Solving Dispersionless Lax Equations

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Abstract

Geogdzhaev's method is used to derive the solution to the initial value problem of any dispersionless Lax equation. The particular case of the dispersionless Boussinesq equation is worked out in detail and possible generalisations are considered.

1 Introduction

There has been much progress recently on Hamiltonian systems of the form:

$$\frac{\partial \lambda_i}{\partial t} = u_i^j(\underline{\lambda})\,\frac{\partial \lambda_j}{\partial x} \tag{1}$$

Whenever such a system admits a change of variables which diagonalises the matrix u_i^j, so that the equation can be reduced to Riemann invariant form,

$$\frac{\partial \lambda_i}{\partial t} = v_i(\underline{\lambda})\,\frac{\partial \lambda_i}{\partial x} \tag{2}$$

and it also possesses non-trivial symmetries in the same diagonal form, then it can be solved exactly. Two important examples of such systems are Whitham's equations [1,2] and Benney's equations [3,4,5,6]. In [7] Tsarev showed how such diagonalisable systems can be solved; he gave the generalised hodograph solution

$$x + v_i(\underline{\lambda})t = w_i(\underline{\lambda}) \tag{3}$$

Here the functions w_i satisfy a set of *linear* equations; these are the compatibility conditions between (2) and a symmetry of it:

$$\frac{\partial \lambda_i}{\partial s} = w_i(\underline{\lambda})\,\frac{\partial \lambda_i}{\partial x} \tag{4}$$

In [8] Geogdzhaev showed how the classical solution to the simplest such system:

$$\frac{\partial u}{\partial t} = u\,\frac{\partial u}{\partial x} \tag{5}$$

the dispersionless KdV equation, could be considered as the appropriate limit of the solution of the KdV via the inverse scattering transform (but see also the work of Lax and Levermore [9]). There and in [10] he also considered the Zakharov reduction of the Benney hierarchy

$$\begin{aligned}\frac{\partial u_i}{\partial t} &= u_i\frac{\partial u_i}{\partial x} + \sum_{j=1}^{N}\frac{\partial h_j}{\partial x}\\ \frac{\partial h_i}{\partial t} &= \frac{\partial (u_i h_i)}{\partial x}\end{aligned} \tag{6}$$

This system may be considered as the quasi-classical, or dispersionless, limit of the N-component vector NLS equation. In [11] we showed how these results could be interpreted in terms of canonical transformations.

Here we consider another reduction of the Benney hierarchy, the dispersionless Lax equations [12].

2 Dispersionless Lax Equations

The Benney hierarchy admits many reductions. Among the simplest are the following, derived by the Gel'fand–Dikii 'fractional power' ansatz. We denote:

$$\Lambda = p^N + A_0 p^{N-2} + \cdots + A_{N-2} \tag{7}$$

and then construct the Hamiltonians, polynomial in p and the $(N-1)$ variables $A_i(x)$:

$$H_m = ((\Lambda)^{m/N})_+ \tag{8}$$

Here the fractional power is to be regarded as a formal series in p, with leading term p^m, and the subscript $_+$ denotes its polynomial part. The equations of motion of the hierarchy are then

$$\frac{\partial \Lambda}{\partial t_m} = \{H_m, \Lambda\} \tag{9}$$

Here the braces denote the canonical Poisson bracket with respect to x and p. We note that the right–hand side is identically zero if N divides m. It is useful to denote $\lambda = \Lambda^{1/N}$. Below we will perform a canonical transformation to variables where λ is the new momentum; then the new Hamiltonian is just λ^m.

It is necessary to suppose that the polynomial $\Lambda(p)$ has N distinct real zeroes, $p_1 > p_2 > \ldots > p_N$; it will have $(N-1)$ distinct real turning points between these, and we may see, from (9), that the values of Λ at these turning points are Riemann invariants. We denote the stationary value of Λ found between p_{k+1} and p_k as Λ_k, and the corresponding value of p as $\tilde{p}_k$. It is convenient to suppose that as $x \to -\infty$ each of these Riemann invariants tends monotonically to zero, so that in this limit $\Lambda \to p^N$, and hence $(p-\lambda) \to 0$.

