Solving a class of linear projection equations*

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Summary. Many interesting and important constrained optimization problems in mathematical programming can be translated into an equivalent linear projection equation

$$u = P_{\Omega}[u - (Mu + q)].$$

Here, $P_{\Omega}(\cdot)$ is the orthogonal projection on some convex set Ω (e.g. $\Omega = \mathbb{R}^n_+$) and M is a positive semidefinite matrix. This paper presents some new methods for solving a class of linear projection equations. The search directions of these methods are straighforward extensions of the directions in traditional methods for unconstrained optimization. Based on the idea of a projection and contraction method [7] we get a simple step length rule and are able to obtain global linear convergence.

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1. Introduction

We consider a class of linear projection equations (abbreviated to LPE)

$$\text{(1)} \qquad \text{(LPE)} \qquad u = P_{\Omega}[u - (Mu + q)],$$

where $M \in \mathbb{R}^{n \times n}$ is a positive semidefinite matrix (i.e. $u^T Mu \geq 0 \;\; \forall u$, but M not necessarily symmetric), $q \in \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a closed convex set and $P_{\Omega}(\cdot)$ denotes the projection on the set Ω. It is well known [2], that the linear projection equation (1) is equivalent to the following linear variational inequality

$$\text{(2)} \qquad \text{(LVI)} \qquad u \in \Omega, \qquad (v - u)^{\mathrm{T}}(Mu + q) \geq 0 \qquad \text{for all } v \in \Omega.$$

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Linear projection equations arise in numerous fields and play a significant role in mathematical programming. The linear complementarity problem

$$\text{(3)} \qquad \text{(LCP)} \qquad u \ge 0, \qquad (Mu + q) \ge 0, \qquad u^{\mathrm{T}}(Mu + q) = 0$$

is equivalent to a special (LPE) with $\Omega = \{u \in \mathbb{R}^n \mid u \ge 0\}$ [15]. Various constrained least squares problems [19] and convex quadratic programming problems can be tranlated into a linear projection equation (1) in which Ω is a general orthant [7].

The complementarity problem has been studied starting with the works of Cottle, Dantzig [3], and Lemke [12,13] and has been developed by many others. There is already a substantial number of algorithms for solving linear projection equations [1, 4-11, 15-19], especially for linear complementarity problems and linear constrained least squares problems. Let

$$\Omega^* = \{u^* \mid u^* \text{ is a solution of (LPE)}\}$$

be the solution set of (1) and

$$e(u) := u - P_\Omega[u - (Mu + q)]. \tag{4}$$

be the "error" by which a given point u fails to satisfy (1). In the projection and contraction method of [7, 8], the vector

$$g(u) = M^{\mathrm{T}}e(u) + (Mu + q) \tag{5}$$

is used as the search direction. The recursion

$$u^{k+1} = P_\Omega[u^k - \rho(u^k)g(u^k)] \tag{6}$$

with

$$\rho(u) = \frac{\|e(u)\|^2}{\|(M^{\mathrm{T}} + I)e(u)\|^2} \tag{7}$$

produces a sequence $\{u^k\} \subset \Omega$, which satisfies

$$\|u^{k+1} - u^*\|^2 \le \|u^k - u^*\|^2 - \rho(u^k)\|e(u^k)\|^2. \tag{8}$$

The main advantages of this method are its simplicity and ability to handle the linear projection equation (1) while some other algorithms (e.g. [14]) can only solve special cases of (1). Each iteration of this method consists essentially of only two matrix vector products and two projections of a vector onto Ω. Therefore the method allows the optimal exploitation of the sparsity of the matrix M and may thus be efficient for large sparse problems [10]. Since the method is easy to parallelise, it may be even more favorable for parallel computation. However, for ill-conditioned problems, the search direction (5) may lead to a very slow convergence.

Our objective in this paper is to find better search directions and thereby to construct more efficient methods for solving problem (1). Throughout this paper we assume that $\Omega^* \neq \emptyset$ and that the projection onto Ω is simple to carry out (e.g. when Ω is a general orthant, a box, a sphere, a cylinder or a subspace).

