be obtained by integrating the possible tangent spaces that we define as follows:

Definition 1.1. *Let $G_q(M)$ be the Grassmann bundle of q-dimensional planes tangent to M. $E \in G_q(M)$ is called a q-dimensional integral element of (1.1) if*

$$\theta^\alpha = 0, \quad d\theta^\alpha = 0 \tag{1.3}$$

on E. By V_q we denote the set of all q-dimensional integral elements, for $q = 1, \ldots, p-1$ and by V_p the set of p-dimensional integral elements on which $\Omega \neq 0$.

Now let $\mathcal{I}$ be the ideal generated by $\theta^\alpha, d\theta^\alpha, \quad \alpha = 1, \ldots, s$, of the ring of differential forms on M and for each integer $q = 0, 1, \ldots$ let $\mathcal{I}^q$ be the submodule of $\mathcal{I}$ consisting of q-forms. Given an integral element $E^q \in V_q$ with a basis $\{e_1, \ldots, e_q\}$ for E^q its polar equations are linear equations for the subspace of all $v \in T_x M$ such that

$$< \phi(x), \quad e_1 \wedge \cdots \wedge e_q \wedge v >= 0, \quad \forall \phi \in \mathcal{I}^{q+1}.$$

This is the subspace of T^*M

$$\{(e_1 \wedge \cdots \wedge e_q) \lrcorner \phi : \phi \in \mathcal{I}^{q+1}\}.$$

A p-dimensional integral element E^p is said to admit a regular flag if there exists a sequence of integral elements

$$(0) = E^0 \subset E^1 \subset \cdots \subset E^p$$

where each E^q, $q = 0, 1, \ldots, p$, is a smooth point of V_q and polar equations are of constant rank. We have

Theorem 1.2 (Cartan-Kähler). *Suppose that M is an analytic (C^ω) manifold and that each 1-form θ^α, ω^i in (1.1)-(1.2) is C^ω. Suppose further that an integral element $(x, E) \in V_p$ admits a regular flag*

$$(0) = E^0 \subset E^1 \subset \cdots \subset E^p = E.$$

Then there exists a (C^ω) integral manifold N passing through x with $T_x N = E$.

The proof is a repeated application of the Cauchy-Kowalewski theorem to construct a sequence of integral namifolds: we construct N^{q+1} from N^q by solving a determined PDE system of Cauchy-Kowalewski type.

(1.1)-(1.2) are said to be involutive if each $(x, E) \in V_p$ admits a regular flag. Given a Pfaffian system (1.1) with independence condition (1.2) on a manifold M, the existence of solutions can be shown by the following two processes:

i) Reduction of the problem to a submanifold $M' \subset M$.

ii) Prolongation to an involutive system.

Let M' be the image of the projection $\pi : V_p \to M$. We assume M' is a manifold. If $M' \neq M$ an integral manifold must lie in M', so we set

$$V'_p := \{(x, E) \in V_p : E \subset T_x M'\}.$$

Let M'' be the image of $\pi : V'_p \to M'$. If $M'' \neq M'$, let

$$V''_p := \{(x, E) \in V'_p : E \subset T_x M''\}.$$

Eventually, we arrive at either empty set or else at $\tilde{M}$ with $\pi : \tilde{V}_p \to \tilde{M}$ surjective.

Now assume $\pi : V_p \to M$ is surjective. The first prolongation of the Pfaffian system (1.1)-(1.2) is the restriction to $M^{(1)} := V_p \subset G_p(M)$ of the canonical Pfaffian system on $G_p(M)$, which is given in terms of local coordinates as follows: Let $x^1, \ldots, x^p, y^1, \ldots, y^{n-p}$ be local coorninates of M with $dx^1 \wedge \cdots \wedge dx^p \neq 0$ on $E \in G_p(M)$. Let $dy^\alpha = \sum_{\rho=1}^p z^\alpha_\rho dx^\rho$, on E. Then z^α_ρ are fibre coordinates of $G_p(M)$ and the canonical system is