The generating function of the canonical transformation we need is

$$S(x, \lambda) = \int_{-\infty}^{x} (p(x', \lambda) - \lambda)dx' + \lambda x \tag{10}$$

(always supposing the potentials A_i tend to zero sufficiently rapidly that this integral converges).We then have

$$p(x, \lambda) = \frac{\partial S}{\partial x} \tag{11}$$

while ξ, the canonical conjugate of λ, is given by

$$\xi = \frac{\partial S}{\partial \lambda} \tag{12}$$

We will be able to rewrite the equations of motion in the new variables below, but first it is important to make the definition of S as precise as possible. We choose λ to be analytic except on the real p-axis, between p_N and p_1. Just above the real p-axis, between p_{k+1} and p_k, λ has the argument $k\pi/N$; just below the cut, it has the argument $-k\pi/N$. In the λ-plane, this segment of the real p-axis thus appears as a *pair* of cuts, stretching along the rays $\arg(\lambda) = \pm k\pi/N$, as far as the branch points λ_k and λ_k^*, where

$$\lambda_k^N = \Lambda_k \tag{13}$$

which was defined above. This choice of λ satisfies

$$\lambda = p + O(1/p) \tag{14}$$

as $|p| \to \infty$. The inverse function $p(x, \lambda)$ is then seen to be analytic for all λ not on the rays

$$\arg(\lambda) = \pm k\pi/N \tag{15}$$

where $k = 1, \ldots, N-1$. On either side of these cuts, $p(x, \lambda)$ is real, between p_{k+1} and p_k; at the branch point λ_k, p takes the value $\tilde{p}_k$. Since the Riemann invariants Λ_k approach zero monotonically as $x \to -\infty$, so do the $2(N-1)$ branch points λ_k and λ_k^*. Therefore, for any λ on the ray $\arg(\lambda) = k\pi/N$, either $p(x,\lambda)$ is analytic for all x, or there is some unique value of x, $x_k^*(\lambda)$ say, such that $\lambda_k(x_k^*) = \lambda$. Since, for $x > x_k^*(\lambda)$, $p(x,\lambda)$ is real, we find that the imaginary part of S is independent of x:

$$Im(S(x,\lambda)) = \int_{-\infty}^{x_k^*(\lambda)} Im(p(x',\lambda) - \lambda)dx' + x_k^*(\lambda)Im(\lambda) \tag{16}$$

3 The Time Evolution

The time dependence of $p(x,\lambda)$ is as follows:

$$\frac{\partial p}{\partial t_m} = \frac{\partial}{\partial x} H_m(p(x,\lambda), x) \tag{17}$$

We thus obtain the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t_m} = H_m(p(x,\lambda), x) - \lambda^m = H_m(\frac{\partial S}{\partial x}, x) - \lambda^m \tag{18}$$

Therefore if $\arg(\lambda) = k\pi/N$ and $x > x_k^*(\lambda)$, $H_m(p(x,\lambda))$ is real, and we hence have:

$$\frac{\partial}{\partial t_m} Im(S(x,\lambda)) = -Im(\lambda^m) \tag{19}$$

It is thus rather more convenient to consider, instead of S, the function

$$\Phi = S + \lambda^m t_m \tag{20}$$

since $Im(\Phi)$ is time-independent on the cuts. Now, since p and Φ are both analytic away from the cuts, so is the expression

$$\Psi = \Phi - H_m(p,x)t_m - px \tag{21}$$

Since the term $H_m t_m + px$ is real on the cuts, evidently $Im(\Psi) = Im(\Phi)$ there. Finally we note that, since $S = \lambda x + O(1/\lambda)$,

$$\Psi = O(1/\lambda) \tag{22}$$

as $|\lambda| \to \infty$.