The paper is organized as follows. In Sect. 2 we illustrate our motivation. The main theorem is proved in Sect. 3. In Sect. 4 some new methods are presented and their contractive properties are shown. In Sect. 5 we prove convergence. Finally, in Sect. 6, we give some extensions and conclusions.

We use the following notation. For $u \in \mathbb{R}^n$, the component i is denoted by u_i. A superscript such as in u^k refers to a specific vector and usually denotes iteration index. The Eucliden norm and the max-norm will be denoted by $\| \cdot \|$ and $\| \cdot \|_\infty$, respectively. Throughout this paper, G denotes a positive definite matrix and $\|u\|_G$ denotes $(u^{\mathrm{T}}Gu)^{\frac{1}{2}}$.

2. Motivation

In order to illustrate our motiviation, we let

$$f(u) = \frac{1}{2}u^{\mathrm{T}}Mu + q^{\mathrm{T}}u$$

and M be symmetric positive definite, and consider the following unconstrained optimization problem

$$\min_{u \in \mathbb{R}^n} f(u). \tag{9}$$

Solving problem (9) is equivalent to finding a zero point of $\nabla f(u)$. The search direction in classical methods for unconstrained optimization is

$$d(u) = Q\nabla f(u) \tag{10}$$

with different matrices Q. If $Q = I$, we obtain the direction of the steepest descent method. Setting $Q = [\nabla^2 f(u)]^{-1}$ yields the direction of Newton's method. When $Q = \sigma I + (\nabla^2 f(u))^{-1}$ or $Q = [\sigma I + \nabla^2 f(u)]^{-1}$ for some nonegative value of $\sigma > 0$, the search direction can be regarded as some combination of steepest descent (σ very large) and Newton's method ($\sigma = 0$).

For the convex constrained optimization problem

$$\min\{f(u) \mid u \in \Omega\}, \tag{11}$$

the Kuhn-Tucker theorem tells us that u^* is a minimum if $u^* \in \Omega$ and it satisfies

$$(u - u^*)^{\mathrm{T}}\nabla f(u^*) \geq 0 \qquad \text{for all } u \in \Omega.$$

This means that u^* is a zero point of the function $e(u)$. Note that in the case of $\Omega = \mathbb{R}^n$, $e(u) = \nabla f(u)$. Since most search directions in unconstrained optimization are constructed from $\nabla f(u)$, a natural question is whether we can build useful search directions for the constrained optimization problem (11) based on $e(u)$. Further, for problem (1), if we take

$$d(u) = Qe(u) \tag{12}$$

as the search direction, which step length should be taken?

3. The main theorem

The following theorem plays an important role in our new methods.

Theorem 1. *Let* $u^* \in \Omega^*$. *Then*

$$(u - u^*)^{\mathrm{T}}(I + M^{\mathrm{T}})e(u) \geq \|e(u)\|^2 + (u - u^*)^{\mathrm{T}} M(u - u^*) \qquad \forall u \in \mathbb{R}^n. \tag{13}$$

Proof. Since $\Omega \subset \mathbb{R}^n$ is a closed convex set and $u^* \in \Omega$, we know by the properties of a projection on a closed convex set [14, Appendix B] that

$$\{v - P_\Omega(v)\}^{\mathrm{T}}\{P_\Omega(v) - u^*\} \geq 0 \qquad \forall v \in \mathbb{R}^n.$$

By setting $v := u - (Mu + q)$ we obtain

$$\{e(u) - (Mu + q)\}^{\mathrm{T}}\{P_\Omega[u - (Mu + q)] - u^*\} \geq 0. \tag{14}$$

Since $P_\Omega(\cdot) \in \Omega$, it follows from (2) that

$$(Mu^* + q)^{\mathrm{T}}\{P_\Omega[u - (Mu + q)] - u^*\} \geq 0. \tag{15}$$

Adding (14) and (15) we get

$$\{e(u) - M(u - u^*)\}^{\mathrm{T}}\{u - u^* - e(u)\} \geq 0 \tag{16}$$

and it follows that

$$\begin{aligned}(u - u^*)^{\mathrm{T}}&(I + M^{\mathrm{T}})e(u) \\ &\geq \|e(u)\|^2 + (u - u^*)^{\mathrm{T}} M(u - u^*).\end{aligned}$$