$$dy^\alpha - \sum_{\rho=1}^{p} z^\alpha_\rho dx^\rho.$$

Now we complete $\{\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p\}$ to a coframe of M by choosing additional $1-$forms $\pi^1, \ldots, \pi^r$. Any element $(x, E) \in G_p(M)$ for which $\omega^1 \wedge \cdots \wedge \omega^p \neq 0$ has equation

$$\theta^\alpha = m^\alpha_\rho \omega^\rho, \quad \pi^\epsilon = \ell^\epsilon_\rho \omega^\rho,$$

where $m^\alpha_\rho, \ell^\epsilon_\rho$ are local fibre coordinates and the summation convention is used for $\rho = 1, \ldots, p$. Then the canonical Pfaffian system with independence condition on $G_p(M)$ is

$$\begin{aligned}
&\theta^\alpha - m^\alpha_\rho \omega^\rho = 0 \quad \text{(summation convention)} \\
&\pi^\epsilon - \ell^\epsilon_\rho \omega^\rho = 0 \quad \text{(summation convention)} \\
&\Omega := \omega^1 \wedge \cdots \omega^p \neq 0.
\end{aligned}$$

Set

$$d\theta^\alpha = \frac{1}{2}a^\alpha_{\epsilon\delta}\pi^\epsilon\wedge\pi^\delta + b^\alpha_{\epsilon\rho}\pi^\epsilon\wedge\omega^\rho + \frac{1}{2}c^\alpha_{\rho\sigma}\omega^\rho\wedge\omega^\sigma \quad \text{(summation convention)}, \quad \mod \theta,$$

where $a^\alpha_{\epsilon\delta} = -a^\alpha_{\delta\epsilon}$ and $c^\alpha_{\rho\sigma} = -c^\alpha_{\sigma\rho}$. Then on an open subset of $G_p(M)$ on which $\Omega \neq 0$ V_p is given by $\theta^\alpha = 0$, and $d\theta^\alpha = 0$, which is equivalent to

$$\begin{aligned} & m^\alpha_\rho = 0 \\ & a^\alpha_{\epsilon\delta}(\ell^\epsilon_\rho\ell^\delta_\sigma - \ell^\epsilon_\sigma\ell^\delta_\rho) + (b^\alpha_{\epsilon\sigma}\ell^\epsilon_\rho - b^\alpha_{\epsilon\rho}\ell^\epsilon_\sigma) + c^\alpha_{\rho\sigma} = 0, \end{aligned} \tag{1.4}$$

$\forall\alpha = 1,\cdots,s$, $\forall\rho,\sigma = 1,\cdots,p$. (1.4) defines locally $M^{(1)} = V_p \subset G_p(M)$. Then the first prolongation of (1.1)-(1.2) on $M^{(1)}$ is

$$\begin{aligned} & \theta^\alpha = 0, \quad a = 1,\cdots,s \\ & \pi^\epsilon - \ell^\epsilon_\rho\omega^\rho = 0, \quad \epsilon = 1,\cdots,r \\ & \Omega \neq 0. \end{aligned}$$

A basic fact in the theory of prolongation is that the integral manifolds of (1.1)-(1.2) are in one-to-one correspondence with the integral manifolds of the first prolongation. Inductively, we define the n-th prolongation of (1.1) by the first prolongation of the $(n-1)$th prolongation. In analytic category one can determine the existence of solutions by a finite number of prolongations thanks to the following theorem:

Theorem 1.3 (Cartan-Kuranishi). *Given a Pfaffian system (1.1)-(1.2) on M there exists a non-negative integer q_0 such that the q-th ($q \geq q_0$) prolongation are involutive under a generic regularity assumption.*

In this paper we study C^∞ systems for the special case where

$$\theta^1,\cdots,\theta^s,\omega^1,\cdots,\omega^p$$

form a coframe for M, that is, the case when $s+p=n$. In this case the following are equivalent:

i) Involutivity
ii) There exists an integral element through each $x \in M$.
iii) $\pi : V_p \to M$ is surjective.
iv) $d\theta^\alpha = 0, \mod \theta$ (Frobenius integrability condition).