4 The Inverse Problem

Since Ψ is analytic away from the cuts, and tends to zero at infinity, we have, by Cauchy's theorem,

$$\Psi(x,\lambda') = -\frac{1}{2\pi i}\oint_\Gamma \frac{\Psi(x,\lambda)}{p(x,\lambda)-p(x,\lambda')}\frac{\partial p}{\partial\lambda}\,d\lambda \tag{23}$$

Here Γ is a contour which encircles the cuts anti–clockwise, but does not enclose λ'. We may write $\Gamma = \Gamma_- - \Gamma_+$, where Γ_+ is in the upper half λ-plane, Γ_- in the lower. Clearly we may replace Ψ by Φ in the integral without changing its value, as the difference between them is an entire function of p. Thus we obtain:

$$\Phi(x,\lambda') - H_m(p(x,\lambda'),x)t_m - p(x,\lambda')x =$$

$$-\frac{1}{2\pi i}\oint_\Gamma \frac{\Phi(x,\lambda)}{p(x,\lambda)-p(x,\lambda')}\frac{\partial p}{\partial\lambda}\,d\lambda \tag{24}$$

Differentiating with respect to λ', and denoting $\partial H_m/\partial p$ as v_m, we get:

$$\frac{\partial\Phi}{\partial\lambda'} - \frac{\partial p}{\partial\lambda'}(x + v_m(p(x,\lambda'),x)t_m) =$$

$$-\frac{\partial p}{\partial\lambda'}\frac{1}{2\pi i}\oint_\Gamma \frac{\Phi}{(p(x,\lambda)-p(x,\lambda'))^2}\frac{\partial p}{\partial\lambda}\,d\lambda =$$

$$-\frac{\partial p}{\partial\lambda'}\frac{1}{2\pi i}\oint_\Gamma \frac{\partial\Phi}{\partial\lambda}\frac{d\lambda}{p(x,\lambda)-p(x,\lambda')} =$$

$$\frac{\partial p}{\partial\lambda'}\frac{1}{\pi}\int_{\Gamma_+}\frac{\partial Im\Phi}{\partial\lambda}\frac{d\lambda}{p(x,\lambda)-p(x,\lambda')} \tag{25}$$

Here we have first integrated by parts, and then collapsed Γ onto the cuts, noting that $\Phi|_{\Gamma_+} = \Phi^*|_{\Gamma_-}$. At the $(N-1)$ points $\lambda_k(x)$, the derivative $\partial\lambda/\partial p$ vanishes, while $\partial\Phi/\partial\lambda$ is bounded. Thus the residue $\partial\Phi/\partial p$ vanishes also. We therefore obtain:

$$x + v_m(\tilde{p}_k(x),x)t_m = -\frac{1}{\pi}\int_{\Gamma_+}\frac{\partial Im\Phi}{\partial\lambda}\frac{d\lambda}{p(x,\lambda)-\tilde{p}_k(x)} \tag{26}$$

Here the left-hand side is precisely Tsarev's generalised hodograph formula, while the right-hand side has no x-dependence, except through the potentials $A_i(x)$. The equations may thus be regarded as a set of $N-1$ equations, depending on the parameters x and t_m, for the $N-1$ potentials A_i.

5 The Dispersionless Boussinesq Equation

This, after the dispersionless KdV, is the simplest system in this class; however it is sufficiently complicated to illustrate the method. The equation of motion is, from (10):

$$\Lambda = p^3 + A_0 p + A_1$$
$$\frac{\partial \Lambda}{\partial t_2} = \{p^2 + \frac{2}{3} A_0, \Lambda\} \tag{27}$$

or, more explicitly:

$$\frac{\partial A_0}{\partial t_2} = 2 \frac{\partial A_1}{\partial x}$$
$$\frac{\partial A_1}{\partial t_2} = -\frac{2}{3} A_0 \frac{\partial A_0}{\partial x} \tag{28}$$

The function $p(x, \lambda)$ is given by:

$$p = \sqrt[3]{(\lambda^3 - A_1 + \sqrt{(\lambda^3 - A_1)^2 + 4(A_0/3)^3})/2}$$
$$+ \sqrt[3]{(\lambda^3 - A_1 - \sqrt{(\lambda^3 - A_1)^2 + 4(A_0/3)^3})/2} \tag{29}$$

The branch cuts for the two functions $\lambda(p)$ and $p(\lambda)$, as well as the contour Γ, are shown below.