□

A similar but weaker result of Theorem 1 was given in [9]. The above proof is an improved version of the one in [9]. We point out that inequality (16) is sharp. This can easily be seen by setting $M = I$. We then obtain $\{e(u) - (u - u^*)\}^{\mathrm{T}}\{u - u^* - e(u)\} = 0$. This implies that also the result of Theorem 1 is tight.

Remark. The methods in [7] and [8] take $g(u)$ as the search direction. It was shown that

$$(u - u^*)^{\mathrm{T}} g(u) \geq e(u)^{\mathrm{T}}(Mu + q).$$

But only under the assumption that $u \in \Omega$ can we prove

$$e(u)^{\mathrm{T}}(Mu + q) \geq \|e(u)\|^2.$$

Therefore, $-g(u)$ is a descent direction of $F(u) = \frac{1}{2}\|u - u^*\|^2$ at $u \in \Omega$. However, here the assertion (13) in Theorem 1 is true for all $u \in \mathbb{R}^n$. Although, as in [9], $-(I + M^{\mathrm{T}})e(u)$ can be taken as a descent direction of $F(u)$ for all $u \in \mathbb{R}^n$, Theorem 1 offers us the possibility to construct better search directions.

4. The methods and their contractive properties

In this section, based on Theorem 1, we give some new methods for solving linear projection equations and show their contractive properties. The iterative scheme of these methods is

$$u^{k+1} = u^k - \rho(u^k)Qe(u^k) \tag{17}$$

with different matrices Q and steplengths $\rho(u)$.

Method 1. (for symmetric $M \geq 0$)

$$Q = I, \qquad \rho(u) = \frac{\|e(u)\|^2}{e(u)^{\mathrm{T}}(I+M)e(u)}.$$

Method 2. (for symmetric $M > 0$)

$$Q = M^{-1}, \qquad \rho(u) = \frac{\|e(u)\|^2}{e(u)^{\mathrm{T}}(I+M^{-1})e(u)}.$$

Method 3. (for symmetric $M > 0$)

$$Q = I + M^{-1}, \qquad \rho(u^k) = \frac{\|e(u)\|^2}{\|(I+M^{-1})e(u)\|_M^2}.$$

Method 4. (for $M \geq 0$ but not necessarily symmetric)

$$Q = (I+M)^{-1}, \qquad \rho(u) = 1.$$

The first method can be viewed as an extention of the steepest descent method for unconstrained optimization, because we take $e(u)$ as the search direction and $e(u)$ is the residue of the projection equation. Obviously, each iteration of this method consists essentially of only a projection to Ω and the computation of Mu and $Me(u)$.

The second method can be viewed as an extention of Newton's method for unconstrained optimization. As in Method 1, each iteration of this method consists essentially of only a projection to Ω and the computation of Mu and $M^{-1}e(u)$.

Method 3 can be regarded as a combination of steepest descent and Newton's method for unconstrained optimization. Method 4 can be viewed as an extention of the Levenberg-Marquardt method for unconstrained optimization.

Theorem 2. *The sequence $\{u^k\}$ generated by each method of methods 1–4 for (LPE) satisfies*

$$\begin{aligned} \|u^{k+1} - u^*\|_G^2 \leq{}& \|u^k - u^*\|_G^2 - \rho(u^k)\|e(u^k)\|^2 \\ & - 2\rho(u^k)\cdot(u^k-u^*)M(u^k-u^*) \qquad \forall u^* \in \Omega^* \end{aligned} \tag{18}$$

where

$$G = \begin{cases} I+M & \textit{in Method 1} \\ (I+M)M & \textit{in Method 2} \\ M & \textit{in Method 3} \\ (I+M^{\mathrm{T}})(I+M) & \textit{in Method 4} \end{cases}$$