§2 Local solvability for generic over-determined PDE systems. In this section we study the existence of solutions for the over-determined PDE systems that admit prolongation to a complete system. We work in C^∞ category.

Let $u = (u^1, \cdots, u^q)$ be a system of real-valued functions of independent variables $x = (x^1, \cdots, x^p)$. Consider a system of partial differential equations of order m

$$\Delta_\lambda(x, u^{(m)}) = 0, \quad \lambda = 1, \cdots \ell, \tag{2.1}$$

where $u^{(m)}$ denotes all the partial derivatives of u of order up to m. We assume that (2.1) is over-determined, that is, $\ell > q$. A multi-index of order r is an unordered r-tuple of integers $J = (j_1, \cdots, j_r)$ with $1 \leq j_s \leq p$. The order of a multi-index J is denoted by $|J|$. By u^α_J we denote the $|J|$-th order partial derivative of u^α with respect to $x^{j_1}, \cdots, x^{j_{|J|}}$. For a smooth function $\Delta(x, u^{(m)})$, the total derivative of Δ with respect to x^i is the function in the arguments $(x, u^{(m+1)})$ defined by the chain rule:

$$D_i\Delta = \frac{\partial \Delta}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{|J| \leq m} \frac{\partial \Delta}{\partial u^\alpha_J} u^\alpha_{Ji},$$

where Ji denotes the multi-index $(j_1, \cdots, j_{|J|}, i)$. Compatibility conditions are those equations obtained from (2.1) by differentiation and algebraic operations, that is, the ideal generated by Δ and the total derivatives of Δ. (2.1) is said to admit a complete prolongation if the compatibility conditions determine all the partial derivatives of u of a sufficiently high order, say $k (k \geq m)$, as functions of derivatives of lower order, namely,

$$u^\alpha_K = H^\alpha_K(x, u^{(k-1)}), \tag{2.2}$$

for all multi-index K with $|K| = k$, and for all $\alpha = 1, \cdots, q$. This is the case where there exists a system of compatibility conditions that satisfies the non-degeneracy hypothesis of the implicit function theorem so that the system is solvable for all u_K's with $|K| = k$ in terms of lower order derivatives. (2.2) is called a complete system of order k. (2.1) shall be said to be generic if it admits prolongation to a complete system (2.2). Now we consider the ring of smooth functions in the arguments $(x, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \cdots)$. For each non-negative integer r let $\Delta^{(r)}$ be the algebraic ideal generated by Δ and the total derivatives of Δ up to order r, where $\Delta = (\Delta_1, \cdots, \Delta_\ell)$ as in (2.1). Suppose that the complete system (2.2) is obtained from $\Delta^{(r)}$. Let

$J^{k-1}(X,U)$ be the space of $(k-1)$th jets $(x, u^{(k-1)})$. Let $\mathcal{S} \subset J^{k-1}(X,U)$ be the common zero set of those functions in the arguments $(x, u^{(k-1)})$ that are elements of $\Delta^{(r)}$. We assume $\mathcal{S}$ is a smooth manifold on which $dx^1 \wedge \cdots \wedge dx^p \neq 0$. Then there exist disjoint sets of indices

$$A := \{(a, I)\} \quad \text{and } B := \{(b, J)\},$$

where $a, b \in \{1, \cdots, q\}$, I and J are multi-indices of order $\leq k-1$ so that $\mathcal{S}$ is the graph