Figure 1: The branch cuts of the function $\lambda(p)$ and the contours $\Gamma_\pm$ in the p-plane.

The two characteristic speeds are:

$$\tilde{p}_1 = \sqrt{-A_0/3}$$

$$\tilde{p}_2 = -\sqrt{-A_0/3} \tag{30}$$

while the corresponding branch points in the upper half λ-plane are:

$$\lambda_1 = \alpha\sqrt[3]{-2(-A_0/3)^{3/2} - A_1}$$

$$\lambda_2 = \alpha^2\sqrt[3]{-2(-A_0/3)^{3/2} + A_1} \tag{31}$$

where α denotes $\exp(i\pi/3)$. We assume that $A_0 < 0$, so that the system is hyperbolic, and further that $4A_0^3 + 27A_1^2 < 0$, so that the cubic has three distinct real roots.

We then obtain:

$$\Phi = \int_{-\infty}^{x} (p(x', \lambda) - \lambda)dx' + \lambda x$$

$$x \pm \sqrt{-A_0/3}\, t_2 = \frac{1}{\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{1}{p(x,\lambda) \mp \sqrt{-A_0/3}}\, d(Im\Phi(\lambda(p,x))) \tag{32}$$

In the inversion formula, we note that Φ is real except between the roots of $\Lambda(p)$.

6 Conclusions

Here we have seen how the Cauchy problem for dispersionless Lax equations may be solved in an effective, explicit way by considering the generating function of a canonical transformation. Although the technical details of the method depend on the specific problem studied, the principle is the same in each case. It is therefore reasonable to hope that similarly effective solutions may be obtained to other open problems in this class. Some important examples are:

1. The Benney equations. These are equivalent to the Vlasov equation:

$$\frac{\partial f}{\partial t_2} + p\frac{\partial f}{\partial x} - \frac{\partial A_0}{\partial x}\frac{\partial f}{\partial p} = 0$$

$$A_n = \int_{-\infty}^{\infty} p^n f\, dp \tag{33}$$

   Here, in some cases, (with $f < 0$) we can obtain an equation like (25), but it is not clear how we can use this to obtain the solution of the problem.

2. The Whitham equations.

$$\frac{\partial \lambda_i}{\partial t_n} = \frac{\omega_n}{\omega_1}(\lambda_i, \underline{\lambda})\frac{\partial \lambda_i}{\partial x} \tag{34}$$

Here the ω_i are meromorphic differentials on the Riemann surface

$$\mu^2 = \prod_0^{2g}(\lambda - \lambda_i) \tag{35}$$

The differential ω_1 plays a role analogous to the momentum p, and we may construct a generating function S as above. The inversion problem is much more difficult, however, as the Cauchy kernel no longer takes a simple form.

3. The Zabolotskaya–Khokhlov equation.

$$\frac{\partial^2 A_0}{\partial t_3 \partial x} + \frac{1}{2}\frac{\partial^2 A_0^2}{\partial x^2} = \frac{\partial^2 A_0}{\partial t_2^2} \tag{36}$$

Although this is intimately connected with the Benney hierarchy, it is not itself a member of the class of systems (1); instead it arises as a consistency condition between different equations in the Benney hierarchy. Also its initial value problem is essentially different; only the moment A_0 is given, throughout the (x, t_2)-plane. It has been shown [13] that the initial value problem for this system is equivalent to the inverse scattering problem for the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t_2} = (\frac{\partial S}{\partial x})^2 + 2A_0(x, t_2) \tag{37}$$

This difficult problem is the classical limit of the inverse problem for the time-dependent Schrödinger equation, solved by Manakov [14].

7 References

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Figure 2: The branch cuts of the function $p(\lambda)$ and the contours $\Gamma_\pm$ in the λ-plane.