Proof. First,

$$\begin{aligned} \|u^{k+1} - u^*\|_G^2 &= \|(u^k - u^*) - \rho(u^k)Qe(u^k)\|_G^2 \\ &= \|u^k - u^*\|_G^2 - 2\rho(u^k)(u^k-u^*)GQe(u^k) \\ &\quad + \rho^2(u^k)e(u^k)^{\mathrm{T}}Q^{\mathrm{T}}GQe(u^k). \end{aligned}$$

Note that in all cases

$$GQ = I + M^{\mathrm{T}}$$

and

$$\rho(u) \cdot e(u)^{\mathrm{T}} Q^{\mathrm{T}} G Q e(u) = \|e(u)\|^2.$$

Using (13) the theorem is proved. □

The sequence $\{u^k\}$ generated by these methods does not necessarily lie in Ω. In general, (because $G \neq I$ and the projection is an orthogonal projection with respect to the Euclidean norm), we can not prove that $\{u^k\}$ satisfies $\|P_\Omega[u^{k+1}] - u^*\|_G \leq \|u^k - u^*\|_G$ even if $u^k \in \Omega$. Note that in all of these methods, the steplength ρ is bounded below. Therefore, there is a $c > 0$, so that the sequence $\{u^k\}$ generated by each of these methods satisfies

$$(19) \qquad \|u^{k+1} - u^*\|_G^2 \leq \|u^k - u^*\|_G^2 - c\|e(u^k)\|^2 \qquad \forall u^* \in \Omega^*.$$

Due to (19) and the fact that each iteration performs a projection onto Ω (in the computation of $e(u)$), we call these methods the projection and contraction methods (PC methods).

5. Convergence

The PC methods in this paper generate an infinite sequence $\{u^k\}$, which is not necessarily contained in the feasible set Ω, but, will be asymptotically feasible as $e(u^k) \to 0$, and, in fact converges to a solution of (LPE).

Theorem 3. *If the sequence $\{u^k\}$ satisfies (19), then it converges to a solution point u^*.*

Proof. See [9], Theorem 3. □

In [9] we have also proved that an inequality of the form (19) implies the global linear convergence in the case that Ω is an orthant.

Theorem 4. *If the sequence $\{u^k\}$ satisfies (19) and $\Omega = \{u \,|\, u \geq 0\}$, then $\{u^k\}$ converges to a solution point $u^* \in \Omega^*$ globally linearly.*

Proof. See [9], Theorem 4. □

The iterative scheme of the fundamental projection method (see [2])

$$(20) \qquad u^{k+1} = P_\Omega[u^k - (Mu^k + q)],$$

belongs to the class of steepest descent methods. In the case that M is positive semidefinite, we denote the largest and the smallest eigenvalue of the matrix M by $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$, respectively. If $0 < \delta \leq \lambda_{\min}(M) \leq \lambda_{\max}(M) \leq 2 - \delta$, then the sequence $\{u^k\}$ generated by the fundemental projection method (20) satisfies

$$
\begin{aligned}
\|e(u^{k+1})\| &= \|u^{k+1} - P_\Omega[u^{k+1} - (Mu^{k+1} + q)]\| \\
&= \|P_\Omega[u^k - (Mu^k + q)] - P_\Omega[u^{k+1} - (Mu^{k+1} + q)]\| \\
&\le \|(I - M)(u^k - u^{k+1})\| \\
&= \|(I - M)e(u^k)\| \\
&\le (1 - \delta)\|e(u^k)\|
\end{aligned} \tag{21}
$$

and

$$
\begin{aligned}
\|u^{k+1} - u^*\| &= \|P_\Omega[u^k - (Mu^k + q)] - u^*\| \\
&= \|P_\Omega[u^k - (Mu^k + q)] - P_\Omega[u^* - (Mu^* + q)]\| \\
&\le \|(I - M)(u^k - u^*)\| \\
&\le (1 - \delta)\|u^k - u^*\|.
\end{aligned} \tag{22}
$$