$$u^b_J = \Phi^b_J(x, u^a_I :), \quad \text{for all } (b, J) \in B. \tag{2.3}$$

We take $(x, u^a_I : (a, I) \in A)$, as local coordinates of $\mathcal{S}$. Observe that if $u = u(x)$ is a solution of (2.1) then its $(k-1)$th jet graph $(x, u^{(k-1)}(x))$ is contained in $\mathcal{S}$ and for each $(a, I) \in A$, we have

$$du^a_I(x) = \begin{cases} \sum_{i=1}^p u^a_{Ii}(x) dx^i & \text{for} |I| \leq k-2, \\ \sum_{i=1}^p H^a_{Ii}(x, u^{(k-1)}) dx^i & \text{for } |I| = k-1, \end{cases}$$

where H's are as in (2.2). Substituting (2.3) for all u^b_J with $(b, J) \in B$ we obtain

$$du^a_I(x) = \sum_{i=1}^p \Psi^a_{Ii}(x, u^\alpha_L(x)) dx^i,$$

where all the indices (α, L) are in A. Thus on $\mathcal{S}$ we define independent 1-forms

$$\theta^a_I := du^a_I - \sum_{i=1}^p \Psi^a_{Ii}(x, u^\alpha_L : (\alpha, L) \in A) dx^i, \tag{2.4}$$

for all $(a, I) \in A$, which defines a plane field of $\mathcal{S}$ of dimension p. Then the smooth solutions of (2.1) are in one-to-one correspondence with the smooth integral manifolds of the Pfaffian system (2.4). Let $\theta = \{\theta^a_I : (a, I) \in A\}$. We set

$$d\theta^a_I \equiv \sum_{i<j} T^a_{Iij}(x, u^\alpha_J : (\alpha, J) \in A) dx^i \wedge dx^j, \qquad \text{mod } \theta.$$

If all of $T := \{T^a_{Iij} : (a, I) \in A\}$ are identically zero then by the Frobenius theorem $\mathcal{S}$ is foliated by integral manifolds. Otherwise, we consider (2.4) restricted to a subset $\mathcal{S}' \subset \mathcal{S}$, where $\mathcal{S}'$ is the common zero set of T^a_{Iij}. We assume that $\mathcal{S}'$ is a smooth manifold on which $dx^1 \wedge \cdots \wedge dx^p \neq 0$. Making use of the following theorem we further check the integrability of (2.4) on $\mathcal{S}'$.

Theorem 2.1. *Let M be a smooth manifold of dimension n. Let $\theta := (\theta^1, \cdots, \theta^s)$ be a set of independent 1-forms on M and $\mathcal{D} :=< \theta >^{\perp}$ be the $(n-s)$ dimensional plane field annihilated by θ. Suppose that N is a submanifold of M of dimension $n-r$ defined by $T_1 = \cdots = T_r = 0$, where T_i are smooth real-valued functions of M such that $dT_1 \wedge \cdots \wedge dT_r \neq 0$ on N. Then the following are equivalent :*

(i) $\mathcal{D}$ is tangent to N.

(ii) $dT_j \equiv 0, \mod \theta^1, \ldots, \theta^s$ on N, for each $j = 1, \ldots, r$.

(iii) $i^*\theta^1, \cdots, i^*\theta^s$ have rank $s-r$, where $i : N \hookrightarrow M$ is the inclusion.

Proof. $(i) \Leftrightarrow (ii)$

Let $P \in N$. Then

$\mathcal{D}_P$ is tangent to N.

$\Leftrightarrow < \theta^1, \cdots, \theta^s >^{\perp} \subset < dT_1, \cdots, dT_r >^{\perp}$ at P.

$\Leftrightarrow dT_i \in < \theta^1, \cdots, \theta^s >$ at P for each $i = 1, \cdots, r$.

$\Leftrightarrow dT_i \equiv 0, \mod \theta$ at P.

$(ii) \Leftrightarrow (iii)$

Assuming (ii), we have $dT_j = \sum_{k=1}^{s} a_{jk}\theta^k$ on N, $j = 1, \ldots, r$. Since $dT_1, \ldots, dT_r$ are linearly independent on N, the matrix (a_{jk}) has rank r. Then we have

$$0 = i^* dT_j = \sum_{k=1}^{s} a_{jk} i^*\theta^k, \; 1 \le j \le r.$$

This means that $i^*\theta^1, \cdots, i^*\theta^s$ have rank no greater than $k-r$. Since N is of dimension $n-r$, $i^*\theta^1, \cdots, i^*\theta^s$ have rank no less than $k-r$. Therefore we obtain (iii).

Conversely, (iii) implies that (after suitable index changes) $i^*\theta^{s-r+l} = \sum_{j=1}^{s-r} b_{lj} i^*\theta^j$, $1 \le l \le r$ for some functions on N. Then $i^*(\theta^{s-r+l} - \sum_{j=1}^{s-r} b_{lj}\theta^j) = 0$, $1 \le l \le r$. Since $dT_1, \ldots, dT_r$ are linearly independent on N, we have $\theta^{s-r+l} - \sum_{j=1}^{s-r} b_{lj}\theta^j = \sum_j c_{lj} dT_j$, $1 \le l \le r$ on N, where c_{ij} are functions on N and the matrix (c_{ij}) is nonsingular. Therefore we can solve dT_j in terms of θ^k's. □

Let p' be the dimension of $\mathcal{S}'$. If $dT^a_{Iij} \equiv 0, \mod \theta$, on $\mathcal{S}'$ then $\mathcal{S}'$ is foliated by integral manifolds, thus we have $(p'-p)$-parameter family of solutions. If $dT^a_{Iij} \neq 0, \mod \theta$, on $\mathcal{S}'$ then we consider the common zero set $\mathcal{S}''$ of the functions $T^a_{Iij\lambda\mu}$ which is defined by

$$dT^a_{Iij} = \sum_{\lambda<\mu} T^a_{Iij\lambda\mu} dx^\lambda \wedge dx^\mu.$$

Repeating the same argument we eventually reach either empty set or a submanifold $\tilde{\mathcal{S}}$ of dimension $\tilde{p}$ such that

$$i^*\theta^a = 0, \qquad \text{for all } a = 1, \dots, s,$$

where $i : \tilde{\mathcal{S}} \to \mathcal{S}$ is the inclusion. In the former case, no solutions exist. In the latter case, there exists a $(\tilde{p} - p)$-parameter family of solutions.

Example 2.2.

$$\begin{cases} u_x = a(x, y, u) \\ u_y = b(x, y, u). \end{cases}$$

This is a complete system of first order for an unknown function u of two variables (x, y). In this case $\mathcal{S}$ is the 0-th jet space of u, that is, $\mathcal{S} = \{(x, y, u)\} = \mathbb{R}^3$ and

$$\theta = du - adx - bdy.$$

Then $d\theta = Tdx \wedge dy, \quad \text{mod } \theta$, where

$$T = a_y + a_u b - b_x - b_u a.$$

If T is identically zero then there exists a solution through any point (x_0, y_0, u_0). If T is a function of (x, y) and not identically equal to zero then $dx \wedge dy = 0$ on $T = 0$ and therefore, there is no solution.

As a special case of the above consider the following

Example 2.3.

$$\begin{cases} u_x = a(x, y) + u^2 \\ u_y = 1. \end{cases} \tag{2.5}$$

If a function $u(x, y)$ is a solution its graph must be contained in $T := a_y + 2u = 0$. This implies that $u = -\frac{1}{2}a_y$, which is indeed a solution if and only if it satisfies (2.5) , namely,

$$\begin{cases} -\frac{1}{2}a_{yx} = a + \frac{1}{4}{a_x}^2 \\ -\frac{1}{2}a_{yy} = 1. \end{cases} \tag{2.6}$$