In the following we prove that the PC methods in this paper have similar properties. Under the assumptions that M is positive definite and $\|M\| \le 2$, the seqeunce $\{\|e(u^k)\|\}$ generated by Method 1 or Method 4 is monotonically decreasing. More precisely, we have the following theorem:

Theorem 5. *Let M be positive semidefinite and symmetric. If $\lambda_{\max}(M) \le 2$, then the sequence $\{e(u^k)\}$ generated by Method 1 or Method 4 for (LPE) satisfies*

$$
\|e(u^{k+1})\| \le \|e(u^k)\|. \tag{23}
$$

Moreover, if $\delta \le \lambda_{\min}(M) \le \lambda_{\max}(M) \le 2 - \delta$ for some $\delta > 0$, then the sequence $\{e(u^k)\}$ satisfies

$$
\|e(u^{k+1})\| \le (1 - \frac{\delta}{3})\|e(u^k)\|. \tag{24}
$$

Proof. First, in Method 1, under the assumptions we have $\frac{1}{3} \le \rho(u^k) \le 1$. By using

$$
u^{k+1} = u^k - \rho(u^k)e(u^k),
$$

we get

$$
\begin{aligned}
\|e(u^{k+1})\| &= \|u^k - \rho(u^k)e(u^k) - P_\Omega[u^{k+1} - (Mu^{k+1} + q)]\| \\
&= \|(1 - \rho(u^k))e(u^k) + u^k - e(u^k) - P_\Omega[u^{k+1} - (Mu^{k+1} + q)]\| \\
&\le \|(1 - \rho(u^k))e(u^k)\| + \|P_\Omega[u^k - (Mu^k + q)] - P_\Omega[u^{k+1} - (Mu^{k+1} + q)]\| \\
&\le (1 - \rho(u^k))\|e(u^k)\| + \|(I - M)(u^k - u^{k+1})\| \\
&\le (1 - \rho(u^k))\|e(u^k)\| + \rho(u^k)\|(I - M)\| \cdot \|e(u^k)\|.
\end{aligned}
$$

Since $0 \le \lambda_{\min}(M) \le \lambda_{\max}(M) \le 2$, it follows that $\|I - M\| \le 1$ and $\|e(u^{k+1})\| \le \|e(u^k)\|$. Moreover, if $\delta \le \lambda_{\min}(M) \le \lambda_{\max}(M) \le 2 - \delta$, then $\|I - M\| \le 1 - \delta$, and it follows that

$$
\|e(u^{k+1})\| \le (1 - \rho(u^k))\|e(u^k)\| + (1 - \delta)\rho(u^k)\|e(u^k)\| \le (1 - \frac{\delta}{3})\|e(u^k)\|.
$$

In Method 4, using $u^{k+1} = u^k - (I + M)^{-1}e(u^k)$ we get

$$
\begin{aligned}
\|e(u^{k+1})\| &= \|u^k - (I+M)^{-1}e(u^k) - P_\Omega[u^{k+1} - (Mu^{k+1}+q)]\| \\
&= \|e(u^k) - (I+M)^{-1}e(u^k) + u^k - e(u^k) - P_\Omega[u^{k+1} - (Mu^{k+1}+q)]\| \\
&\le \|(I+M)^{-1}Me(u^k)\| + \|P_\Omega[u^k - (Mu^k+q)] - P_\Omega[u^{k+1} - (Mu^{k+1}+q)]\| \\
&\le \|(I+M)^{-1}Me(u^k)\| + \|(I-M)(u^k - u^{k+1})\| \\
&= \|(I+M)^{-1}Me(u^k)\| + \|(I-M)(I+M)^{-1}e(u^k)\|.
\end{aligned}
$$

Let $M = U^{\mathrm{T}}\Sigma U$ be the Schur normal form of M with $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$. Then

$$
\begin{aligned}
\|e(u^{k+1})\| &\le \|(I+M)^{-1}Me(u^k)\| + \|(I-M)(I+M)^{-1}e(u^k)\| \\
&= \|(I+\Sigma)^{-1}\Sigma Ue(u^k)\| + \|(I-\Sigma)(I+\Sigma)^{-1}Ue(u^k)\|.
\end{aligned}
$$