We see that (2.6) is equivalent to

$$dT \wedge \theta = 0, \quad \text{on } T = 0.$$

Example 2.4.

$$\begin{cases} u_x + u_y = 1 \\ u_{yy} = a(x, y, u, u_x). \end{cases} \tag{2.7}$$

Differentiating the first equation of (2.7) with respect to x and y, respectively, and solving these with the second equation of (2.7) for all the second order partial derivatives of u we obtain

$$u_{xx} = a, \quad u_{xy} = -a, \quad u_{yy} = a.$$

Introduce new variables p and q for u_x and u_y, respectively, and consider the submanifold $\mathcal{S}$ of dimension 4 given by $p + q = 1$ in the first jet space $J^1(\mathbb{R}^2, \mathbb{R}) = \{(x, y, u, p, q)\}$. The first jet graph of a solution is an integral manifold that is contained in $\mathcal{S}$ of the Pfaffian differential system

$$\begin{cases} \theta_0 := du - pdx - qdy \\ \theta_1 := dp - adx + ady \\ \theta_2 := dq + adx - ady. \end{cases}$$

On $\mathcal{S}$ we have $\theta_2 = -\theta_1$. Thus our problem is finding integral manifolds of the following Pfaffian system on $\mathcal{S}$:

$$\theta : \begin{cases} \theta_0 := du - pdx - (1-p)dy \\ \theta_1 := dp - adx + ady. \end{cases}$$

with the independence condition $dx \wedge dy \neq 0$. To check the Frobenius integrability condition we compute $d\theta$ modulo θ. We have

$$d\theta_1 = T dx \wedge dy,$$

where

$$T(x, y, u, p) = a_x + a_u p + a_p a + a_y + a_u(1-p) - a_p a.$$

If T vanishes identically on $\mathcal{S}$ then $\mathcal{S}$ is foliated by 1-jet graph of solutions, thus we have 2-parameter family of solutions. Otherwise, we consider a submanifold $\mathcal{S}'$ of $\mathcal{S}$ given by $T = 0$. $\mathcal{S}'$ is of dimension 3. Generically $\mathcal{S}'$ is a submanifold of $\mathcal{S}$ of the form $p = p(x, y, u)$. If

$$dT = 0, \quad \mathrm{mod}\{\theta_0, \theta_1\}, \text{ on } \mathcal{S}' \tag{2.8}$$

$\mathcal{S}'$ is foliated by 1-jet graph of solutions, thus we have 1-parameter family of solutions. By Theorem 2.1 (2.8) is equivalent to the pull-back to $\mathcal{S}'$ of θ_0 and that of θ_1 are linearly dependent everywhere. If (2.8) does not hold we let

$$dT = T'dx \wedge dy, \quad \mod \{\theta_0, \theta_1\}.$$

Let $\mathcal{S}''$ be the submanifold of $\mathcal{S}'$ given by $T' = 0$. Then $\mathcal{S}''$ is of dimension 2. $\mathcal{S}''$ is graph of a solution if and only if

$$dT' = 0, \quad \mod \{\theta_0, \theta_1\}, \text{ on } \mathcal{S}''. \tag{2.9}$$

Thus (2.9) is a necessary and sufficient condition for a solution to exist.

§**3 Some local problems in CR geometry.** In this section we propose problems of determining the CR embeddibility into spheres and the existence of infinitesimal CR automorphisms. We recall the basic definitions first: Let M be a differentiable manifold of dimension $2n+1$. A CR structure of hypersurface type on M is a subbundle $\mathcal{V}$ of the complexified tangent bundle $T_{\mathbb{C}}M$ having the following properties :
i) each fiber is of complex dimension n,
ii) $\mathcal{V} \cap \bar{\mathcal{V}} = \{0\}$,
iii) $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ (integrability).
Given a CR structure $\mathcal{V}$ the Levi form $\mathcal{L}$ is defined by

$$\mathcal{L}(L_1, L_2) := \sqrt{-1}[L_1, \bar{L_2}], \quad \mod \quad (\mathcal{V} + \bar{\mathcal{V}}).$$

$\mathcal{L}$ is a hermitian form on $\mathcal{V}$ with values in $T_{\mathbb{C}}M/(\mathcal{V}+\bar{\mathcal{V}})$. M is said to be strictly pseudoconvex if $\mathcal{L}$ is definite. A real hypersurface in a complex manifold has natural CR structure induced from the complex structure of the ambient space. A complex valued function f is called a CR function if f is annihilated by $\bar{\mathcal{V}}$. Let $\{L_1, \cdots, L_n\}$ be a set of complex vector fields that generates $\mathcal{V}$. Then f is a CR function if and only if

(3.1) $\bar{L}_i f = 0, \quad i = 1, \cdots, n$ (tangential Cauchy-Riemann equations).

A system of CR functions $(f_1, \cdots, f_{n+1})$ with $df_1 \wedge \cdots \wedge df_{n+1} \neq 0$ is a CR immersion into C^{n+1}.
Let $(N, \mathcal{V}')$ be a CR manifold of dimension $2N+1, \quad N \geq n$, with the CR

structure bundle $\mathcal{V}'$. A mapping $F : M \to M'$ is called a CR mapping if F preserves the CR structure, that is,

$$F_* \mathcal{V} \subset \mathcal{V}'.$$

A CR manifold of dimension $2N-1$ embedded in a CR manifold of dimension $2N+1$ is called a CR hypersurface.

Webster proved in [Web] that every CR hypersurface in the sphere S^{2N+1}, $(N \geq 4)$ is rigid. This implies that if M is a CR hypersurface in S^{2N+1} then a CR mapping of M into S^{2N+1} is determined by its finite jet at a point. That is, there exists a non-negative integer k so that if two CR mappings f and g have the same kth jet at a point then $f = g$. Our question is whether the $(k+1)$th jet of embeddings depends continuously on the k-th jet as functions given a priori independently of particular embeddings:

Problem 3.1. *Let M be a smooth CR manifold of dimension $2N-1$, $(N \geq 4)$. A mapping $F = (f^1, \cdots, f^{N+1}) : M \to S^{2N+1}$ is a CR mapping if F satisfies (3.1) with $i = 1, \cdots, N-1$ and*

$$f^1 \bar{f}^1 + \cdots + f^{N+1} \bar{f}^{N+1} = 1. \tag{3.2}$$

Does the system (3.1)-(3.2) admit prolongation to a complete system of finite order?

In [H1] the author constructed a complete system for CR mappings of a strongly pseudo-convex CR manifold into a CR submanifold of higher dimension under a certain generic assumption. Problem 3.1 asks whether one can remove the generic assumptions if the CR mapping is an embedding into a sphere as a hypersurface.

Problem 3.2 (CR embeddibility into spheres). *Discuss the existence of solutions for the complete system constructed in Problem 3.1. Express the Frobenius integrability conditions in terms of the Chern-Moser invariants of M.*

Results for some special cases have been obtained in [Kim] and [Oh]. A real vector field X on a CR manifold $(M, \mathcal{V})$ is an infinitesimal CR automorphism if the flow maps ϕ_t of X are the local CR diffeomorphisms for each t with $|t| \leq \epsilon$. A smooth vector field X is an infinitesimal CR automorphism if and only if the Lie derivative of a section L of $\mathcal{V}$ with respect to X is again a section of $\mathcal{V}$, that is, $[X, L] \in \mathcal{V}$. We set

$$[X, L_i] = \alpha_i^j L_j, \quad \text{(summation convention)}, \tag{3.3}$$

for some functions α_i^j for each $i = 1, \dots, n$.