Since $0 \le \sigma_i \le 2$, it follows that

$$
\frac{\sigma_i}{1+\sigma_i} + \frac{|1-\sigma_i|}{1+\sigma_i} \le 1
$$

and

$$
\begin{aligned}
\|e(u^{k+1})\| &\le \|(I+\Sigma)^{-1}\Sigma Ue(u^k)\| + \|(I-\Sigma)(I+\Sigma)^{-1}Ue(u^k)\| \\
&\le \|Ue(u^k)\| = \|e(u^k)\|.
\end{aligned}
$$

Moreover, if $\delta \le \sigma_i \le 2-\delta$, it is straightforward to prove that

$$
\frac{\sigma_i}{1+\sigma_i} + \frac{|1-\sigma_i|}{1+\sigma_i} \le 1 - \frac{\delta}{3}
$$

and it follows that

$$
\|e(u^{k+1})\| \le (1 - \frac{\delta}{3})\|e(u^k)\|. \qquad \square
$$

The following theorem contrasts the convergence proofs for the fundamental projection methods, in that no advance knowledge of the largest and smallest eigenvalue of M is required.

Theorem 6. *Let M be positive definite, then all four PC methods are globally linearly convergent.*

Proof. From (18) (Theorem 2) we have

$$
\|u^{k+1} - u^*\|_G^2 \le \|u^k - u^*\|_G^2 - 2c(u^k - u^*)^{\mathrm{T}}M(u^k - u^*).
$$

Since M is positive definite, there exists a $\tau > 0$, such that

$$
(u - u^*)^{\mathrm{T}}M(u - u^*) \ge \tau\|u - u^*\|_G^2.
$$

It follows that

$$
\|u^{k+1} - u^*\|_G^2 \le (1 - 2c\tau)\|u^k - u^*\|_G^2,
$$

which implies that $\{u^k\}$ converges to u^* globally and linearly. $\square$

6. Extensions and conclusions

Let $\alpha > 0$ be a constant. It is easy to see that problem (1) is equivalent to the following problem

$$(25) \qquad (\mathrm{LPE}_\alpha) \qquad u = P_\Omega[u - \alpha(Mu + q)].$$

Therefore, instead of taking M and q, we can use αM and αq in our PC methods.

In addition, for some parameter γ, $0 < \gamma < 2$, with the same direction $d(u)$ and its relevant steplength $\rho(u)$, the iterative scheme

$$(26) \qquad u^{k+1} = u^k - \gamma\rho(u^k)d(u^k)$$

produces a sequence $\{u^k\}$, which satisfies

$$(27) \qquad \|u^{k+1} - u^*\|_G^2 \leq \|u^k - u^*\|_G^2 - \gamma(2-\gamma)\rho(u^k)\|e(u^k)\|^2$$

and thus also converges to a solution point u^*. A close look at the inequalities used in the proof of Theorem 1 shows that the best choice of γ should be ≥ 1.

The search directions of the presented methods are the extensions of those in unconstrained optimization. If $\Omega = \mathbb{R}^n$, then the search direction in [7] is

$$g(u) = (M^{\mathrm{T}} + I)(Mu + q).$$

In this special case, the search directions in our new methods are

$$\begin{aligned} d_1(u) &= (Mu + q), && \text{(in Method 1)} \\ d_2(u) &= M^{-1}(Mu + q), && \text{(in Method 2)} \\ d_3(u) &= (I + M^{-1})(Mu + q), && \text{(in Method 3)} \\ d_4(u) &= (I + M)^{-1}(Mu + q), && \text{(in Method 4)} \end{aligned}$$

respectively. Although we have only proved linear convergence, by comparing analogous direction for unconstrained optimization we are convinced that the directions $d(u)$ developed in this paper are better than the direction $g(u)$ in the original PC methods [7,8]. We believe that the use of Newton-like directions will lead to a substantial improvement in computational efficiency.

Developing these methods for nonlinear problems is a topic of further research .

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