Theorem 3.3. *Let M^{2n+1} be a C^ω CR manifold of nondegenerate Levi form. Then the defining equation (3.3) of the infinitesimal CR automorphisms admits prolongation to a complete system of order 3. Therefore, a C^3 infinitesimal CR automorphism is in fact C^ω. Moreover, the set of infinitesimal CR automorphisms of M forms a finite dimensional Lie algebra.*

This is a well known fact in the theory of CR structures due to Cartan, Tanaka [T1] and Chern-Moser [CM]. A direct proof is found in [H2].

Problem 3.4 (Existence of infinitesimal CR automorphisms). *Express the Frobenius integrability of the complete system for the infinitesimal CR automorphisms in terms of the Chern-Moser invariants.*

For the cases of three dimensional CR manifolds of non-degenerate Levi form H.D. Lee [Lee] proved the following: If the manifold is a hyperquadric then there exists 8-parameter family of CR automorphisms. Otherwise, there exists at most 3-dimensional CR automorphisms.

References

[**BCGGG**] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, *Exterior differential systems*, Springer-Verlag, New-York, 1986.

[**BS**] D. Burns and S. Schnider, *Real hypersurfaces in complex manifolds*, Proc. Symp. Pure Math., Amer. Math. Soc. **30** (1977), 141-168.

[**CH1**] C. K. Cho and C. K. Han, *Compatibility equations for isometric embeddings of Riemannian manifolds*, Rocky Mt. J. Math. **23** (1993), 1231-1252.

[**CH2**] C. K. Cho and C. K. Han, *Finiteness of infinitesimal deformations of CR mappings of CR manifolds of nondegenerate Levi form*, J. Korean Math. Soc. **139** (2002), 91-102.

[**CM**] S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219-271.

[**GJ**] P. Griffiths and G. Jensen, *Differential systems and isometric embeddings*, Ann. of Math. Studies, No. 114, Princeton U. Press, Princeton, NJ, 1987.

[**H1**] C. K. Han, *Complete differential system for the mappings of CR manifolds of nondegenerate Levi forms*, Math. Ann. **309** (1997), 229-238.

[**H2**] C. K. Han, *Equivalence problem and complete system of finite order*, J. Korean Math. Soc. **37** (2000), 225-243.

[**Ja**] H. Jacobowitz,, *An introduction to CR structures*, Amer. Math. Soc., Providence, RI, 1990.

[**JChH**] J. S. Cho and C. K. Han, *Complete prolongation and the Frobenius integrability for overdetermined systems of partial differential equations*, J. Korean Math. Soc. **39** (2002), 237-252.

[**Kim**] S. Y. Kim, *Local embeddibility of pseudo-hermitian manifolds into spheres,*, preprint.

[**Kura**] M. Kuranishi, *On E. Cartan's prolongation theorem of exterior differential systems*, Amer. J. Math. **79** (1957), 1-47.

[**Lee**] H. D. Lee, *Existence of infinitesimal automorphisms in the three dimensional CR manifolds of nondegenerate Levi form,*, preprint.

[**Oh**] J. W. Oh, *CR embeddibility of ellipsoids into spheres*, preprint.

[**Olver**] P. J. Olver, *Applications of Lie groups to differential equations*, Springer-Verlag, New York, 1993.

[**Spi**] M. Spivak, *A comprehensive introduction to differential geometry, Vol. 2*, Publish or Perish, 1970.

[**T1**] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables*, J. Math. Soc. Japan **14** (1962), 397-429.

[**T2**] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japanese J. Math. **2** (1976), 131-190.

[**Web**] S. Webster, *The rigidity of CR hypersurfaces in a sphere*, Indiana U. Math. J. **28** (1979), 405-416.

[**YY**] K. Yamaguchi and T. Yatsui, *Geometry of higher order differential equations of finite type associated with symmetric spaces,*, Hokkaido U. Preprint Series in Math. **524** (May 2001).

C.K. Han : Department of Mathematics, Seoul National University, Seoul 151-742, Korea. E-mail address : ckhan@math.snu.ac.kr